On the Siegel–Tatuzawa theorem

by

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1. Introduction. Let $d$ be the discriminant of a quadratic field $k$ and let

$$
\chi(n) = \left( \frac{d}{n} \right) \quad \text{(Kronecker's symbol)}.
$$

It is well known that if $L(s, \chi)$ has no zero in the interval $(1 - c_1 \log |d|, 1)$ then $L(1, \chi) > c_2 \log |d|$, where $c_1$ and $c_2$ are positive constants and $c_2$ depends upon $c_1$ (see Lemma 1). If, however, $L(s, \chi)$ has a real zero close to 1, the only non-trivial lower bounds that are known for $L(1, \chi)$ are ineffective. Siegel, for example, has shown that for any $\varepsilon > 0$

$$
L(1, \chi) > \frac{c(\varepsilon)}{|d|^{1/4}},
$$

where $c(\varepsilon)$ is an ineffective constant depending upon $\varepsilon$ [3], while Tatuzawa has shown [5] that if $1/1.2 > \varepsilon > 0$ and $|d| > e^{1/4}$ then with at most one exception

$$
L(1, \chi) > \frac{.655 \varepsilon}{|d|^{1/4}}.
$$

The main objective of this paper is to arrive at a result somewhat stronger than Tatuzawa's. Using a technique of Goldfeld [1], we prove:

Theorem 1. Let $d$ and $\chi$ have the meaning defined above and let $1/(6 \log 10) > \varepsilon > 0$. If $|d| > e^{1/4}$ then with at most one exception the following two expressions hold:

$$
L(1, \chi) > \min \left[ \frac{1}{7.735 \log |d|^4}, \frac{\varepsilon}{.349 |d|^{1/4}} \right],
$$

$$
L(1, \chi) > \min \left[ \frac{1}{7.735 \log |d|^4}, \frac{\varepsilon}{(1 + \varepsilon \log |d|)^4 (.596 |d|^{1.3658})} \right].
$$
We also show:

**Theorem 2.** Let \(1/1000 > \epsilon > 0\) and suppose the exceptional quadratic field in the above theorem is imaginary with class number \(h_0\), For any other discriminant \(d, |d| > \epsilon^{-2}\)

\[
L(1, \chi) > \min \left[ 1, \frac{7.738}{\log |d|}, \frac{1}{15.350 h_0 (\log |d|)^{1.5134 h_0}} \right].
\]

This implies large values for all \(L(1, \chi)\) if there exists just one imaginary quadratic field with a large discriminant and small class number.

2. The proof of Theorem 1 depends upon several lemmas.

**Lemma 1.** Let \(a, b, c\) be as above, \(|d| > 10^4\). If \(L(s, \chi) \neq 0\) on the interval \((\beta, 1)\) and \(1 - \beta < 11.657 \log |d|^{-1}\), then

\[
L(1, \chi) > 1.507 (1 - \beta).
\]

If \(L(s, \chi) \neq 0\) on the interval \((0, 1)\) then

\[
L(1, \chi) > 1 \cdot 1.503 \log |d|.
\]

**Proof.** Let \(a = -\frac{1}{2} - \beta\) where \(0 < \beta < 1\) and let \(x = |d|^{\frac{1}{2}}\), \(A > 0\). If \(\zeta_k(s)\) is the zeta function of \(k\), then by the functional equation

\[
\zeta_k \left[ \frac{1}{2} + it \right] = \left( \frac{|d|}{(2\pi)^{1/2}} \right)^{-1} \left( \frac{t}{|d|^{-1/2} + it} \right)^{z_k} \left( \frac{t}{|d|^{-1/2} + it} \right)^{1/2} \zeta_k \left[ \frac{1}{2} + it \right].
\]

We note first that

\[
|\zeta_k \left[ \frac{1}{2} + it \right]| \leq |\zeta_k \left[ \frac{1}{2} \right]| \leq \zeta_k(2) \leq \zeta(2)^k = \pi^k/36
\]

and also that

\[
|\Gamma \left[ \frac{1}{2} + it \right]| \leq |\Gamma \left[ \frac{1}{2} + it \right]| \left| \frac{1}{2} + it \right|^k
\]

and

\[
|\Gamma \left[ \frac{1}{2} + it \right]| \left| \frac{1}{2} + it \right|^k \leq \frac{1}{2} \left| \frac{1}{2} + it \right|^k \left| \frac{1}{2} + it \right|^k.
\]

Using the above we can show

\[
\text{(1)} \quad \frac{6!}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta_k(s + \beta) x^s ds}{s \prod_{n=1}^{k} (s + n)} \leq \frac{250}{|d|^{1/2 + \beta}}.
\]

Now by the standard argument ([12], p. 31)

\[
\frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \frac{x^s ds}{s \prod_{n=1}^{k} (s + n)} = \frac{1}{6i} \sum_{n=1}^{k} \frac{(n-1)(-1)^n}{n(6-n) n^{1/2}} > 0 \quad \text{if} \quad x > 1,
\]

\[
0 \quad \text{if} \quad 0 < x < 1.
\]

Since for \(\text{Res} > 1\)

\[
\zeta_k(\epsilon) = \sum (N a)^{-\epsilon},
\]

it follows that

\[
I = \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \frac{\zeta_k(s + \beta) x^s ds}{s \prod_{n=1}^{k} (s + n)} = \sum (N a)^{-\epsilon} \left( \frac{1}{6i} - \sum_{n=1}^{k} \frac{(n-1)(-1)^n}{n(6-n) n^{1/2}} \left( N a \right)^{1/2} \right),
\]

where the right-hand sum goes over all ideals \(\alpha\) of \(Q(\sqrt{d})\) with norm \(\leq a\). Now \(n^2\) is the norm of an ideal for every integer \(n\) and every term of the right-hand side is \(> 0\), so if we choose \(A > .58\)

\[
(2) \quad 6! I \geq \sum_{n=1}^{100} \frac{1}{n^{1/2} - 15 n^2} > 1.635.
\]

On the other hand, moving the line of integrations to \(\text{Res} = 1\)

\[
(3) \quad I = \frac{1}{2\pi i} \int_{1-\infty}^{1+i\infty} \frac{\zeta_k(s + \beta) x^s ds}{s \prod_{n=1}^{k} (s + n)} + \frac{L(1, \chi) \log(1 - \beta) + \zeta_k(\beta)}{6!} - \frac{\zeta_k(-2 + \beta) x^{-\epsilon}}{2.4!}.
\]

Choose \(\epsilon > 0\) and let \(\beta\) be a real zero of \(L(\alpha, \chi)\) with \(1 - \beta = \epsilon |d|^{-1}\) if such a zero exists, and let \(\beta = 1 - \epsilon |d|^{-1}\) otherwise. Then \(-\zeta_k(-2 + \beta) < 0\) and \(\zeta_k(\beta) < 0\). Also, as \(1 - \beta < \epsilon |d|^{-1}\)

\[
x^{-\beta} < e^{\epsilon}, \quad A \left( \frac{1}{2} + \beta \right) \geq \frac{A}{\log |d|} - A \epsilon.
\]

This, together with (1), (2), and (3) implies

\[
\frac{L(1, \chi)}{1 - \beta} > \frac{1.612}{e^{\epsilon}} - \frac{.250}{(10^{1/2})^{2 - \epsilon}}.
\]

Letting \(A = .92\) in the first case, and letting \(\epsilon = 1.06\) and \(A = .88\) in the second gives the result.

**Lemma 2.** Let \(K\) be an algebraic number field with discriminant \(D_K\). Then \(\zeta_K(s)\) has at most one real simple zero \(\beta\) with

\[
1 - \beta < \frac{1}{2.9142 \log |D_K|}.
\]
Proof. In [4], Lemma 3, we see that if $\mathcal{S}$ is any subset of the real zeros of $\zeta_K(s)$ then for any $\sigma > 1$

\[
\sum_{\rho \in \mathcal{S}} \frac{1}{\sigma - \rho} < \frac{1}{\sigma - 1} + \frac{1}{2} \log |D_K|.
\]

Let $\sigma = 1 + 2/[1 + \sqrt{2}] \log |D_K|$ and suppose there exist two real zeros $\rho$, with $\rho > 1 - 1/\log |D_K|$. By (4), $y < 2.9142$.

**Lemma 3.** Let $a, a', |a| \geq |a'| \geq 10^6$ be the discriminants of two quadratic fields and let $L(s, \chi)$, $L(s, \chi')$ be the corresponding $L$-series. If $L(s, \chi)$ has a real zero $\beta'$, then

\[
1 - \beta' > \frac{1}{5.828 \log |a|} \quad \text{or} \quad L(1, \chi) > \frac{1}{7.735 \log |a'|}.
\]

Proof. Let $K = \mathbb{Q}(\sqrt{a}, \sqrt{a'})$. Then $\zeta_K(s) = \zeta(s)L(s, \chi)L(s, \chi') \times L(s, \chi'')$. If $L(\chi, \chi') \neq 0$ on the interval $(1 - 1/11.657 \log |d|, 1)$ then by Lemma 1 the result follows. If $L(\beta, \chi) = 0$ for some $\beta$ in that interval then both $\beta$ and $\beta'$ are zeros of $\zeta_K(s)$ so by Lemma 2

\[
1 - \beta' > \frac{1}{2.9142 \log |D_K|} \quad \text{or} \quad 1 - \beta > \frac{1}{2.9142 \log |D_K|}.
\]

But $|D_K(a', b')|$ and $|a| \geq |d| > 2.9142 \log |D_K| < 11.657 \log |d|$. Thus the lower bound for $1 - \beta$ must hold.

We have now shown that the best we can hope for as a general lower bound for $L(1, \chi)$ is $7.735 \log |d|^{-1}$. In what follows we will take the first $a'$ to come along with $L(1, \chi')$ smaller than this and use it to find a lower bound for all $L(1, \chi)$ with $|d| \geq |d'|$. In fact it will turn out that the smaller $L(1, \chi')$ is, the better the results we will get for all other discriminants.

3. Proof of Theorem 1. Let $(6 \log 10)^{-1} > \epsilon > 0$ and let $d'$ be the discriminant of smallest absolute value such that $|d'| > e^{6\epsilon}$ and

\[
L(1, \chi') < \frac{1}{7.735 \log |d'|}.
\]

By Lemma 1, $L(s, \chi')$ has a real zero, $\beta'$, and

\[
1 - \beta' > \frac{1}{11.657 \log |d'|}.
\]

Let $d$ be another discriminant such that $|d| > e^{6\epsilon}$. By our choice of $d'$ we can assume $|d| \geq |d'|$. Let $K, D_K, \zeta_K(s)$ be as in Lemma 3 and let $\alpha = -\frac{1}{2} - \beta'$ and $\sigma = |D_K|^4, A > .8$. Using the functional equation for $\zeta_K(s)$ we can show as in Lemma 1 that

\[
\frac{91}{2 \pi^4} \int_{-\infty}^{+\infty} \frac{\zeta_K(s + \beta') x^s ds}{s \prod_{n=1}^{s}(s+n)} \lesssim \frac{.099}{|D_K|^{A(e^{|d'|} - 1)}}
\]

and also that

\[
0 < \frac{91 \zeta_K(-2 + \beta') x^{-2 - 1}}{2 \pi^4} \lesssim \frac{.000237}{|D_K|^{A(e^{|d'|} - 1)}}
\]

Proceeding as in Lemma 1, as

\[
\frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \frac{x^s ds}{\pi \prod_{n=1}^{s}(s+n)} = \frac{1}{91} \left( \frac{1 - 1}{x} \right) \text{ if } x > 1,
\]

\[
0 \text{ if } 0 < x < 1
\]

it follows that

\[
I = \frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \frac{\zeta_K(s + \beta') x^s ds}{s \prod_{n=1}^{s}(s+n)} = \frac{1}{91} \sum_{N \leq \epsilon} \frac{1}{(N \alpha)^{2}} \left( 1 - \frac{N \alpha}{x} \right)
\]

where the right-hand sum is over all ideals $a$ of $K$ with norm $\leq \epsilon$. For every integer $n, n^4$ is the norm of an ideal, so

\[
91! I \geq \sum_{n=1}^{\epsilon} \frac{1}{n^4} \left( 1 - \frac{n}{x} \right) \geq 1.080.
\]

Moving the line of integration to $\text{Res} = a$, we get

\[
I = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\zeta_K(s + \beta') x^s ds}{s \prod_{n=1}^{s}(s+n)} = \frac{L(1, \chi') L(1, \chi') \zeta_K(a_1 - 1 - \beta')}{s \prod_{n=1}^{s}(s+n)}
\]

But $\zeta_K(\beta') = 0$ and $-\zeta_K(-1 + \beta') < 0$, so letting $A = 2 |(1 + \beta')|, (7), (8), (9), (10)$ give us

\[
0.981 (1 - \beta') \leq L(1, \chi') L(1, \chi') L(1, \chi') \zeta_K(a_1 - 1 - \beta')
\]

To use (11), we need to bound $L(1, \chi', \chi')$ and $a_1 - 1 - \beta'$ from above and $1 - \beta'$ from below. We know from Tatuza (5), Lemmas 4 and 5 that
if \( \chi \) is a real non-principal character mod \( k \) then

\[
L(1, \chi) < \log M(\chi) + C + \frac{1}{2M(\chi)},
\]

where \( C \) is Euler's constant and \( M(\chi) = \max_n \sum_{i=1}^n \chi(i) \). Also, if \( \chi_1 \) and \( \chi_2 \) are primitive characters mod \( k_1 \) and \( k_2 \) respectively, then

\[
M(\chi_1 \chi_2) \leq \frac{1}{2\pi} \sqrt{k_1 k_2} \left( \log k_1 k_2 + 2 \log \log k_1 k_2 + \log 4 + 6 + \frac{\pi}{k_1 k_2} \right).
\]

As \( |d|, |d'| > 10^6 \), (12) and (13) imply

\[
L(1, \chi\chi') < 0.5891 \log |d'd'|.
\]

By (6),

\[
\alpha^{1-\beta'} = |D_{K'}|^{(1-\beta')} \leq |d'd'|^{\log \log |d'd'|} \leq |d'd'|^{1.18\log |d'd'|}.
\]

Finally, \( 1 - \beta' \) is easy to bound from below, for as we can assume \( L(1, \chi) < (7.735 \log |d|)^{-1} \), Lemma 3 implies

\[
1 - \beta' < \frac{1}{5.828 \log |d'd'|}.
\]

Combining (9), (11), (14), (15) and (16) we get

**Theorem 1'**. If \( \log |d| > \log |d'| > 10^6 \)

\[
L(1, \chi) > \frac{1}{7.735 \log |d'|}.
\]

This is a description of the lower bound in terms of \( d' \). To introduce \( \epsilon < 0 \) we note that the expression decreases as \( |d'| \) decreases and we can substitute \( e^{1/2} \) for \( |d'| \). To demonstrate this, let \( y = \log |d|/\log |d'| \). If \( y < 4.63 \) the above expression gives

\[
L(1, \chi) > \frac{1}{7.735 \log |d'|}.
\]

Hence we may assume \( y > 4.63 \). In this case

\[
2 \log (y+1) < 0.6139 (y+1)
\]

and

\[
L(1, \chi) > \frac{1}{1.101 \log |d'|} |d'|^{1.181 \log |d'|}.
\]

For fixed \( k > 0 \), \( |k|^k \chi^k \) decreases until it reaches a minimum at \( x = k \). In this case \( x = \log |d'| \) and \( k = 0.752 \log |d'| \), so as \( \log |d| > 4.63 \log |d'| > (7.735)^{-1} \log |d'| \) we may substitute \( \epsilon^{-1} \) for \( \log |d'| \).

This, together with the lower bound for \( |d'| \), establishes the first part of Theorem 1.

For the second part we notice that if \( \log |d'| \) is fixed then

\[
(\log |d'|)^{1+\frac{\log |d|}{\log |d'|}} |d'|^{1.18 \log |d'|}
\]

decreases to a minimum when \( \log |d'| = 1.248 \log |d'| \). Thus it is again safe to substitute \( \epsilon^{-1} \) for \( \log |d'| \).

**4. Proof of Theorem 2.** Let \( \epsilon > 0 \) and let \( d', |d'| > \epsilon^2 \) be the exceptional discriminant of the previous theorem (where we have replaced \( \epsilon^{-1} \) by \( 2 \log \epsilon^{-1} \)). Suppose that \( d' < 0 \). Then

\[
L(1, \chi') = \frac{\pi h_\epsilon}{\sqrt{|d'|}},
\]

where \( h_\epsilon \) is the class number of \( Q(\sqrt{d'}) \). If \( \beta' \) is the real zero of \( L(\epsilon, \chi') \), then as \( L(1, \chi') < (7.735 \log |d'|)^{-1} \) it follows from Lemma 1 that

\[
1 - \beta' < \frac{L(1, \chi')}{1.507}.
\]

Then

\[
\alpha^{1-\beta'} = |D_{K'}|^{(1-\beta')} \leq |d'd'|^{\log \log |d'd'|} \leq |d'd'|^{1.148 |\log |d'd'||^2}.
\]

Proceeding as in Theorem 1 but using (17) and (18) instead of (5) and (15) we see that

\[
L(1, \chi) > \frac{V|d'|}{12.624 h_\epsilon (\log |d'|)^{4/3} |d'|^{5.344 \log V|d'|}}.
\]

As we can again assume that \( \log |d|/\log |d'| > 4.63 \), \( \log |d'| < 2.16 \log |d| \) and

\[
L(1, \chi) > \frac{V|d'|}{15.360 h_\epsilon (\log |d'|)^{4/3} |d'|^{5.344 \log V|d'|}}.
\]

It is clear that the above expression remains true if we substitute a smaller value for \( V|d'| \), so as \( V|d'| > \epsilon^{-1} \), Theorem 2 follows.
On the coprimality of certain multiplicative functions

by

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1. Introduction. An integer-valued multiplicative function \( f \) is said to be polynomial-like if there exists a polynomial \( W \) with coefficients in \( \mathbb{Z} \) (the set of all integers) such that

\[
  f(p) = W(p) \quad \text{for all primes } p;
\]

it will not be necessary for us to impose any corresponding condition on \( f(p^n) \) for \( n \geq 2 \). Obvious examples of functions in this class are Euler's function \( \varphi \) and the divisor functions

\[
  \sigma_r(n) = \sum_{d \mid n} d^r
\]

for \( r \) a non-negative integer.

In an earlier paper [8] we investigated the sum

\[
  \sigma_f(x) = \sum_{\substack{n \leq x \mid \gcd(n,f(n)=1)}} 1,
\]

and for \( f \) a polynomial-like multiplicative function such that the polynomial \( W \) in (1) has degree \( l \) and satisfies \( W(0) \neq 0 \), we obtained the asymptotic formula

\[
  \sigma_f(x) \sim \begin{cases} 
  C \alpha \log \log \log x^{-1} & \text{if } l > 0, \\
  C \alpha & \text{if } l = 0
\end{cases}
\]

as \( x \to \infty \), where \( C, \alpha, \lambda \) are positive constants with \( \lambda \) rational and \( \lambda \leq 1 \). When \( W(0) = 0 \), we deduced easily that

\[
  \sigma_f(x) = O(x^{1/2}),
\]

and indeed for some \( f \) in this category, one can obtain, by a minor adaptation of the argument in [8], an asymptotic formula of the type in (4) but with \( x \) replaced by \( x^a \) for some rational \( a \) with \( 0 < a \leq 1/2 \). The proof of (4) is elementary, although complicated, and depends in part on a double...