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On the Siegel–Tatuzawa theorem

by

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1. Introduction. Let d be the discriminant of a quadratic field k and let

$$\chi(n) = \left(\frac{d}{n}\right) \quad \text{(Kronecker's symbol).}$$

It is well known that if $L(s, \chi)$ has no zero in the interval $(1-c_1/\log|d|, 1)$ then $L(1, \chi) > c_2/\log|d|$, where c_1 and c_2 are positive constants and c_2 depends upon c_1 (see Lemma 1). If, however, $L(s, \chi)$ has a real zero close to 1, the only non trivial lower bounds that are known for $L(1, \chi)$ are ineffective. Siegel, for example, has shown that for any $\varepsilon > 0$

$$L(1, \chi) > \frac{c(\varepsilon)}{|d|^{\varepsilon}},$$

where $c(\varepsilon)$ is an ineffective constant depending upon ε [3], while Tatuzawa has shown [5] that if $1/11.2 > \varepsilon > 0$ and $|d| > e^{1/\varepsilon}$ then with at most one exception

$$L(1, \chi) > \frac{.655\,\varepsilon}{|d|^{\varepsilon}}.$$

The main objective of this paper is to arrive at a result somewhat stronger than Tatuzawa's. Using a technique of Goldfeld [1], we prove:

THEOREM 1. *Let d and χ have the meaning defined above and let $1/(6\log 10) > \varepsilon > 0$. If $|d| > e^{1/\varepsilon}$ then with at most one exception the following two expressions hold:*

$$L(1, \chi) > \min\left[\frac{1}{7.735\log|d|}, \frac{\varepsilon}{(.349)\,|d|^{\varepsilon}}\right],$$

$$L(1, \chi) > \min\left[\frac{1}{7.735\log|d|}, \frac{\varepsilon}{(1+\varepsilon\log|d|)^2(.596)\,|d|^{.138\varepsilon}}\right].$$

We also show:

THEOREM 2. *Let $1/1000 > \varepsilon > 0$ and suppose the exceptional quadratic field in the above theorem is imaginary with class number h_0. For any other discriminant d, $|d| > \varepsilon^{-2}$*

$$L(1,\chi) > \min\left[\frac{1}{7.735\log|d|},\ \frac{\varepsilon^{-1}}{15.350\,h_0(\log|d|)^2|d|^{3.344h_0\varepsilon}}\right].$$

This implies large values for all $L(1,\chi)$ if there exists just one imaginary quadratic field with a large discriminant and small class number.

2. The proof of Theorem 1 depends upon several lemmas.

LEMMA 1. *Let d, k, χ be as above, $|d| > 10^6$. If $L(s,\chi) \neq 0$ on the interval $(\beta, 1)$ and $1-\beta < (11.657\log d)^{-1}$ then,*

$$L(1,\chi) > 1.507(1-\beta).$$

If $L(s,\chi) \neq 0$ on the interval $(0,1)$ then

$$L(1,\chi) > \frac{1}{1.502\log|d|}.$$

Proof. Let $a = -\frac{3}{2}-\beta$ where $0 < \beta < 1$ and let $x = |d|^A$, $A > 0$. If $\zeta_k(s)$ is the zeta function of k, then by the functional equation

$$\zeta_k\left(-\tfrac{3}{2}+it\right) = \left(\frac{|d|}{(2\pi)^2}\right)^{2-it}\left(\frac{\Gamma\left(\frac{5}{2}-it\right)}{\Gamma\left(-\frac{3}{2}+it\right)}\right)^{r_2}\left(\frac{\Gamma\left(\frac{5}{4}-it/2\right)}{\Gamma\left(-\frac{3}{4}+it/2\right)}\right)^{r_1}\zeta_k\left(\tfrac{5}{2}-it\right).$$

We note first that

$$\left|\zeta_k\left(\tfrac{5}{2}-it\right)\right| \leqslant \left|\zeta_k\left(\tfrac{5}{2}\right)\right| \leqslant \zeta_k(2) \leqslant \zeta(2)^2 = \pi^4/36$$

and also that

$$\left|\Gamma\left(\tfrac{5}{2}-it\right)\Gamma^{-1}\left(-\tfrac{3}{2}+it\right)\right| = \left|\tfrac{3}{2}+it\right|^2\left|\tfrac{1}{2}+it\right|^2$$

and

$$\left|\Gamma\left(\tfrac{5}{4}-it/2\right)\Gamma^{-1}\left(-\tfrac{3}{4}+it/2\right)\right|^2 = \tfrac{1}{16}\left|\tfrac{3}{2}+it\right|^2\left|\tfrac{1}{2}+it\right|^2.$$

Using the above we can show

$$\text{(1)}\qquad \left|\frac{6!}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\zeta_k(s+\beta)x^s\,ds}{s\prod_{n=2}^{6}(s+n)}\right| \leqslant \frac{.250}{|d|^{A\left(\frac{3}{2}+\beta\right)-2}}.$$

Now by the standard argument ([2], p. 31)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s\,ds}{s\prod_{n=2}^{6}(s+n)} = \begin{cases}\dfrac{1}{6!}-\displaystyle\sum_{n=2}^{6}\frac{(n-1)(-1)^n}{n!(6-n)!\,x^n} > 0 & \text{if}\quad x > 1,\\ 0 \quad \text{if} \quad 0 < x < 1.\end{cases}$$

Since for $\operatorname{Re} s > 1$

$$\zeta_k(s) = \sum (N\mathfrak{a})^{-s},$$

it follows that

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta_k(s+\beta)x^s\,ds}{s\prod_{n=2}^{6}(s+n)} = \sum_{N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-\beta}\left(\frac{1}{6!}-\sum_{n=2}^{6}\frac{(n-1)(-1)^n}{n!(6-n)!}\left(\frac{N\mathfrak{a}}{x}\right)^n\right),$$

where the right-hand sum goes over all ideals $\mathfrak{a}$ of $Q(\sqrt{d})$ with norm $\leqslant x$. Now n^2 is the norm of an ideal for every integer n and every term of the right-hand side is > 0, so if we choose $A > .88$

$$\text{(2)}\qquad 6!\,I \geqslant \sum_{n=1}^{100}\left(\frac{1}{n^2}-\frac{15n^2}{x^2}\right) > 1.635.$$

On the other hand, moving the line of integrations to $\operatorname{Re} s = a$

$$\text{(3)}\qquad I = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\zeta_k(s+\beta)x^s\,ds}{s\prod_{n=1}^{6}(s+n)} + \frac{L(1,\chi)x^{1-\beta}}{(1-\beta)\prod_{n=2}^{6}(n+1-\beta)} + \frac{\zeta_k(\beta)}{6!} - \frac{\zeta_k(-2+\beta)x^{-2}}{2\cdot 4!}.$$

Choose $c > 0$ and let β be a real zero of $L(s,\chi)$ with $1-\beta < c/\log|d|$ if such a zero exists, and let $\beta = 1-c/\log|d|$ otherwise. Then $-\zeta_k(-2+\beta) < 0$ and $\zeta_k(\beta) \leqslant 0$. Also, as $1-\beta \leqslant c/\log|d|$

$$x^{1-\beta} < e^{Ac} \quad\text{and}\quad A\left(\tfrac{3}{2}+\beta\right) \geqslant \tfrac{5}{2}A - \frac{Ac}{\log|d|}.$$

This, together with (1), (2), and (3) implies

$$\frac{L(1,\chi)}{1-\beta} > \frac{1.612}{e^{Ac}} - \frac{.250}{(10^6)^{\frac{5}{2}A-2}}.$$

Letting $A = .92$ in the first case, and letting $c = 1.06$ and $A = .88$ in the second gives the result.

LEMMA 2. *Let K be an algebraic number field with discriminant D_K. Then $\zeta_K(s)$ has at most one real simple zero β with*

$$1-\beta < \frac{1}{2.9142\log|D_K|}.$$

Proof. In [4], Lemma 3, we see that if S is any subset of the real zeros of $\zeta_K(s)$ then for any $\sigma > 1$

$$\sum_{\varrho \in S} \frac{1}{\sigma - \varrho} < \frac{1}{\sigma - 1} + \frac{1}{2}\log|D_K|. \tag{4}$$

Let $\sigma = 1 + 2/[(1+\sqrt{2})\log|D_K|]$ and suppose there exist two real zeros ϱ, with $\varrho > 1 - 1/y\log|D_K|$. By (4), $y < 2.9142$.

LEMMA 3. *Let d, d', $|d| \geqslant |d'| \geqslant 10^6$ be the discriminants of two quadratic fields and let $L(s, \chi)$, $L(s, \chi')$ be the corresponding L-series. If $L(s, \chi')$ has a real zero β', then*

$$1 - \beta' > \frac{1}{5.828\log|dd'|} \quad \text{or} \quad L(1, \chi) > \frac{1}{7.735\log|d|}.$$

Proof. Let $K = Q(\sqrt{d}, \sqrt{d'})$. Then $\zeta_K(s) = \zeta(s)L(s, \chi)L(s, \chi')\times$ $\times L(s, \chi\chi')$. If $L(s, \chi) \neq 0$ on the interval $(1 - 1/11.657\log|d|, 1)$ then by Lemma 1 the result follows. If $L(\beta, \chi) = 0$ for some β in that interval then both β and β' are zeros of $\zeta_K(s)$ so by Lemma 2

$$1 - \beta' > \frac{1}{2.9142\log|D_K|} \quad \text{or} \quad 1 - \beta > \frac{1}{2.9142\log|D_K|}.$$

But $D_K|(dd')^2$ and $|d| \geqslant |d'|$ so $2.9142\log|D_K| \leqslant 11.657\log|d|$. Thus the lower bound for $1-\beta'$ must hold.

We have now shown that the best we can hope for as a general lower bound for $L(1, \chi)$ is $(7.735\log|d|)^{-1}$. In what follows we will take the first d' to come along with $L(1, \chi')$ smaller than this and use it to find a lower bound for all $L(1, \chi)$ with $|d| \geqslant |d'|$. In fact it will turn out that the smaller $L(1, \chi')$ is, the better the results we will get for all other discriminants.

3. Proof of Theorem 1. Let $(6\log 10)^{-1} > \varepsilon > 0$ and let d' be the discriminant of smallest absolute value such that $|d'| > e^{1/\varepsilon}$ and

$$L(1, \chi') < \frac{1}{7.735\log|d'|}. \tag{5}$$

By Lemma 1, $L(s, \chi')$ has a real zero, β', and

$$1 - \beta' < \frac{1}{11.657\log|d'|}. \tag{6}$$

Let d be another discriminant such that $|d| > e^{1/\varepsilon}$. By our choice of d' we can assume $|d| \geqslant |d'|$. Let K, D_K, $\zeta_K(s)$ be as in Lemma 3 and let $\alpha = -\frac{3}{2} - \beta'$ and $x = |D_K|^A$, $A > .8$. Using the functional equation for $\zeta_K(s)$ we can show as in Lemma 1 that

$$\left|\frac{9!}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\zeta_K(s+\beta')x^s\,ds}{s\prod_{n=1}^{9}(s+n)}\right| \leqslant \frac{.099}{|D_K|^{A\left(\frac{3}{2}+\beta'\right)-2}} \tag{7}$$

and also that

$$0 < \frac{9!\,\zeta_K(-2+\beta')x^{-2}}{2\cdot 7!} \leqslant \frac{.000237}{|D_K|^{2A+\beta'-5/2}}. \tag{8}$$

Proceeding as in Lemma 1, as

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s\,ds}{s\prod_{n=1}^{9}(s+n)} = \begin{cases} \dfrac{1}{9!}\left(1 - \dfrac{1}{x}\right)^9 & \text{if} \quad x > 1, \\ 0 \quad \text{if} \quad 0 < x < 1 \end{cases}$$

it follows that

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta_K(s+\beta')x^s\,ds}{s\prod_{n=1}^{9}(s+n)} = \frac{1}{9!}\sum_{N\mathfrak{a}<x} \frac{1}{(N\mathfrak{a})^{\beta'}}\left(1 - \frac{N\mathfrak{a}}{x}\right)^8$$

where the right-hand sum is over all ideals $\mathfrak{a}$ of K with norm $\leqslant x$. For every integer n, n^4 is the norm of an ideal, so

$$9!\,I \geqslant \sum_{n=1}^{5} \frac{1}{n^4}\left(1 - \frac{n^4}{x}\right)^8 \geqslant 1.080. \tag{9}$$

Moving the line of integration to $\mathrm{Re}\,s = \alpha$,

$$I = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\zeta_K(s+\beta')x^s\,ds}{s\prod_{n=1}^{9}(s+n)} + \frac{L(1, \chi)L(1, \chi')L(1, \chi\chi')x^{1-\beta'}}{(1-\beta')\prod_{n=1}^{9}(n+1-\beta')} + \frac{\zeta_K(\beta')}{9!} - \frac{\zeta_K(-1+\beta')x^{-1}}{8!} + \frac{\zeta_K(-2+\beta')x^{-2}}{2\cdot 7!}. \tag{10}$$

But $\zeta_K(\beta') = 0$ and $-\zeta_K(-1+\beta') < 0$, so letting $A = 2/(\frac{3}{2}+\beta')$, (7), (8), (9), and (10) give us

$$0.981(1-\beta') \leqslant L(1, \chi)L(1, \chi')L(1, \chi\chi')x^{1-\beta'}. \tag{11}$$

To use (11), we need to bound $L(1, \chi\chi')$ and $x^{1-\beta'}$ from above and $1-\beta'$ from below. We know from Tatuzawa ([5], Lemmas 4 and 5) that if χ is a real non-principal character mod k then

$$L(1,\chi) < \log M(\chi) + C + \frac{1}{2M(\chi)}, \tag{12}$$

where C is Euler's constant and $M(\chi) = \max_n \sum_{i=1}^{n} \chi(i)$. Also, if χ_1 and χ_2 are primitive characters mod k_1 and k_2 respectively, then

$$M(\chi_1\chi_2) \leqslant \frac{1}{2\pi}\sqrt{k_1 k_2}\left(\log k_1 k_2 + 2\log\log k_1 k_2 + \log 4 + 6 + \frac{\pi}{\sqrt{k_1 k_2}}\right). \tag{13}$$

As $|d|, |d'| > 10^6$, (12) and (13) imply

$$L(1,\chi\chi') < .589\log|dd'|. \tag{14}$$

By (6),

$$x^{1-\beta'} = |D_K|^{\Delta(1-\beta')} \leqslant |dd'|^{2\Delta(1-\beta')} \leqslant |dd'|^{.138/\log|d'|}. \tag{15}$$

Finally, $1-\beta'$ is easy to bound from below, for as we can assume $L(1,\chi) < (7.735\log|d|)^{-1}$, Lemma 3 implies

$$1-\beta' < \frac{1}{5.828\log|dd'|}. \tag{16}$$

Combining (9), (11), (14), (15) and (16) we get

THEOREM 1′. *If* $|d| \geqslant |d'| > 10^6$

$$L(1,\chi) > \frac{1}{.520\log|d'|\left(1+\dfrac{\log|d|}{\log|d'|}\right)^2 |dd'|^{.138/\log|d'|}}.$$

This is a description of the lower bound in terms of d'. To introduce ε we note that the expression decreases as $|d'|$ decreases and we can substitute $e^{1/\varepsilon}$ for $|d'|$. To demonstrate this, let $y = \log|d|/\log|d'|$. If $y < 4.63$ the above expression gives

$$L(1,\chi) > \frac{1}{7.735\log|d|}.$$

Hence we may assume $y > 4.63$. In this case

$$2\log(y+1) < .6139(y+1)$$

and

$$L(1,\chi) > \frac{1}{1.101(\log|d'|)\,|d|^{.752/\log|d'|}}.$$

For fixed $k > 0$, $xe^{k/x}$ decreases until it reaches a minimum at $x = k$. In this case $x = \log|d'|$ and $k = .752\log|d|$, so as $\log|d| \geqslant 4.63\log|d'| > (.752)^{-1}\log|d'|$ we may substitute ε^{-1} for $\log|d'|$.

This, together with the lower bound for $|d|$, establishes the first part of Theorem 1.

For the second part we notice that if $\log|d|$ is fixed then

$$(\log|d'|)\left(1+\frac{\log|d|}{\log|d'|}\right)^2 |d|^{.138/\log|d'|}$$

decreases to a minimum when $\log|d'| = 1.248\log|d|$. Thus it is again safe to substitute ε^{-1} for $\log|d'|$.

4. Proof of Theorem 2. Let $\varepsilon > 0$ and let d', $|d'| > \varepsilon^{-2}$ be the exceptional discriminant of the previous theorem (where we have replaced ε^{-1} by $2\log\varepsilon^{-1}$). Suppose that $d' < 0$. Then

$$L(1,\chi') = \frac{\pi h_0}{\sqrt{|d'|}}, \tag{17}$$

where h_0 is the class number of $Q(\sqrt{d'})$. If β' is the real zero of $L(s,\chi')$, then as $L(1,\chi') < (7.735\log|d'|)^{-1}$ it follows from Lemma 1 that

$$1-\beta' < \frac{L(1,\chi')}{1.507}.$$

Then

$$x^{1-\beta'} = |D_K|^{\Delta(1-\beta')} \leqslant |dd'|^{2\Delta(1-\beta')} \leqslant 1.148\,|d|^{3.344h_0/\sqrt{|d'|}}. \tag{18}$$

Proceeding as in Theorem 1 but using (17) and (18) instead of (5) and (15) we see that

$$L(1,\chi) > \frac{\sqrt{|d'|}}{12.624\,h_0(\log|dd'|)^2|d|^{3.344h_0/\sqrt{|d'|}}}.$$

As we can again assume that $\log|d|/\log|d'| > 4.63$, $\log|dd'| < 1.216\log|d|$ and

$$L(1,\chi) > \frac{\sqrt{|d'|}}{15.350\,h_0(\log|d|)^2|d|^{3.344h_0/\sqrt{|d'|}}}.$$

It is clear that the above expression remains true if we substitute a smaller value for $\sqrt{|d'|}$, so as $\sqrt{|d'|} > \varepsilon^{-1}$, Theorem 2 follows.

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On the coprimality of certain multiplicative functions

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1. Introduction. An integer-valued multiplicative function f is said to be *polynomial-like* if there exists a polynomial W with coefficients in $\mathbf{Z}$ (the set of all integers) such that

$$f(p) = W(p) \quad \text{for all primes } p; \tag{1}$$

it will not be necessary for us to impose any corresponding condition on $f(p^a)$ for $a \geqslant 2$. Obvious examples of functions in this class are Euler's function φ and the divisor functions

$$\sigma_\nu(n) = \sum_{d|n} d^\nu \tag{2}$$

for ν a non-negative integer.

In an earlier paper [8] we investigated the sum

$$\Sigma_f(x) = \sum_{\substack{1 \leqslant n \leqslant x \\ (n, f(n)) = 1}} 1, \tag{3}$$

and for f a polynomial-like multiplicative function such that the polynomial W in (1) has degree l and satisfies $W(0) \neq 0$, we obtained the asymptotic formula

$$\Sigma_f(x) \sim \begin{cases} Cx(\log\log\log x)^{-\lambda} & \text{if} \quad l > 0, \\ Cx & \text{if} \quad l = 0 \end{cases} \tag{4}$$

as $x \to \infty$, where C, λ are positive constants with λ rational and $\lambda \leqslant 1$ When $W(0) = 0$, we deduced easily that

$$\Sigma_f(x) = O(x^{1/2}),$$

and indeed for some f in this category, one can obtain, by a minor adaptation of the argument in [8], an asymptotic formula of the type in (4) but with x replaced by x^a for some rational a with $0 < a \leqslant \frac{1}{2}$. The proof of (4) is elementary, although complicated, and depends in part on a double