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On the residue mod 2 and mod 4 of $p(n)$

by

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For $n \geqslant 0$ let $p(n)$ denote the number of unrestricted partitions of n, $q(n)$ denote the number of partitions of n into different odd parts, and $r(n)$ denote the number of solutions with $n_i \geqslant 0$ of the equation

$$n = \Delta(n_1) + 4\Delta(n_2) + 16\,\Delta(n_3) + \ldots$$

where $\Delta(n) = \frac{1}{2}n(n+1)$.

We prove the following results, namely

THEOREM 1.

$$p(n) \equiv q(n) \equiv r(n) \bmod 2.$$

THEOREM 2.

$$p(n) \geqslant q(n) \geqslant r(n).$$

THEOREM 3.

$$r(0) = 1, \quad r(2) = 0,$$

and for $n \geqslant 0$,

$$r(4n) = r(n) + \sum_{k\geqslant 1} \{r(n-(8k^2-k)) + r(n-(8k^2+k))\},$$

$$r(4n+1) = r(n) + \sum_{k\geqslant 1} \{r(n-(8k^2-3k)) + r(n-(8k^2+3k))\},$$

$$r(4n+3) = r(n) + \sum_{k\geqslant 1} \{r(n-(8k^2-5k)) + r(n-(8k^2+5k))\},$$

$$r(4n+6) = r(n) + \sum_{k\geqslant 1} \{r(n-(8k^2-7k)) + r(n-(8k^2+7k))\}.$$

THEOREM 4.

$$p(n) \equiv q(n) + 2\sum_{k\geqslant 1} q(n-2k^2) \bmod 4.$$

THEOREM 5.

$$p(n) \equiv r(n) + 2\sum_{\substack{k\geqslant 1\\ t(k)\,\mathrm{even}}} r(n-2k^2) \bmod 4,$$

where $t(k)$ is the highest power of 2 which divides k.

THEOREM 6.

$$q(n) \equiv r(n)+2 \sum_{\substack{k\geqslant 1\\ t(k)\text{even}}} r(n-8k^2)\bmod 4.$$

Remarks. 1. The congruences modulo 2 for $p(n)$ implied by Theorems 1 and 3, namely

$$p(4n) \equiv p(n)+\sum_{k\geqslant 1}\{p(n-(8k^2-k))+p(n-(8k^2+k))\}$$

and so on, were first found by MacMahon [2] and were later employed by Parkin and Shanks [3] (who also gave proofs) in an investigation of the parity of $p(n)$ for n up to 2×10^6.

2. Theorem 2 greatly understates the facts. Indeed $p(n)$ is far greater than $r(n)$. The first few values of $p(n)$, $q(n)$ and $r(n)$ are given in the following table

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77
$q(n)$	1	1	0	1	1	1	1	1	2	2	2	2	3
$r(n)$	1	1	0	1	1	1	1	1	0	0	2	0	1

Proof of Theorem 1.

$$\sum_{n\geqslant 0} p(n)x^n = \prod_{n\geqslant 1}\frac{1}{(1-x^n)} = \prod_{n\geqslant 1}\left(\frac{1+x^n}{1-x^{2n}}\right)$$

$$= \prod_{n\geqslant 1}(1+x^{2n-1})\prod_{n\geqslant 1}\left(\frac{1+x^{2n}}{1-x^{2n}}\right) = \sum_{n\geqslant 0} q(n)x^n\prod_{n\geqslant 1}\left(\frac{1+x^{2n}}{1-x^{2n}}\right)$$

$$\equiv \sum_{n\geqslant 0} q(n)x^n \bmod 2$$

(since $\dfrac{1+x^n}{1-x^n}\equiv 1\bmod 2$), so $p(n)\equiv q(n)\bmod 2$.

$$\sum_{n\geqslant 0} q(n)x^n = \prod_{n\geqslant 0}(1+x^{2n-1}) = \prod_{n\geqslant 1}(1+x^{4n-3})(1+x^{4n-1})$$

$$= \Big[1+\sum_{n\geqslant 1}(x^{2n^2-n}+x^{2n^2+n})\Big]\Big/\prod_{n\geqslant 1}(1-x^{4n}) \text{ by Jacobi}$$

$$= \sum_{n\geqslant 0} x^{\Delta(n)}\Big/\prod_{n\geqslant 1}(1-x^{4n}) = \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} p(n)x^{4n}$$

$$\equiv \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} q(n)x^{4n}\bmod 2$$

from which it follows by iteration that, mod 2,

$$\sum_{n\geqslant 0} q(n)x^n \equiv \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} x^{4\Delta(n)}\sum_{n\geqslant 0} x^{16\Delta(n)}\ldots = \sum_{n\geqslant 0} r(n)x^n,$$

so $q(n)\equiv r(n)\bmod 2$.

Proof of Theorem 2. Clearly $p(n)\geqslant q(n)$.

Let us write $\sum_{n\geqslant 0} a_n x^n \geqslant \sum_{n\geqslant 0} b_n x^n$ if $a_n\geqslant b_n$ for every n. Then

$$\sum_{n\geqslant 0} q(n)x^n = \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} p(n)x^{4n} \geqslant \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} q(n)x^{4n},$$

from which it follows by iteration that

$$\sum_{n\geqslant 0} q(n)x^n \geqslant \sum_{n\geqslant 0} r(n)x^n,$$

or,

$$q(n)\geqslant r(n).$$

Proof of Theorem 3. We have

$$\sum_{n\geqslant 0} r(n)x^n = \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} x^{4\Delta(n)}\sum_{n\geqslant 0} x^{16\Delta(n)}\ldots = \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} r(n)x^{4n}.$$

If we now substitute

$$\sum_{n\geqslant 0} x^{\Delta(n)} = \sum_{n\geqslant 0}(x^{\Delta(8n)}+x^{\Delta(8n+7)})+\sum_{n\geqslant 0}(x^{\Delta(8n+1)}+x^{\Delta(8n+6)})+$$

$$+\sum_{n\geqslant 0}(x^{\Delta(8n+2)}+x^{\Delta(8n+5)})+\sum_{n\geqslant 0}(x^{\Delta(8n+3)}+x^{\Delta(8n+4)})$$

$$= \Big[1+\sum_{n\geqslant 1}(x^{32n^2-4n}+x^{32n^2+4n})\Big]+x\Big[1+\sum_{n\geqslant 1}(x^{32n^2-12n}+x^{32n^2+12n})\Big]+$$

$$+x^3\Big[1+\sum_{n\geqslant 1}(x^{32n^2-20n}+x^{32n^2+20n})\Big]+x^6\Big[1+\sum_{n\geqslant 1}(x^{32n^2-28n}+x^{32n^2+28n})\Big]$$

and compare coefficients, the result follows.

Proof of Theorem 4.

$$\sum_{n\geqslant 0} p(n)x^n = \sum_{n\geqslant 0} q(n)x^n\cdot\prod_{n\geqslant 1}\left(\frac{1+x^{2n}}{1-x^{2n}}\right)$$

$$= \sum_{n\geqslant 0} q(n)x^n\Big/\Big\{1+2\sum_{n\geqslant 1}(-1)^n x^{2n^2}\Big\}$$

([1], eq. (2.2.12))

$$\equiv \sum_{n\geqslant 0} q(n)x^n\Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\} \bmod 4,$$

$$\text{(since } \Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\}\Big\{1+2\sum_{n\geqslant 1}(-1)^n x^{2n^2}\Big\} \equiv 1+4\sum_{n\geqslant 1} x^{8n^2} \equiv 1 \bmod 4)$$

from which the result follows.

Proof of Theorem 5. Modulo 4, we have

$$\sum_{n\geqslant 0} p(n)x^n \equiv \Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\}\sum_{n\geqslant 0} q(n)x^n$$
$$= \Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\}\sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} p(n)x^{4n},$$

from which it follows by iteration that, modulo 4,

$$\sum_{n\geqslant 0} p(n)x^n \equiv \Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\}\sum_{n\geqslant 0} x^{\Delta(n)}\Big\{1+2\sum_{n\geqslant 1} x^{8n^2}\Big\}\sum_{n\geqslant 0} x^{4\Delta(n)}\ldots$$
$$= \Big\{1+2\sum_{n\geqslant 1} x^{2n^2}\Big\}\Big\{1+2\sum_{n\geqslant 1} x^{8n^2}\Big\}\ldots\sum_{n\geqslant 0} r(n)x^n$$
$$\equiv \Big\{1+2\Big(\sum_{n\geqslant 1} x^{2n^2}+\sum_{n\geqslant 1} x^{8n^2}+\sum_{n\geqslant 1} x^{32n^2}+\ldots\Big)\Big\}\sum_{n\geqslant 0} r(n)x^n$$
$$= \Big\{1+2\sum_{n\geqslant 1}\big(t(n)+1\big)x^{2n^2}\Big\}\sum_{n\geqslant 0} r(n)x^n$$
$$\equiv \Big\{1+2\sum_{\substack{n\geqslant 1\\ t(n)\,\text{even}}} x^{2n^2}\Big\}\sum_{n\geqslant 0} r(n)x^n,$$

from which the result follows.

Proof of Theorem 6. Modulo 4, we have

$$\sum_{n\geqslant 0} q(n)x^n = \sum_{n\geqslant 0} x^{\Delta(n)}\sum_{n\geqslant 0} p(n)x^{4n}$$
$$\equiv \sum_{n\geqslant 0} x^{\Delta(n)}\Big\{1+2\sum_{\substack{n\geqslant 1\\ t(n)\,\text{even}}} x^{8n^2}\Big\}\sum_{n\geqslant 0} r(n)x^{4n}$$
$$= \Big\{1+2\sum_{\substack{n\geqslant 1\\ t(n)\,\text{even}}} x^{8n^2}\Big\}\sum_{n\geqslant 0} r(n)x^n$$

from which the result follows.

I am indebted to the referee for his very nice proof of Theorem 4.

References

[1] George E. Andrews, *The theory of partitions*, Addison-Wesley, 1976.
[2] P. A. MacMahon, *Note on the parity of the number which enumerates the partitions of a number*, Proc. Camb. Phil. Soc. 20 (1921), pp. 281–283.
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