[3] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. of Math. 15 (1913–1914), pp. 30–70.
[4] Chao Ko, *On the diophantine equation* $x^2 = y^n + 1$, $xy \neq 0$, Scientia Sinica (Notes), 14 (1964), pp. 457–460.
[5] W. Ljunggren, *Some theorems on indeterminate equations of the form* $\frac{x^n-1}{x-1} = y^q$, Norsk Mat. Tidsskr. 25 (1943), pp. 17–20.
[6] T. Nagell, *Sur l'impossibilité de l'equation indéterminée* $z^p + 1 = y^2$, Norsk Mat. Forenings Skrifter 1 (1921), Nr. 3.

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On Waring's problem

by

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1. Introduction. Among the various estimates known for $G(k)$ in Waring's problem, the most significant (for large k) are the following:

(1) $$G(k) < k(2\log k + 4\log\log k + 2\log\log\log k + 13) \quad \text{for} \quad k \geqslant 170000$$

and

(2) $$G(k) \leqslant k(3\log k + 5.2) \quad \text{for} \quad k \geqslant 15.$$

These are due to Vinogradov [12] and Chen [1] respectively. Although (1) is better than (2) for sufficiently large k, for a large number of values of k, (2) is a better estimate than (1).

In this paper, we improve on (2) and prove the following:

Theorem 1. $G(k) \leqslant k(3\log k + \log 108) < k(3\log k + 4.7)$. (The improvement being by essentially $k/2$.)

For special (small) values of k Theorem 1 can be improved by modifying the method. For $k \leqslant 10$, H. Davenport [3], [4] and V. Narasimhamurti [10] obtained improvements on the estimates given by T. Estermann [7]. R. J. Cook [2] later showed that

(3) $$G(9) \leqslant 96 \quad \text{and} \quad G(10) \leqslant 121.$$

Theorem 2 is an improvement on (3). The paper of R. C. Vaughan [11] containing the following results appeared since the results of this paper were obtained. A brief comparison of the methods is made towards the end of the paper.

(4) $$G(9) \leqslant 91,\ G(10) \leqslant 107,\ G(11) \leqslant 122,\ G(12) \leqslant 137,\ G(13) \leqslant 153,$$
$$G(14) \leqslant 168,\quad G(15) \leqslant 184,\quad G(16) \leqslant 200,\quad G(17) \leqslant 216.$$

In this paper, we prove the following:

Theorem 2. $G(9) \leqslant 90$, $G(10) \leqslant 106$.

THEOREM 3. $G(11) \leqslant 121$, $G(12) \leqslant 136$, $G(13) \leqslant 152$, $G(14) \leqslant 167$, $G(15) \leqslant 183$, $G(16) \leqslant 199$, $G(17) \leqslant 215$, $G(18) \leqslant 231$, $G(19) \leqslant 248$, $G(20) \leqslant 264$.

Only the new and necessary arguments required in the proofs are given and standard details are avoided.

2. Notation and estimate of a certain trigonometric sum. s positive numbers $\lambda_1, \ldots, \lambda_s$ are called admissible exponents for kth powers in accordance with the definition in [6]. Let $U_s(k; X, Y)$ denote the number of integers n with $X \leqslant n \leqslant Y$ that are representable as a sum of s non-negative kth powers.

Let N be a large positive integer, δ a small positive constant and ε a sufficiently small positive number. Write

(5) $$2P = N^{1/k}, \quad P_0 = \sqrt{P}, \quad \tau = P^{k-1+\delta},$$

(6) $$\mu_i = \left(1-\frac{1}{k}\right)^i \quad (i = 1, 2, \ldots).$$

It is known that 1, $\mu_1, \mu_2, \ldots, \mu_{s-1}$ $(s \geqslant 2)$ are admissible exponents, and that

(7) $$U_s(k; P^k, s(2P)^k) \gg P^{k\gamma-\varepsilon} \gg N^{\gamma-\varepsilon}$$

where

(8) $$\gamma = \gamma^{(s)}(k) = \frac{1+\mu_1+\mu_2+\ldots+\mu_{s-1}}{k}.$$

From (7) (if $\gamma < 1$), $U_s(k; P_0^{k-\delta}, s2^k P_0^{k-\delta}) \gg P_0^{(k-\delta)(\gamma-\varepsilon)} \gg P_0^{k\gamma-\delta}$. Hence, for large N, there exists a set $\mathscr{U} = \{u_1, \ldots, u_{U_0}\}$ where each u_i is of the form $\sum_{i=1}^{s} x_i^k$ and satisfying

(9) $$P_0^{k-\delta} < u_i < s2^k P_0^{k-\delta} \quad (i = 1, \ldots, U_0)$$

with

(10) $$U_0 \gg P_0^{k\gamma-\delta}.$$

Let v run through the primes with

(11) $$\frac{1}{2} P_0^{1-\delta/2} \leqslant v \leqslant P_0^{1-\delta/2}$$

and denote the set of these primes by $\mathscr{P}$. The number of elements V in $\mathscr{P}$ satisfies

(12) $$V \gg P_0^{1-\delta/2-\varepsilon} \gg P_0^{1-\delta}.$$

Write

(13) $$Q(\alpha) = \sum_{v\in\mathscr{P}} \sum_{u\in\mathscr{U}} e(\alpha v^k u).$$

The next lemma follows by the same arguments as in case II of Lemma 2 in Ch. IV (pp. 66–67) of [13] (on replacing the inequality $X < P_0^n$ by $X \ll P_0^{k-\delta}$).

LEMMA 1. *Let* $\alpha = a/q+\beta$ *with* $P^{1/4} < q \leqslant P_0^k$, $|\beta| \leqslant q^{-1}P_0^{-k}$. *Then*
$$Q(\alpha) \ll (P_0 q^{-1}+1) q^{\varepsilon} \{U_0 \min(P_0, q) P_0^k\}^{1/2}.$$

LEMMA 2. *If* $\alpha = a/q+\beta$ *with* $P_0 < q \leqslant P_0^k$, $|\beta| \leqslant q^{-1}P_0^{-k}$, *then*
$$Q(\alpha) \ll Q(0) P^{-\frac{1}{4}\{1-k(1-\gamma)\}+\delta}.$$

Proof. By Lemma 1, (10), (12) and (13),
$$Q(\alpha) \ll (U_0 P_0^{k+1+\delta})^{1/2} \ll (VU_0)(V^{-1}U_0^{-1/2}) P_0^{(k+1+\delta)/2}$$
$$\ll Q(0) P_0^{-1+\delta} P_0^{(-k\gamma+\delta)/2} P_0^{(k+1+\delta)/2} \ll Q(0) P_0^{-\frac{1}{2}\{1-k(1-\gamma)\}+2\delta}.$$

The result now follows from (5).

We modify the proof given for case I of the corresponding lemma in Vinogradov [13], and prove the following

LEMMA 3. *If* $\alpha = a/q+\beta$ *with* $q \leqslant P_0$, $q^{-1}P^{-k+1-\delta} < |\beta| \leqslant q^{-1}P_0^{-k}$, *then*
$$Q(\alpha) \ll Q(0) P^{-\frac{1}{4}\{1-k(1-\gamma)\}+k\delta}.$$

Proof. Corresponding to the inequalities in the proofs of lemmas 10b and 10c of [13], Ch. I (pp. 29–31) we have (taking $\eta(y) = 1$, and noting that $Q(\alpha)$ replaces S),

(14) $$|Q(\alpha)|^2 \ll U_0 (P_0^{k-\delta})^{-1} \sum_{v_1\in\mathscr{P}} \sum_{v_2\in\mathscr{P}} \min\left\{(P_0^{k-\delta})^2, \frac{1}{\|\varphi(v_1)-\varphi(v_2)\|^2}\right\}$$

where $\varphi(v) = (av^k + q\beta v^k)/q$.

Now instead of using Lemma 9 in Ch. I of [13], we argue as follows. Let $T(\alpha)$ denote the double sum on the right-hand side of the inequality (14).

Case (a). If $v_1 = v_2$ (the number of such possibilities being V), the contribution to T of the corresponding terms is

(15) $$\ll VP_0^{2(k-\delta)}.$$

Case (b). Let $v_1 \neq v_2$ but $v_1^k \equiv v_2^k \pmod q$, so that
$$\|\varphi(v_1)-\varphi(v_2)\| = \|\beta(v_1^k - v_2^k)\|.$$

Now

$$|v_1^k - v_2^k| \geq |v_1 - v_2|\, v_1^{k-1} \gg P_0^{(k-1)(1-\delta/2)} \quad \text{by (11).}$$

Also, by hypothesis,

$$|\beta| \geq q^{-1}P^{-k+1-\delta} \gg q^{-1}P_0^{2(-k+1-\delta)} \quad (\text{cf. (5)}).$$

Hence

$$|\beta(v_1^k - v_2^k)| \gg q^{-1}P_0^{-k+1-k\delta}.$$

Furthermore,

(16) $$\beta(v_1^k - v_2^k) \ll q^{-1}P_0^{-k}P_0^{k(1-\delta/2)},$$

so that

$$\beta(v_1^k - v_2^k) = o(1).$$

Thus

(17) $$\|\beta(v_1^k - v_2^k)\| \gg q^{-1}P_0^{-k+1-k\delta}.$$

Now (since the number of divisors of q is $\ll q^{\varepsilon}$) it can be proved in a standard way that for a given v_1, the number of v_2's satisfying $v_1^k \equiv v_2^k \pmod q$, and (11) is

$$\ll \left(1 + \frac{P_0^{1-\delta/2}}{q}\right) q^{\varepsilon} \ll \frac{P_0^{1+\varepsilon}}{q} \quad (\text{since } q \leq P_0).$$

It now follows from (17) that the sum of the corresponding terms in T is

(18) $$\ll V\left(\frac{P_0^{1+\varepsilon}}{q}\right) q^2 P_0^{2(k-1+k\delta)}$$
$$\ll VP_0^{2k+(2k+1)\delta} \quad (\text{using } q \leq P_0).$$

Case (c). Let $v_1^k \not\equiv v_2^k \pmod q$ $(q \geq 2)$. Then, since $(a, q) = 1$,

$$a(v_1^k - v_2^k) \equiv t \pmod q \quad \text{with} \quad 1 \leq t < q.$$

By (16),

$$q\beta(v_1^k - v_2^k) \ll P_0^{-k\delta/2}, \quad \text{so that} \quad q\beta(v_1^k - v_2^k) = o(1).$$

Hence

$$\|\varphi(v_1) - \varphi(v_2)\| = \left\|\frac{t + o(1)}{q}\right\| \quad (\text{with } q > t \geq 1)$$
$$\geq 1/q.$$

(This does not lead to a good estimate for large q, but we are only considering $q \leq P_0$.) Hence, the sum of the corresponding terms in T is

(19) $$\ll V^2 q^2 \ll V^2 P_0^2 \ll VP_0^3 \quad (\text{cf. (11)}).$$

It now follows from (14), (15), (18) and (19) that

$$|Q(\alpha)|^2 \ll U_0 P_0^{-k+\delta} V P_0^{2k+(2k+1)\delta} \ll (U_0V)^2 (U_0V)^{-1} P_0^{k+(2k+2)\delta}.$$

Hence, since $U_0V = Q(0)$, we have from (10) and (12),

$$|Q(\alpha)|^2 \ll \big(Q(0)\big)^2 P_0^{-k\gamma+\delta} P_0^{-1+\delta} P_0^{k+(2k+2)\delta},$$

so that

$$Q(\alpha) \ll Q(0) P_0^{-\frac{1}{2}\{1-k(1-\gamma)\}+2k\delta}.$$

The lemma now follows since from (5), $P^{1/2} \ll P_0 \ll P^{1/2}$.

3. Further notation. With μ_i defined by (6), let

$$f_i = f_i(\alpha) = \sum_{P^{\mu_i} \leq x \leq 2P^{\mu_i}} e(\alpha x^k), \qquad f = f(\alpha) = \sum_{P \leq x \leq 2P} e(\alpha x^k),$$

$$S(a, q) = \sum_{x=1}^{q} e_q(ax^k), \qquad J(X, Y; \beta) = \frac{1}{k} \sum_{X^k \leq y \leq Y^k} y^{1/k-1} e(\beta y),$$

$$J_i = J_i(\beta) = J(P^{\mu_i}, 2P^{\mu_i}; \beta), \qquad J = J(\beta) = J(P, 2P; \beta),$$

$$g_i = g_i\left(\frac{a}{q} + \beta\right) = q^{-1}S(a, q)J_i(\beta), \qquad g = g\left(\frac{a}{q} + \beta\right) = q^{-1}S(a, q)J(\beta).$$

Since $\left(1 - \frac{1}{k}\right)^k$ is an increasing function of k, it is a numerical verification that

(20) $$\mu_k = \left(1 - \frac{1}{k}\right)^k > \frac{1}{4} + \frac{1}{2k} \quad (k \geq 6).$$

With $\gamma^{(s)}(k)$ given by (8) let the integers s_1, s_2 be chosen to satisfy

(21) $$\gamma^{(s_2)}(k) + \frac{\gamma^{(s_1)}(k) - 1}{4} + \frac{1}{4k} > 1$$

with minimal $s_1 + 2s_2$.

Let

(22) $$F(\alpha) = f(\alpha)\Big\{\prod_{i=1}^{s_2-1} f_i(\alpha)\Big\}, \qquad \tau = P^{k-1+\delta},$$

and

(23) $$r(N) = \int_{\tau^{-1}}^{1+\tau^{-1}} F^2(\alpha) Q(\alpha) e(-N\alpha)\, d\alpha,$$

where $Q(\alpha)$ is defined by (13) with $s = s_1$ (so that every u in the definition of $Q(\alpha)$ is a sum of s_1 non-negative kth powers).

The interval

$$\tau^{-1} < \alpha < 1 + \tau^{-1} \tag{24}$$

is divided as follows. For $0 < q \leqslant P_0$, let $\mathfrak{m}_{a,q}$ denote the interval consisting of those α with

$$\alpha = \frac{a}{q} + \beta, \quad a \leqslant q, \quad (a, q) = 1, \quad |\beta| \leqslant q^{-1}\tau^{-1} \tag{25}$$

and $\overline{\mathfrak{m}}_{a,q}$ the complement of $\mathfrak{m}_{a,q}$ in (24).

The $\mathfrak{m}_{a,q}$'s are disjoint, and their union is denoted by $\mathfrak{m}$. The complement of $\mathfrak{m}$ in (24) is denoted by m; so that by (23),

$$r(N) = \int_{\mathfrak{m}} F^2(\alpha)Q(\alpha)e(-N\alpha)\,d\alpha + \int_{m} F^2(\alpha)Q(\alpha)e(-N\alpha)\,d\alpha. \tag{26}$$

4. Integral over m.

LEMMA 4. *If $\alpha \in m$,*

$$Q(\alpha) \ll Q(0)P^{-\frac{1}{4}\{1-k(1-\gamma^{(s_1)}(k))\}+k\delta} \ll Q(0)N^{-\frac{1}{4k}+\frac{1-\gamma^{(s_1)}(k)}{4}+\delta}.$$

Proof. Every real number α can be represented in the form

$$\alpha = \frac{a}{q} + \beta, \quad 0 < q \leqslant P_0^k, \quad |\beta| \leqslant q^{-1}P_0^{-k}.$$

Thus, if $\alpha \in m$, it must satisfy the hypothesis of either Lemma 2 or Lemma 3 (since $\mathfrak{m}_{a,q}$'s are defined by (25) with $q \leqslant P_0$).

Hence result follows from these two lemmas and (5).

LEMMA 5.

$$\int_0^1 |F(\alpha)|^2 d\alpha \ll P^{1+\mu_1+\ldots+\mu_{s_2-1}+\varepsilon} \ll N^{-\gamma^{(s_2)}(k)+\varepsilon}F^2(0).$$

Proof. The proof follows from (8) with $s = s_2$ since $1, \mu_1, \ldots, \mu_{s_2-1}$ are admissible exponents for kth powers, and the integral is the number of solutions of

$$x^k + \Big(\sum_{i=1}^{s_2-1} x_i^k\Big) = y^k + \Big(\sum_{i=1}^{s_2-1} y_i^k\Big)$$

with

$$\left.\begin{matrix} P \leqslant x \leqslant 2P \\ P \leqslant y \leqslant 2P \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} P^{\mu_i} \leqslant x_i \leqslant 2P^{\mu_i} \\ P^{\mu_i} \leqslant y_i \leqslant 2P^{\mu_i} \end{matrix}\right\} \quad i = 1, \ldots, s_2-1.$$

LEMMA 6.

$$\int_m F^2(\alpha)Q(\alpha)e(-N\alpha)\,d\alpha \ll N^{-1-\delta}F^2(0)Q(0).$$

Proof. The integral is

$$\ll \{\max_{\alpha\in m}|Q(\alpha)|\}\int_0^1 |F(\alpha)|^2 d\alpha$$

$$\ll F^2(0)Q(0)N^{-\gamma^{(s_2)}(k)-\frac{1}{4}+\frac{1-\gamma^{(s_1)}(k)}{4}+2\delta}$$

by Lemmas 4 and 5. Result now follows from (21).

5. Integral over $\mathfrak{m}$.

LEMMA 7. *If $|\beta| \leqslant \frac{1}{2}$, then*

$$g_i\Big(\frac{a}{q}+\beta\Big) \ll q^{-1/k}\min(P^{\mu_i}, P^{\mu_i(1-k)}|\beta|^{-1}) \tag{27}$$

and

$$g\Big(\frac{a}{q}+\beta\Big) \ll q^{-1/k}\min(P, P^{1-k}|\beta|^{-1}). \tag{28}$$

Proof. Lemma 5 of [3].

The next lemma is the main theorem in [9].

LEMMA 8.

$$\sum_{1\leqslant x\leqslant P} e_q(ax^k) - \frac{P}{q}S(a, q) \ll q^{1/2+\varepsilon}.$$

LEMMA 9.

$$f_i\Big(\frac{a}{q}+\beta\Big) - g_i\Big(\frac{a}{q}+\beta\Big) \ll q^{1/2+\varepsilon}\max(1, P^{k\mu_i}|\beta|).$$

Proof. The proof follows by a partial summation with Lemma 8 (with P^{μ_i} in place of P).

LEMMA 10. *On* $\mathfrak{m}_{a,q}$ $(q \leqslant P_0)$,

$$f_i - g_i \ll q^{1/2+\varepsilon} \quad (1 \leqslant i \leqslant k) \tag{29}$$

and

$$f - g \ll q^{3/4+\varepsilon}. \tag{30}$$

Proof. (30) is Lemma 8 of [3]. Since $|\beta| \leqslant q^{-1}P^{-k+1-\delta}$,

$$P^{k\mu_i}|\beta| \ll P^{k(1-1/k)}q^{-1}P^{-k+1-\delta} \ll 1 \quad (i \geqslant 1).$$

Hence (29) follows from Lemma 9.

LEMMA 11. *On* $\mathfrak{m}_{a,q}$ $(q \leqslant P_0)$,

$$\max(|f_i|, |g_i|) \ll q^{-1/k}P^{\mu_i} \quad (1 \leqslant i \leqslant k) \tag{31}$$

and

(32) $$\max(|f|, |g|) \ll q^{-1/k} \min\{P, P^{1-k}|\beta|^{-1}\} \quad (k \geqslant 4).$$

Proof. By (20), for $1 \leqslant i \leqslant k$,

$$q^{-1/k}P^{\mu_i} \gg q^{-1/k}P^{1/4+1/2k+\delta} \gg P_0^{-1/k}P_0^{1/2+1/k+\delta} \gg P_0^{1/2+\delta} \gg q^{1/2+\varepsilon}.$$

Hence (31) follows from (27) and (29). Similarly, (32) follows from (28) and (30) (since $|\beta|^{-1} \gg qP^{k-1+\delta}$).

LEMMA 12. *On* $\mathfrak{m}_{a,q}$,

33) $$f^2-g^2 \ll q^{3/4+\varepsilon}\{q^{-1/k}\min(P, P^{1-k}|\beta|^{-1})\}$$

and

(34) $$f_1^2f_2^2\ldots f_k^2-g_1^2g_2^2\ldots g_k^2 \ll q^{1/2+\varepsilon}q^{-(2k-1)/k}P^{2(\mu_1+\mu_2+\ldots+\mu_k)}P^{-\left(\frac{1}{4}+\frac{1}{2k}\right)}.$$

Proof. $f^2-g^2 \ll |f-g||f+g|$, so that (33) follows from (30) and (32) Now

(35) $$f_1^2f_2^2\ldots f_k^2-g_1^2g_2^2\ldots g_k^2$$
$$= (f_1^2-g_1^2)f_2^2f_3^2\ldots f_k^2+\Big\{\sum_{i=1}^{k-2} g_1^2g_2^2\ldots g_i^2(f_{i+1}^2-g_{i+1}^2)f_{i+2}^2\ldots f_k^2\Big\}+$$
$$+g_1^2g_2^2\ldots g_{k-1}^2(f_k^2-g_k^2),$$

and for $1 \leqslant i \leqslant k$,

$$f_i^2-g_i^2 \ll |f_i-g_i|\{\max(|f_i|, |g_i|)\}$$
$$\ll q^{1/2+\varepsilon}q^{-1/k}P^{\mu_i} \quad \text{(by (29) and (31))}.$$

Thus, estimating the absolute value of each term on the right-hand side of (35) by using Lemma 11, the result follows since

$$\mu_i > \frac{1}{4}+\frac{1}{2k} \quad (\text{for } 1 \leqslant i \leqslant k).$$

LEMMA 13.

$$\sum_{q\leqslant P_0}\sum_a \int_{\mathfrak{m}_{a,q}} |f^2f_1^2f_2^2\ldots f_k^2-g^2g_1^2g_2^2\ldots g_k^2|\,d\alpha \ll P^{2(1+\mu_1+\ldots+\mu_k)-k(1+\delta)}.$$

Proof. By Lemmas 11 and 12,

$$f^2f_1^2f_2^2\ldots f_k^2-g^2g_1^2g_2^2\ldots g_k^2$$
$$= (f^2-g^2)f_1^2f_2^2\ldots f_k^2+g^2(f_1^2f_2^2\ldots f_k^2-g_1^2g_2^2\ldots g_k^2)$$
$$\ll q^{3/4+\varepsilon}q^{-1/k}\{\min(P, P^{1-k}|\beta|^{-1})\}q^{-2}P^{2(\mu_1+\ldots+\mu_k)}+$$
$$+q^{-2/k}\{\min(P^2, P^{2(1-k)}|\beta|^{-2})\}q^{1/2+\varepsilon}q^{-(2k-1)/k}P^{2(\mu_1+\ldots+\mu_k)-\frac{1}{4}-\frac{1}{2k}}.$$

Now (with $\alpha = a/q+\beta$),

$$\int_{\mathfrak{m}_{a,q}} \min(P, P^{1-k}|\beta|^{-1})\,d\beta \ll P^{1-k+\varepsilon} \quad \text{and}$$
$$\int_{\mathfrak{m}_{a,q}} \min(P^2, P^{2(1-k)}|\beta|^{-2})\,d\beta \ll P^{2-k}.$$

Hence, the integral of the lemma is

$$\ll \sum_{q\leqslant P_0}\sum_a \{q^{-\frac{5}{4}-\frac{1}{k}+\varepsilon}P^{2(1+\mu_1+\ldots+\mu_k)-1-k+\varepsilon}+q^{-\frac{3}{2}-\frac{1}{k}+\varepsilon}P^{2(1+\mu_1+\ldots+\mu_k)-k-\frac{1}{4}-\frac{1}{2k}}\}.$$

Also

$$\sum_{q\leqslant P_0}\sum_a q^{-\frac{5}{4}-\frac{1}{k}+\varepsilon} \ll P_0^{3/4} \ll P^{3/8} \quad \text{and} \quad \sum_{q\leqslant P_0}\sum_a q^{-\frac{3}{2}-\frac{1}{k}+\varepsilon} \ll P_0^{1/2} \ll P^{1/4}.$$

The lemma now follows.

LEMMA 14.

$$\sum_{q\leqslant P_0}\sum_a \int_{\overline{\mathfrak{m}}_{a,q}} |g^2g_1^2g_2^2\ldots g_k^2|\,d\alpha \ll P^{2(1+\mu_1+\ldots+\mu_k)-k(1+\delta)}.$$

Proof. Using the estimates

$$g \ll q^{-1/k}P^{1-k}|\beta|^{-1}, \quad g_i \ll q^{-1/k}P^{\mu_i} \quad (1 \leqslant i \leqslant k)$$

(from Lemma 7),

$$\int_{\overline{\mathfrak{m}}_{a,q}} |g^2g_1^2\ldots g_k^2|\,d\alpha \ll P^{2(\mu_1+\ldots+\mu_k)}P^{2(1-k)}q^{-2(k+1)/k}\int_{q^{-1}\tau^{-1}}^{\infty}\beta^{-2}\,d\beta,$$

and

$$\int_{q^{-1}\tau^{-1}}^{\infty}\beta^{-2}\,d\beta \ll q\tau \ll qP^{k-1+\delta}.$$

Hence, the integral of the lemma is

$$\ll \sum_{q\leqslant P_0}\sum_a q^{-\left(\frac{k+2}{k}\right)}P^{2(1+\mu_1+\ldots+\mu_k)}P^{-k-1+\delta}.$$

Result now follows since

$$\sum_{q\leqslant P_0}\sum_a q^{-\left(\frac{k+2}{k}\right)} \ll P_0 \ll P^{1/2}.$$

Using the trivial estimates $f_i(\alpha) \ll f_i(0)$ for $k+1 \leqslant i \leqslant s_2-1$, and $Q(\alpha) \ll Q(0)$ (and noting $P^{-k(1+\delta)} \ll N^{-1-\delta}$), Lemmas 13 and 14 respectively give

(36) $$\sum_{q\leqslant P_0}\sum_{a}\int_{\mathfrak{m}_{a,q}}|f^2f_1^2\dots f_k^2-g^2g_1^2\dots g_k^2||f_{k+1}^2\dots f_{s_2-1}^2||Q(\alpha)|\,d\alpha \ll N^{-1-\delta}F^2(0)Q(0),$$

and

(37) $$\sum_{q\leqslant P_0}\sum_{a}\int_{\mathfrak{m}_{a,q}}|g^2g_1^2\dots g_k^2||f_{k+1}^2\dots f_{s_2-1}^2||Q(\alpha)|\,d\alpha \ll N^{-1-\delta}F^2(0)Q(0)$$

(since $P^{2(1+\mu_1+\dots+\mu_k)} \ll f^2(0)f_1^2(0)\dots f_k^2(0)$).

If $A(n,q) = \sum_a \{q^{-1}S(a,q)\}^{2k+2}e_q(-an)$, the singular series that we have to consider is $\sum_{q=1}^{\infty}A(n,q)$. It can be proved in a standard way that this is absolutely convergent (see for example, Lemma 11 in Ch. II of [13]), and that $\sum_{q>P_0}A(n,q) \ll N^{-\delta}$ (for $n \ll N$). Also, the positiveness of the singular series depends on the solubility of the usual p-adic condition to be satisfied by n. (Here the n's will be of the form $N-X-Y-v^ku$ (u as in the definition of $Q(\alpha)$ cf. (13)), and each of X, Y is of the form $\sum_{i=k+1}^{s_2-1}x_i^k$ with $P^{\mu_i}\leqslant x_i\leqslant 2P^{\mu_i}$.)

If $\Gamma(k)\leqslant 2k+2$, this is satisfied by every n. If $2k+2<\Gamma(k)\leqslant 4k$ (it is known that $\Gamma(k)\leqslant 4k$), as in [1], we need to impose certain congruence conditions (mod $4k$) on the x's in the definitions of $f_i(\alpha)$ for $k<i\leqslant s_2-1$. These conditions will not affect the bounds for $\gamma^{(s_2)}(k)$ in (8) by more than $N^{-\varepsilon}$; so that the lemmas proved for integrals over m and $\mathfrak{m}$ still remain valid. (This problem does arise in the case $k=16$ since $\Gamma(16)=64$, but not for the other values of k in $9\leqslant k\leqslant 20$ since from [8], $\Gamma(9)=13$, $\Gamma(10)=12$, $\Gamma(11)=11$, $\Gamma(12)=16$, $\Gamma(13)=6$, $\Gamma(14)=14$, $\Gamma(15)=15$, $\Gamma(17)=6$, $\Gamma(18)=27$, $\Gamma(19)=4$, $\Gamma(20)=25$.)

It now follows in a standard way from (26), (36), (37), Lemma 6 and the positiveness and convergence of the singular series that

(38) $$r(n) \gg N^{-1}F^2(0)Q(0).$$

6. Proof of Theorem 1. Since $\gamma^{(s_2)}(k) = 1-\left(1-\frac{1}{k}\right)^{s_2}$ and $\gamma^{(s_1)}(k) = 1-\left(1-\frac{1}{k}\right)^{s_1}$, choosing s_1, s_2 (as in [1]) with

(39) $$s_2 = \left[\frac{\log 6k}{-\log\left(1-\frac{1}{k}\right)}+1\right],\quad s_1 = \left[\frac{\log 3k}{-\log\left(1-\frac{1}{k}\right)}+1\right],$$

we see that the condition (21) is satisfied. It is verified from (39) that

(40) $$2s_2+s_1 \leqslant k\{3\log k+\log 108\}.$$

Since $r(N)$ does not exceed the number of representations of N as sums of $2s_2+s_1$ positive kth powers, it now follows from (38) and (40) that

$$G(k)\leqslant k\{3\log k+\log 108\},$$

proving Theorem 1.

(The improvement in this paper over [1] depends on the removal of a factor like $\{\sum_{P_0\leqslant x\leqslant 2P_0} e(\alpha x^k)\}^{[k/2]}$ which was introduced in [1].)

7. Further lemmas for Theorems 2 and 3. The next two lemmas correspond to Theorem 1 and its corollary in [6].

LEMMA 15. *If* $\theta = 1-k^{-1}$, $\lambda_0 = 1$, $\lambda_1 = \sigma$, $\lambda_i = \sigma\theta^{i-1}$ $(2\leqslant i\leqslant s-1)$ *with* $0<\sigma\leqslant 1$, $k\sigma-(k-1)\leqslant \sigma\theta^{s-2}$, *then* $\lambda_0, \lambda_1, \dots, \lambda_{s-1}$ *are admissible exponents.*

LEMMA 16. *In addition to the hypothesis of Lemma* 1, *let*

(41) $$\sigma = (k-1)/(k-\theta^{s-2}),$$

and

(42) $$\alpha = \alpha^{(s)}(k) = \frac{\lambda_0+\lambda_1+\dots+\lambda_{s-1}}{k} = 1-\frac{\left(1-\frac{1}{k}\right)^{s-1}\left(1-\frac{2}{k}\right)}{1-\frac{1}{k}\left(1-\frac{1}{k}\right)^{s-2}}.$$

Then,

(43) $$U_s(k; P^k, s2^kP^k) \gg P^{k\alpha-\varepsilon}.$$

LEMMA 17. *With α given by* (42), *let*

(44) $$\beta = \max_{h\leqslant k-2}\frac{1}{k}\left\{1+\frac{(2^h-1)(k-1)+(h+1)}{2^h-1+\alpha}\alpha\right\}.$$

Then,

(45) $$U_{s+1}(k; P^k, (s+1)2^kP^k) \gg P^{k\beta-\varepsilon}.$$

Proof. This is essentially Theorem 2 of [5] (since $1/k<\alpha<1$).

The next lemma can be proved by the same method as in Theorems 1 and 2 of [5], but needs some modifications which are crucial to the proofs of Theorems 2 and 3 in this paper.

LEMMA 18. *With $\lambda_0, \lambda_1, \dots, \lambda_{s-1}$ and α as in Lemmas* 15 *and* 16, *let Lemma* 17 *be applied r times successively to give*

(46) $$U_{r+s}(k; P^k, (r+s)2^kP^k) \gg P^{k\gamma'-\varepsilon}.$$

Then, there exist numbers $\lambda_0', \lambda_1', \ldots, \lambda_{r+s-1}'$ forming admissible exponents with

(47) $$\lambda_0' = 1, \quad 0 < \lambda_i' < \lambda_{i-1}' \quad (i = 1, \ldots, r+s-1).$$

Furthermore,

(48) $$\gamma' = \frac{\lambda_0' + \lambda_1' + \ldots + \lambda_{r+s-1}'}{k} \quad \text{and} \quad \lambda_i' > \left(1 - \frac{1}{k}\right)^i \quad (i = 1, \ldots, r+s-1).$$

(The λ''s are computable.)

Proof. It is sufficient to consider $r = 1$, for then the lemma would follow inductively for $r \geqslant 2$. It is important to make the following change in the proof of Theorem 1 of [5]. In place of equation (1) in [5] (with the same λ) we start with

(49) $$x^k + u_i = y^k + u_j,$$

where $u_i = x_0^k + \ldots + x_{s-1}^k$, $u_j = y_0^k + \ldots + y_{s-1}^k$, subject to

$$\begin{aligned} &P \leqslant x \leqslant 2P, \quad P^{\lambda\lambda_i} \leqslant x_i \leqslant 2P^{\lambda\lambda_i}, \\ &P \leqslant y \leqslant 2P, \quad P^{\lambda\lambda_i} \leqslant y_i \leqslant 2P^{\lambda\lambda_i} \end{aligned} \quad (i = 0, 1, \ldots, s-1).$$

Note that the u_i's or u_j's (unlike in [5]) need not all be distinct, but from Lemma 17, the number of solutions of (49) with $x = y$ (and hence with $u_i = u_j$) is

(50) $$\ll P^{1+\lambda(\lambda_0+\lambda_1+\ldots+\lambda_{s-1})+\varepsilon},$$

which replaces the corresponding estimate PU in [5]. However, $U \gg P^{\lambda(\lambda_0+\lambda_1+\ldots+\lambda_{s-1})-\varepsilon}$, and hence the estimate (50) is weaker by only $P^{2\varepsilon}$. The arguments for the case $x \neq y$ will be the same as in [5], but here again the estimate will be weaker by only $P^{l\varepsilon}$ for some constant l. We take $\lambda_i' = \lambda\lambda_i$. Since $\lambda > 1-1/k$ and $\sigma > 1-1/k$ (from (41)), (48) follows.

In the rest of the paper, we abbreviate for $U_s(k; P^k, s2^kP^k)$ by $U_s(k)$. For the values of k under consideration, up to a certain value of s, the bounds for $U_s(k)$ given by (42) are better than those given by (44). Thereafter (for larger s), (44) gives better bounds. However, for $13 \leqslant k \leqslant 20$, these improvements do not seem to be sufficient to get better estimates for $G(k)$. Hence, for the sake of computational convenience, we use only (42) for these values of k. We choose two integers $s_1 = s_1(k)$, $s_2 = s_2(k)$ with

(51) $$U_{s_1}(k) \gg N^{\gamma_1-\varepsilon}, \quad U_{s_2}(k) \gg N^{\gamma_2-\varepsilon}$$

as follows:

(a) $k = 9$. With $k = 9$, $s = 7$, (42) gives $\alpha = \alpha^{(7)}(9) > 0.591135$.

Also (44) with $k = 9$ gives (by taking $h = 5, 6$ and 7 respectively)

(52) $$\beta \geqslant \frac{31 + 255\alpha}{9(31+\alpha)},$$

(53) $$\beta \geqslant \frac{63 + 512\alpha}{9(63+\alpha)},$$

and

(54) $$\beta \geqslant \frac{127 + 1025\alpha}{9(127+\alpha)}.$$

Using (52) once; and then (53) thrice; and then (54) 15 and 21 times respectively we get

$$U_{26}(9) \gg N^{\gamma_1-\varepsilon}, \quad U_{32}(9) \gg N^{\gamma_2-\varepsilon}$$

(with $s_1 = 26$, $s_2 = 32$; $\gamma_1 = 0.961709$, $\gamma_2 = 0.981956$).

Taking $h = k-2$ (for $k = 10, 11$ and 12), we have

(55) $$\beta \geqslant \frac{255 + 2305\alpha}{10(255+\alpha)} \quad (k = 10),$$

(56) $$\beta \geqslant \frac{511 + 5121\alpha}{11(511+\alpha)} \quad (k = 11),$$

(57) $$\beta \geqslant \frac{1023 + 11265\alpha}{12(1023+\alpha)} \quad (k = 12).$$

(b) Taking $k = 10$, $s = 14$ in (42), we get $\alpha^{(14)} > 0.79074$. Then, we use (55) 20 and 22 times to get γ_1, γ_2 (with $s_1 = 34$, $s_2 = 36$).

(c) $k = 11$, $s = 20$ in (42) gives $\alpha^{(20)} > 0.863996$. Then (56) is used 17 and 22 times to get γ_1, γ_2 (with $s_1 = 37$, $s_2 = 42$).

(d) With $k = 12$, $s = 26$ in (42), $\alpha^{(26)} > 0.904366$. (57) is now used 14 and 21 times (with $s_1 = 40$, $s_2 = 47$) to get γ_1, γ_2.

(e) For $13 \leqslant k \leqslant 20$, (42) is used with the values of s_1, s_2 given in the tables to get γ_1, γ_2 satisfying (51).

8. Proofs of Theorems 2 and 3. Although the proofs of Theorems 2 and 3 are essentially the same as that of Theorem 1, it is necessary to make a slight change. In place of μ_i defined by (6), we have λ_i from Lemma 16 or λ_i' from Lemma 18 ($i \geqslant 1$). In either case, we have $\mu_1 > 1-1/k$ and $(1-1/k)^2 < \mu_2 < 1-1/k$. Hence, in place of (29) (for $i = 1$), we use $f_1 - g_1 \ll q^{3/4+\varepsilon}$ (which is Lemma 8 of [3]). From this also, it is an easy deduction that on $\mathfrak{m}_{a,q}$ $(q \leqslant P_0)$,

k	s_1	γ_1	s_2	γ_2	$2s_2+s_1$
9	26	0.961709	32	0.981956	90
10	34	0.976306	36	0.980953	106
11	37	0.973909	42	0.983953	121
12	40	0.972075	48	0.986185	136
13	48	0.980299	52	0.985704	152
14	51	0.978882	58	0.987439	167
15	57	0.981779	63	0.98796	183
16	61	0.981765	69	0.989125	199
17	65	0.981754	75	0.990054	215
18	71	0.983719	80	0.990271	231
19	78	0.986067	85	0.99046	248
20	80	0.984209	92	0.991503	264

$$\max(|f_1|, |g_1|) \ll q^{-1/k}P^{\mu_1} \quad \text{and} \quad f_1-g_1 \ll q^{\frac{1}{2}+\varepsilon}P^{\mu_1}P^{-\frac{1}{4}-\frac{1}{2k}},$$

as required in the proof of Lemmas 12 and 13.

We also note that the set corresponding to $\mathscr{U}$ in § 2 is constructed in a slightly different way. However, this does not affect the method since we use only the estimate for U_0, the number of elements in $\mathscr{U}$. The rest of the arguments remain valid. Thus, if the integers s_1, s_2 are such that (corresponding to (21))

$$\gamma_2+\frac{\gamma_1-1}{4}+\frac{1}{4k}>1, \tag{58}$$

then $G(k)\leqslant 2s_2+s_1$.

(58) is satisfied by the values given in the tables, and the theorems follow. It is estimated (by comparing (8) and (42)) that the estimate for $G(k)$ given by Theorem 1 can be improved by about 10 for further values of k.

Remark. The use of a factor like $\sum_{p_1,p_2,r} e(\alpha p_1^k p_2^k r^k)$ in [11] (see (2.1)) is dispensed with in this paper. While (2.20) of [11] gives slightly better bounds (for certain values of k and s) for $U_s(k)$ than the corollary to Theorem 1 in [6], the improvements do not seem to be sufficient to obtain better bounds for $G(k)$ (in this paper).

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References

[1] Jing-jun Chen, *On Waring's problem for n-th powers*, Acta Math. Sinica 8 (1958), pp. 849–853.
[2] R. J. Cook, *A note on Waring's problem*, Bull. London Math. Soc. 5 (1973), pp. 11–12.
[3] H. Davenport, *On Waring's problem for fourth powers*, Ann. of Math. 40 (1939), pp. 731–747.
[4] — *On Waring's problem for fifth and sixth powers*, Amer. J. Math. 64 (1942), pp. 199–207.
[5] — *On sums of positive integral k-th powers*, ibid. 64 (1942), pp. 189–198.
[6] H. Davenport and P. Erdös, *On sums of positive integral k-th powers*, Ann. of Math. 40 (1939), pp. 533–536.
[7] T. Estermann, *On Waring's problem for fourth and higher powers*, Acta Arith. 2 (1937), pp. 197–211.
[8] G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum VIII*, Proc. London Math. Soc. 28 (1928), pp. 518–542.
[9] L. K. Hua, *On exponential sums*, Science Record (N. S) 1 (1) (1957), pp. 1–4.
[10] V. Narasimhamurti, *On Waring's problem for 8th, 9th and 10th powers*, J. Indian Math. Soc. 5 (1941), p. 122.
[11] R. C. Vaughan, *Homogeneous additive equations and Waring's problem*, Acta Arith. 33 (1977), pp. 231–253.
[12] I. M. Vinogradov, *On an upper bound for G(n)*, Izv. Akad. Nauk SSSR. Ser. Mat. 23 (1959), pp. 637–642.
[13] — *The method of trigonometrical sums in the theory of numbers*, English translation by K. F. Roth and A. Davenport, Interscience Publishers.

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