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Contributions to the Erdös–Szemerédi theory of sieved integers

by

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In honour of Professor Paul Erdös on the occasion of his sixty-fifth birthday

1. Introduction. In an attempt to put things like gaps between square-free integers in a general setting, Erdös initiated in [1], researches on the following problem.

DEFINITION. In order to state the problem it is better to define a constant $G(B)$ $(0 \leqslant G(B) \leqslant \infty)$ associated with any given sequence $B = \{b_n\}$ $(n = 1, 2, \ldots)$ of integers satisfying $2 \leqslant b_1 < b_2 < \ldots$ thus: If all natural numbers with a finite number of possible exceptions are divisible by some integer or other of the sequence B then we put $G(B) = \infty$. Otherwise we have an infinite sequence $\{q_n\}$ $(n = 1, 2, \ldots)$ of all natural numbers q_n not divisible by any b_m. Let $1 = q_1 < q_2 < \ldots$ In this case we put

$$G(B) = \limsup_{n\to\infty} \left(\frac{\log(q_{n+1}-q_n)}{\log(1+q_n)}\right).$$

($G(B)$ may be called the sieving power of B or the gap constant associated with B.) We put for $x \geqslant 2$, $B(x) = \sum_{b_n \leqslant x} 1$.

PROBLEM. What conditions on B will imply $G(B) < 1$?

P. Erdös proved in [1] the following

THEOREM 1. *If $(b_i, b_j) > 1$ is impossible unless $i = j$, and if $\sum_{n=1}^{\infty} b_n^{-1}$ is convergent then the supremum of $G(B)$ as B runs over all such sequences is less than* 1.

Theorem 1 was improved by E. Szemerédi who proved in [7],

THEOREM 2. *Under the conditions of Theorem* 1, $G(B) \leqslant \frac{1}{2}$.

By a modification of Szemerédi's method and using the results available from small sieve we improve Szemerédi's theorem to

THEOREM 3. *For each fixed prime p let us denote by $r(p)$ the number of numbers b_n divisible by p. Then the following two conditions together imply $G(B) \leqslant \theta/(1+\theta)$.*

(i) $\limsup_{p\to\infty}\left(\dfrac{\log(r(p)+2)}{\log p}\right) = A_0 < \infty.$

(ii) $\lim_{y\to\infty}\Big(\sum_{y\leqslant b_n\leqslant y^2} b_n^{-\theta}\Big) = 0,$

where θ is a positive constant not exceeding 1.

Remarks. Note that the condition "$(b_i, b_j) > 1$ is impossible unless $i=j$" is equivalent to "$r(p)\leqslant 1$". Also the convergence of $\sum_{n=1}^{\infty} b_n^{-1}$ trivially implies condition (ii) with $\theta = 1$. Hence Theorem 3 with $\theta = 1$ gives a slightly more general form of Theorem 2. Probably under the conditions of Theorem 3 and $\theta = 1$, it is true that $G(B) = 0$. But this is beyond the reach of any known techniques and looks very difficult.

Theorems 4 and 5 below give some new information on $G(B)$ when the conditions of Erdös, in Theorem 1, are slightly strengthened. The proofs of Theorems 4 and 5 are based on a new idea and some applications of a theorem of van der Corput.

THEOREM 4. *Let the condition* (i) *of Theorem 3 be satisfied with $A_0 = 0$. In addition let the following two conditions be satisfied:*

(iii) $\sum_{n=1}^{\infty} d(b_n)\,b_n^{-1} < \infty$

(*where $d(b_n)$ is the number of natural numbers which divide b_n*);

(iv) $\lim_{y\to\infty}\Big(\sum_{y\leqslant b_n\leqslant y^2}(\log b_n)^{\mu} b_n^{-1}\Big) = 0$

(*where* $\mu = (\log\frac{4}{3})(\log\frac{3}{2})^{-1}$).

Then $G(B)\leqslant\frac{37}{75}$.

Remark. If in the course of our proof (instead of F. V. Atkinson's results) we use Wen-Yin-Lin's improvement [9] of the results of F. V. Atkinson (see [8], p. 270) then our proof leads to $G(B)\leqslant 8/17$ under conditions (i) with $A_0 = 0$, (iii) and (iv).

THEOREM 5. *Let the condition* (i) *of Theorem 3 be satisfied with $A_0 = 0$. In addition let the following condition be satisfied for a fixed positive constant θ not exceeding* 1 *and for all positive constants C*:

(v) $\sum_{n=1}^{\infty}\left(\exp\left(\dfrac{C\log b_n}{\log\log(b_n+20)}\right)\right) b_n^{-\theta} < \infty.$

Then $G(B)\leqslant (1+1/\theta+(2^g-1)^{-1})^{-1}$, *where g is the smallest positive integer* $\geqslant 1+1/\theta$.

Remark 1. Note that when $\theta = 1$ and $\theta = \frac{1}{2}$ Theorem 5 gives $G(B)\leqslant\frac{3}{7}$ and $G(B)\leqslant\frac{7}{22}$ respectively.

Remark 2. Note that the result $G(B)\leqslant\dfrac{\theta}{1+\theta} = \left(1+\dfrac{1}{\theta}\right)^{-1}$, of Theorem 3, is improved to $G(B)\leqslant\left(1+\dfrac{1}{\theta}+\dfrac{1}{2^g-1}\right)^{-1}$ at the cost of strengthening very slightly the condition (ii) and also condition (i).

Remark 3. The condition with $\theta = \frac{1}{2}+\varepsilon$ includes the difference between consecutive square-free integers (though slightly more general). Theorem 3 gives (for ε arbitrary) $G(B)\leqslant\frac{1}{3}$ and Theorem 5 gives $G(B)\leqslant\frac{7}{22}$. In the special case of square free integers, i.e. B = sequence of squares of all primes, $G(B)\leqslant\frac{1}{3}$ is due to K. F. Roth, T. Estermann and H. Davenport, $G(B)\leqslant\frac{1}{4}$ due to K. F. Roth. (For these references see Roth's paper [6].) However, H.-E. Richert [5] has shown in this special case that $G(B)\leqslant\frac{2}{9}$. Further slight improvements are also known by more special and more complicated methods. Similar remarks apply to cube-free integers and so on [3] due to H. Halberstam and K. F. Roth. In concluding the introduction, we remark that our method can be applied in conjunction with some other methods (see [3] and [6]) to prove certain results of which we state one.

THEOREM 6. *Let B consist only of primes and squares of primes and let condition* (v) *of Theorem 5 be satisfied with $\theta = \frac{2}{3}$. Then $G(B)\leqslant\frac{1}{3}$.*

Remark. Note that Theorem 5 gives $G(B)\leqslant\frac{14}{37}$ and that $\frac{1}{3}<\frac{14}{37}$.

2. Proof of Theorem 3. We borrow the following result from the small sieve (see the book [2] by H. Halberstam and H.-E. Richert) which we state as our first lemma.

LEMMA 1. *In the interval $(x, x+x^{\delta})$ ($\delta>0$, $\varepsilon>0$ positive constants and $x\geqslant x_0(\varepsilon,\delta)$) there exist $\gg x^{\delta}(\log x)^{-1}$ integers all of whose prime factors exceed x^{ε}, provided only that ε is fixed to be a small constant depending on δ. Also the number of such integers is $O(x^{\delta}(\log x)^{-1})$.*

Our next lemma is meant to be proved as an easy exercise by the reader.

LEMMA 2. *Let a be a positive constant less than* 1 *and $\varepsilon = a/N$ be a small positive constant, where N is an integer. Let s be a real variable and put* (*for* $x\geqslant x_0(\varepsilon)$),

$$F(s) = \Big(\sum_{x^{\varepsilon}\leqslant p\leqslant x^{\varepsilon+\varepsilon^{20}}} p^{-s}\Big)^N = \sum\frac{a_q}{q^s}$$

where the sum on the right side extends only over those q for which $a_q\neq 0$. (*We will denote such integers with $a_q\neq 0$ by q.*) *Then $1\leqslant a_q = O(1)$ and $F(1)\gg 1$.*

LEMMA 3. *Let η be a positive constant satisfying $\varepsilon \leqslant \alpha + \eta < 1$. Then for every integer q of Lemma 2 there are, in the interval $(x, x+x^{\alpha+\eta})$, $\gg q^{-1}x^{\alpha+\eta}$ integers of the form nq where n runs over a block of consecutive integers. Of these integers n there are $\gg q^{-1}x^{\alpha+\eta}(\log x)^{-1}$ integers (which we denote by m) all of whose prime factors exceed x^{ζ} where ζ is a positive constant depending on ε and η.*

Remark. The first part is trivial and the second part follows from Lemma 1. The details are left to the reader.

LEMMA 4. *The number of distinct integers of the form mq which lie in the interval $(x, x+x^{\alpha+\eta})$ is $\gg x^{\alpha+\eta}(\log x)^{-1}$.*

A straightforward application of Lemma 1 gives

LEMMA 5. *Of the distinct integers of Lemma 4, the number N_i of numbers divisible by b_i is $O\left(x^{\alpha+\eta}(\log x)^{-1}b_i^{-1}+x^{\varepsilon^{100}}\right)$. Also this number is zero if $b_i \leqslant \min(x^{\varepsilon}, x^{\zeta})$. It is also zero if $b_i \geqslant 2x$.*

From Lemmas 4 and 5, Theorem 3 follows thus: Now we have only to show that $\sum N_i = o\left(x^{\alpha+\eta}(\log x)^{-1}\right)$. We split up the sums over N_i according as $(b_i, q) > 1$ or not (the latter case cannot occur if $N_i \geqslant 1$ and $b_i > 2x^{1-\alpha}$). So necessarily $(b_i, q) > 1$ and number of such b_i are not too many by our assumption on $r(p)$. This proves Theorem 3 provided we choose α by $\theta(1-\alpha) = \alpha$, since ε and η are arbitrary subject to $\eta \geqslant \varepsilon$. For an alternative proof of Theorem 3 by using analytic methods see a paper by M. J. Narlikar [4].

3. Proofs of Theorems 4 and 5. The main idea in the proof of Theorems 4 and 5 is to consider the sums related to

$$\sum_{x \leqslant n_1 n_2 q \leqslant x+h} 1$$

where h is a certain positive power of x, (n_1, n_2, q) run over all triplet of integers where q is as before and n_1, n_2 run over certain sets of consecutive integers. Then we have to use a theorem of van der Corput (used by K. F. Roth [6]) to deal with such sums and related ones. Finally we deduce Theorems 4 and 5. We give a brief sketch of the details.

Let $h = x^{\alpha_0+\eta}$ where $\eta \geqslant \varepsilon$ is a positive constant such that $\alpha_0 + \eta < \frac{3}{4}$. Let β be a positive constant, S a fixed positive integer and b a positive integer is prime to S and less than x^{ε} where ε is the positive constant of Lemma 2. Our first object is to obtain, as $x \to \infty$, an asymptotic formula for (it will be convenient to set $\beta = \theta(1-\alpha)(1+\theta)^{-1}$)

$$Q_b = \sum_{\substack{x \leqslant n_1 n_2 q \leqslant x+h,\, x^{\beta} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}} \\ (n_1, S) = (n_2, S) = 1,\, n_1 n_2 \equiv 0 \pmod{b}}} 1$$

where the sum is over all possible triplets (n_1, n_2, q) satisfying the stated conditions. Clearly

$$Q_b = \sum_{d_1|S} \sum_{d_2|S} \mu(d_1)\mu(d_2) \sum_{a=1}^{b} Q_b(a)$$

where

$$Q_b(a) = \sum_{\substack{x \leqslant n_1 n_2 d_1 d_2 q \leqslant x+h,\, x^{\alpha} \leqslant q \leqslant x^{\alpha(1+\varepsilon^{19})} \\ n_1 \equiv a \pmod{b},\, n_1 n_2 \equiv 0 \pmod{b} \\ x^{\beta} \leqslant n_1 d_1 \leqslant x^{\beta+\varepsilon^{300}}}} 1.$$

Actually it is better to write the sum over a in the definition of Q_b as

$$\sum_{d|b} \sum_{\substack{a=1 \\ (a,b)=d}}^{b} Q_b(a) = \sum_{d|b} \sum_{\substack{a=1 \\ (a,b/d)=1}}^{bd^{-1}} Q_b(ad).$$

In the last sum the congruence conditions in the sum for $Q_b(ad)$ read $n_1 \equiv ad \pmod{b}$, $n_2 \equiv 0 \pmod{bd^{-1}}$. Now

$$Q_b(ad) = \sum_{q} \sum_{x^{\beta}d_1^{-1} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}}d_1^{-1}} I_1,$$

where

$$I_1 = \sum_{x(n_1 d_1 d_2 q b d^{-1})^{-1} \leqslant n_2 \leqslant (x+h)(n_1 d_1 d_2 q b d^{-1})^{-1}} 1$$
$$= h(n_1 d_1 d_2 q b d^{-1})^{-1} + \psi\left(\frac{\varrho_1}{n_1}\right) - \psi\left(\frac{\varrho_2}{n_1}\right),$$

where $\psi(u) = u - [u] - \frac{1}{2}$, $\varrho_1 = x(d_1 d_2 q b d^{-1})^{-1}$ and $\varrho_2 = (x+h) \times \times (d_1 d_2 q b d^{-1})^{-1}$. Denoting by ϱ either of the quantities ϱ_1 or ϱ_2 we see that $(\log \varrho)(\log x)^{-1}$ is approximately $1 - \alpha = \beta(1+\theta^{-1})$. We now proceed to estimate

$$I_2 = \sum_{q} \sum_{n_1 \equiv ad \pmod{b},\, x^{\beta}d_1^{-1} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}}d_1^{-1}} \psi(\varrho/n_1).$$

THEOREM (van der Corput). *Let $\psi(v)$ denote $v - [v] - \frac{1}{2}$. Let a, b be integers $R > 0$, $k \geqslant 2$ a positive integer, $K = 2^k$. Let the function $f(u)$ be defined for $a \leqslant u \leqslant b$ such that the k-th derivative $f^{(k)}(u)$ exists throughout the interval (with obvious modifications at the end points) and either $f^{(k)}(u) \geqslant R^{-1}$ or $f^{(k)}(u) \leqslant -R^{-1}$ throughout the interval. Then*

$$\left|\sum_{n=a}^{b} \psi\left(f(n)\right)\right| \leqslant 5400\, KP\{P^{-2/K}\log(3+P) + R^{-1/(K-1)} + (RP^{-k})^{2/K}\},$$

where $P = R|f^{(k-1)}(b) - f^{(k-1)}(a)|$.

Remark 1. This is used by Roth in his work "On gaps between square-free integers" to make an essential improvement on $G(B) \leqslant \frac{1}{4}$ in the special case of square-free integers.

Remark 2. Let $a \geqslant 30$, $b \leqslant 2a$, and $f(n) = \varrho/(An+B)$, where A and B are integers which in absolute value do not exceed a^{ε}, and $A \neq 0$. Then for fixed k, $\log R$ is very nearly $\log(a^{k+1}/\varrho)$, $\log P$ is very nearly $\log a$, and $\log(RP^{-k})$ is nearly $\log(a/\varrho)$. Thus we record the following lemma which is a special case of the above theorem.

Lemma 6. *We have for any fixed integer $k \geqslant 2$,*

$$\sum_{n_1 \equiv ad \pmod b,\, x^{\beta} d_1^{-1} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}} d_1^{-1}} \psi\left(\frac{\varrho}{n_1}\right) = O\left(x^{1000k\varepsilon+\beta}\left\{x^{-2\beta/K} + \left(\frac{x^{\beta(1+\theta^{-1})}}{x^{\beta(k+1)}}\right)^{-1/(K-1)} + \left(\frac{x^{\beta}}{x^{\beta(1+\theta^{-1})}}\right)^{2/K}\right\}\right).$$

Next choosing k such that $\beta k \geqslant \beta(1+\theta^{-1})$ and observing that the first and last terms in the flower brackets are small we deduce from Lemma 6,

Lemma 7. *We have*

$$I_2 = O\left(x^{2000k\varepsilon+\beta\left(1-\frac{1}{K-1}\right)+\alpha}\right).$$

From this lemma follows

Lemma 8. *We have*

$$Q_b = J\varepsilon^{300} d^*(b) b^{-1} h(\log x) \prod_{p|S}\left(1-\frac{1}{p}\right)^2 + O(hx^{-2\varepsilon}) + O\left(x^{\alpha+\beta\left(1-\frac{1}{K-1}\right)+3000k\varepsilon}\right)$$

where

$$d^*(b) = \sum_{d|b} \varphi(d)/d \quad \textit{and} \quad J = \sum 1/q.$$

Proof. We have only to simplify the main term in Q_b namely

$$h \sum_{d_1|S, d_2|S, q} \mu(d_1)\mu(d_2) \sum_{d|b} \sum_{\substack{a=1 \\ (a,b/d)=1}}^{bd^{-1}} \sum_{\substack{x^{\beta} d_1^{-1} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}} d_1^{-1} \\ n_1 \equiv ad \pmod b}} (n_1 d_1 d_2 q b d^{-1})^{-1}.$$

Here the innermost sum does not change much if we replace $n_1 \equiv ad \pmod b$ by $n_1 \equiv 0 \pmod b$, and then replace both the conditions of summation on n_1 by a single condition $x^{\beta} d_1^{-1} b^{-1} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}} d_1^{-1} b^{-1}$ with obvious changes. This sum over n_1 is (on using the well known formula $\sum_{n \leqslant x} 1/n = \log x + \gamma + O(1/x)$) $\varepsilon^{300} \log x$ with a small error. Thus we are led to

$$h \sum_{d_1|S} \sum_{d_2|S} \frac{\mu(d_1)\mu(d_2)}{d_1 d_2} \varepsilon^{300} \log x \sum_{d|b} \sum_{\substack{a=1 \\ (a,b/d)=1}}^{bd^{-1}} (b^2 d^{-1})^{-1}.$$

Here the sum over d is $b^{-2} \sum_{d|b} d\varphi(b/d)$ which is plainly $b^{-1} d^*(b)$. This completes the proof of Lemma 8.

Let N_0 denote a large positive integer constant and put $S = \prod p$ where the product runs over all primes which divide $\prod_{n \leqslant N_0} b_n$. Note that the conditions of the Theorems 4 and 5 imply the convergence of $\sum b_n^{-1} d^*(b_n)$. Hereafter we agree to omit those b_n which are divisible by some one or other of the primes dividing S. The condition $(b_n, S) = 1$ is satisfied for all $n > N_0$. It is now a simple matter to reduce from Lemma 8 that

$$Q = Q_1 - \sum_{n > N_0, b_n < x^{\varepsilon}} Q_{b_n} \gg h \log x,$$

provided we agree to impose the condition $\alpha_0 > \alpha + \beta\left(1 - \frac{1}{K-1}\right) + 3000k\varepsilon$.

Note that the only condition on the integer constant k which we have imposed is $k \geqslant 1 + \theta^{-1}$.

Under these conditions we state

Lemma 9. *Let m run through distinct integers of the form $n_1 n_2$ where $x^{\beta} \leqslant n_1 \leqslant x^{\beta+\varepsilon^{300}}$. Then*

$$\sum_{x \leqslant mq \leqslant x+h,\ b_n \nmid mq \text{ for every } b_n < x^{\varepsilon}} d(m) \gg h \log x,$$

where $\beta = \theta(1-\alpha)(1+\theta)^{-1}$ and $h = x^{\alpha_0+\eta}$.

We would now like to subtract from the sum in Lemma 9 the contributions from those integers mq which are divisible by some one or other of the b_n satisfying $x^{\varepsilon} \leqslant b_n \leqslant 2x$. Note that the number of representations of an integer n_3 in the form mq is bounded provided $x \leqslant n_3 \leqslant x+h$ and so $\sum_{mq = n_3} d(m) = O(d(n_3))$.

Lemma 10. *Let n_3 run over distinct integers of the form $n_1 n_2 q$ where $x \leqslant n_3 \leqslant x+h$, and which appear in the sum in Lemma 9. Let N_{b_n} denote the number of integers divisible by b_n for a fixed $b_n \geqslant x^{\varepsilon}$. Then either $N_{b_n} = 0$ or*

$$N_{b_n} \leqslant \frac{h}{b_n} + 2.$$

In the latter case the contribution from n_3 with $n_3 \equiv 0 \pmod{b_n}$ for fixed b_n to the sum in Lemma 9 is uniformly

$$O\left(\left(\frac{h}{b_n}+1\right)\exp\left(\frac{C_1 \log b_n}{\log\log b_n}\right)\right)$$

where C_1 is a constant depending on ε.

Proof. The proof follows from $d(n) = O\left(\exp\left(\frac{3\log n}{\log\log n}\right)\right)$.

It is clear that the term involving h contributes $o(h\log x)$. The other term contributes at most

$$x^{\varepsilon} \sum_{N_{b_n}\neq 0,\, x^{\varepsilon}\leqslant b_n\leqslant 2x} 1.$$

The contributions to this sum from those b_n with $(b_n, q) > 1$ is very small. In fact the contributions from those b_n with $(b_n, n_1 q) > 1$ is $O(x^{\beta+\varepsilon})$, while the contributions from those b_n with $(b_n, n_1 q) = 1$ is $O(x^{\theta(1-\alpha-\beta)+200\varepsilon})$. We select β such that $\beta = \theta(1-\alpha-\beta)$, i.e. $\beta = \theta(1-\alpha)(1+\theta)^{-1}$. Next we select α such that $\alpha+\beta\left(1-\frac{1}{K-1}\right) = \beta$, i.e. $\alpha = \theta(1-\alpha)(1+\theta)^{-1}\times$ $\times(K-1)^{-1}$, i.e. $1-\beta(1+\theta^{-1}) = \beta(K-1)^{-1}$. This proves Theorem 5 since ε is arbitrary, $\eta \geqslant \varepsilon$ is arbitrary and $\alpha_0 \geqslant \beta$ is arbitrary.

To prove Theorem 4 we go back to Lemma 9. In the notation of Lemma 10 we have from Lemma 9, the result that

LEMMA 11. *We have*

$$\sum_{x\leqslant n_3\leqslant x+h} d(n_3) \geqslant h\log x.$$

In standard notation it is easy to check that for every positive integer n,

$$\big(d(n)\big)^{1+\lambda} \leqslant d_3(n) \quad \text{where} \quad \lambda = (\log\tfrac{3}{2})(\log 2)^{-1}.$$

Now F. V. Atkinson has proved that if $x^{37/75+\varepsilon} \leqslant h \leqslant x$ then

$$\sum_{x\leqslant n\leqslant x+h} d_3(n) = O\big(h(\log x)^2\big).$$

In Lemma 11 the contributions from those n_3 with $d(n_3) \geqslant Y$ is $O\big(Y^{-\lambda}h(\log x)^2\big)$ provided the conditions on h are satisfied. We choose Y such that this contribution is less than $h\log x$ times a sufficiently small constant. This requires $Y = (\log x)^{1/\lambda}$ times a large constant and we get

LEMMA 12. *We have*

$$\sum_{x\leqslant n_4\leqslant x+h} 1 \geqslant h(\log x)^{1-\lambda^{-1}},$$

where $\lambda = (\log\frac{3}{2})(\log 2)^{-1}$ and n_4 runs over those n_3 with $d(n_3)$ not exceeding a large constant times $(\log x)^{\lambda^{-1}}$. Clearly $\mu = -1+\lambda^{-1}$.

The contributions from those n_4 with $n_4 \equiv 0 \pmod{b_n}$, $x^{\varepsilon} \leqslant b_n \leqslant 2x$ can be treated as before and this leads to Theorem 4. We leave the details to the reader. In the course of proving Theorems 4 and 5 we have used at the final stage

LEMMA 13. *We have,*

$$B(x) = O(x^{\theta+\varepsilon}).$$

This can be left as a trivial exercise and this completes the proofs of Theorems 4 and 5.

Added in proof. The result of Atkinson quoted before Lemma 12, has been improved in a beautiful paper by P. Shiu (*A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. 313 (1980), pp. 161–170). One of his results reads: If ε is any positive constant < 1 and $x^{\varepsilon} < h < x$, then $\sum_{x\leqslant n\leqslant x+h} d_3(n) = O(h(\log x)^2)$. Accordingly, using this result in place of Atkin's in our proof, we can improve Theorem 4 to $G(B) < 3/7$ in place of $G(B) < 37/75$.

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