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The metrical theory of continued fractions to the nearer integer

by

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Let I be $[-\frac{1}{2}, \frac{1}{2}]$ and let $z_0 \in I$ be nonzero. Let $a_1 = a_1(z_0)$ be the sign of z_0 and let $b_1 = b_1(z_0)$ be $[a_1/z_0]$ if $\{a_1/z_0\} \leqslant \frac{1}{2}$ and $[a_1/z_0]+1$ otherwise. Then $z_1 = (a_1/z_0) - b_1$ is in I. This process may now be applied to z_1 to yield a_2, b_2, and $z_2 \in I$. It is clear that for any irrational $z_0 \in I$ this process may be continued indefinitely and thus a unique sequence of digits a_n, b_n is formed for each such z_0. The resulting expression

$$z_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots$$

will be called the (semiregular) *continued fraction to the nearer integer* for z_0. The $b_1, b_2, \ldots$ are the *partial denominators* and the $a_1, a_2, \ldots$ are the *partial numerators*. If this continued fraction is terminated at the nth term ($n \geqslant 1$), the resulting rational number, the *n-th convergent*, may be expressed in the form

$$\frac{A_n}{B_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}$$

where the A_n and B_n satisfy the recursion formulas

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \qquad B_n = b_n B_{n-1} + a_n B_{n-2}$$

and where $A_{-1} = 1$, $A_0 = 0$, $B_{-1} = 0$, and $B_0 = 1$.

Let $N(x)$ be the nearer integer function given by $N(x) = [x]$ if $\{x\} \leqslant \frac{1}{2}$ and $[x]+1$ if $\{x\} > \frac{1}{2}$. Then the transformation $T: I \to I$ given by $Tx = \operatorname{sign}(x)\left(\frac{1}{x} - N\left(\frac{1}{x}\right)\right)$ generates the continued fraction to the nearer integer algorithm in the sense that $T^n z_0 = z_n$ for $n \geqslant 0$ where T^0 is the identity transformation of I and T^n is the n-fold composition of T with itself for $n \geqslant 1$. That T is indecomposable with respect to the Lebesgue measure follows by translating Knopp's Theorem ([2]) for regular continued fractions. Let A be $\frac{1}{2}(1+\sqrt{5})$ and let $p(t)$ be $1/(A+1+t)$ if $-\frac{1}{2} \leqslant t < 0$ and $1/(A+t)$ if $0 \leqslant t \leqslant \frac{1}{2}$. Then if E is any Lebesgue measurable set, the measure

$$\nu(E) = \frac{1}{\log A} \int\limits_E p(t)\,dt$$

is invariant under T, as may be checked by direct calculation. These results are included in [6] and also are announced by Rieger ([5]).

Let $x \in I$ and let $\nu_n(x)$ for $n \geqslant 0$ denote the Lebesgue measure of the set of $z_0 \in I$ such that $-\frac{1}{2} \leqslant T^n z_0 < x$. The estimate $\nu_n'(x) = \nu'(x)\left(1+O(q^n)\right)$, where $0 < q < \frac{1}{2}$ is a constant, follows from the general

THEOREM. *Let $f_0(x)$ be any twice differentiable function on I such that $f_0(-\frac{1}{2}) = 0$ and $f_0(\frac{1}{2}) = 1$. Let the sequence of functions $f_0(x)$, $f_1(x), \ldots$ be defined recursively by*

$$f_{n+1}(x) = \begin{cases} \displaystyle\sum_{k=3}^{\infty}\left(f_n\left(\frac{1}{k-\frac{1}{2}}\right)-f_n\left(\frac{1}{k+x}\right)\right)+\left(f_n\left(\frac{-1}{k-x}\right)-f_n\left(\frac{-1}{k-\frac{1}{2}}\right)\right) & \\ \qquad\qquad\qquad\qquad\qquad \text{if} \quad -\frac{1}{2} \leqslant x < 0; & \\ f_n(\frac{1}{2})-f_n\left(\frac{1}{2+x}\right)+f_n\left(\frac{-1}{2+x}\right)-f_n(-\frac{1}{2})+ & \\ \displaystyle +\sum_{k=3}^{\infty}\left(f_n\left(\frac{1}{k-\frac{1}{2}}\right)-f_n\left(\frac{1}{k+x}\right)\right)+\left(f_n\left(\frac{-1}{k+x}\right)-f_n\left(\frac{-1}{k-\frac{1}{2}}\right)\right) & \\ \qquad\qquad\qquad\qquad\qquad \text{if} \ 0 \leqslant x \leqslant \frac{1}{2}; & \end{cases}$$

and let $g_n(x)$ for $n \geqslant 0$ be defined by $f_n'(x) = g_n(x)\nu_n'(x)$. If $g_0(-\frac{1}{2}) = g_0(\frac{1}{2})$ and $|g_0'(x)| \leqslant M_0$ on I for some positive constant M_0 then $|g_{n+1}'(x)| \leqslant M_n q$ on I where M_n is the maximum of $|g_n'(x)|$ on I and $0 < q < \frac{1}{2}$ is a constant independent of n.

This result is proven in [6] using the methods of Szüsz ([7]) and several numerical estimations are required. Using this Gauss–Kuzmin type theorem, I show in the following sections the results for continued fractions to the nearer integer corresponding to Khintchine's Theorem (§ 1) and Levy's Theorem (§ 2) for regular continued fractions. I will use these results to calculate the relative frequency of digits in the continued fraction to the nearer integer expansion of almost all numbers and will calculate the geometric mean of the partial denominators. In the course of

these demonstrations, I will indicate how to extend these results to include central limit theorems and the law of the iterated logarithm.

I would like to express my thanks to Professor Peter Szüsz for his inspiration and guidance throughout the preparation of this paper and my thesis ([6]).

1. Let $t = \dfrac{a_1}{b_1} + \dfrac{a_2}{b_2} + \ldots$ be the continued fraction to the nearer integer representation of the number t. Let $E(t\colon \mathrm{P})$ be the set of numbers $t \in I$ with the property P and let $m(E)$ be the Lebesgue measure of the set E. The phrase "almost all" will mean "except for a set of Lebesgue measure zero."

THEOREM 1. *Let $f(k)$ be any number theoretic function such that $f(k) = O(k^{\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(b_k) = \frac{1}{\log A}f(2)\log\frac{5A+3}{5A+2}+$$
$$+\frac{1}{\log A}\sum_{i=3}^{\infty} f(i)\log\left(\frac{A(i-\frac{1}{2})+1}{A(i+\frac{1}{2})+1}\cdot\frac{(A+1)(i+\frac{1}{2})-1}{(A+1)(i-\frac{1}{2})-1}\right)$$

for almost all t.

This follows directly from the next two lemmas.

LEMMA 1.1.

$$\int\limits_{-1/2}^{1/2}\sum_{k=1}^{n} f(b_k)\,dt = \frac{n}{\log A}f(2)\log\frac{5A+3}{5A+2}+$$
$$+\frac{n}{\log A}\sum_{i=3}^{\infty} f(i)\log\left(\frac{A(i-\frac{1}{2})+1}{A(i+\frac{1}{2})+1}\cdot\frac{(A+1)(i+\frac{1}{2})-1}{(A+1)(i-\frac{1}{2})-1}\right)+O(1).$$

Proof.

$$m\big(E(t\colon b_k = 2)\big) = \int\limits_{-1/2}^{-1/(2+\frac{1}{2})} d\nu_k + \int\limits_{1/(2+\frac{1}{2})}^{1/2} d\nu_k = \frac{1}{\log A}\log\frac{5A+3}{5A+2}+O(q^k),$$

and for $i \geqslant 3$,

$$m\big(E(t\colon b_k = i)\big) = \int\limits_{-1/(i-\frac{1}{2})}^{-1/(i+\frac{1}{2})} d\nu_k + \int\limits_{1/(i+\frac{1}{2})}^{1/(i-\frac{1}{2})} d\nu_k$$
$$= \frac{1}{\log A}\log\left(\frac{A(i-\frac{1}{2})+1}{A(i+\frac{1}{2})+1}\cdot\frac{(A+1)(i+\frac{1}{2})-1}{(A+1)(i-\frac{1}{2})-1}\right)+O\left(\frac{q^k}{i^2}\right).$$

Thus

$$\int_{-1/2}^{1/2}\sum_{k=1}^{n} f(b_k)\,dt = \sum_{k=1}^{n}\left(\frac{f(2)}{\log A}\log\frac{5A+3}{5A+2}+O(q^k)\right)+$$

$$+\sum_{k=1}^{n}\sum_{i=3}^{\infty}\left(\frac{f(i)}{\log A}\log\left(\frac{A(i-\frac12)+1}{A(i+\frac12)+1}\cdot\frac{(A+1)(i+\frac12)-1}{(A+1)(i-\frac12)-1}\right)+O\left(\frac{q^k}{i^2}\right)\right)$$

and the result follows.

LEMMA 1.2. *Let E_n denote the integral of Lemma 1.1. Then*

$$m\left(E\left(t:\ \left|\sum_{k=1}^{n} f(b_k)-E_n\right|>\delta n\right)\right)<c/\delta^2 n$$

where c is a constant and $\delta>0$.

Proof. This follows from Chebyshev's Inequality provided that

$$\int_{-1/2}^{1/2}\left(\sum_{k=1}^{n} f(b_k)-E_n\right)^2 dt = O(n).$$

But the integral is just

$$\int_{-1/2}^{1/2}\sum_{k=1}^{n} f^2(b_k)\,dt - E_n^2+\sum_{r=2}^{\infty}\sum_{s=2}^{\infty} f(r)f(s)\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n} m\big(E(t:\ b_k=r,\ b_i=s)\big).$$

Since

$$(*)\qquad m\big(E(t:\ b_k=r,\ b_i=s)\big) = m\big(E(t:\ b_k=r)\big)\,m\big(E(t:\ b_i=s)\big)\big(1+O(q^{|k-i|})\big),$$

this last sum is $E_n^2+O(n)$ and the whole expression becomes

$$\frac{n}{\log A}\big(1+O(1)\big)\left(f^2(2)\log\frac{5A+3}{5A+2}+\right.$$

$$\left.+\sum_{i=3}^{\infty} f^2(i)\log\left(\frac{A(i-\frac12)+1}{A(i+\frac12)+1}\cdot\frac{(A+1)(i+\frac12)-1}{(A+1)(i-\frac12)-1}\right)\right)+O(n).$$

Application of the condition $f(k)=O(k^{1/2-\varepsilon})$ to the above expression gives the desired result.

COROLLARY 1 (Frequency of digits). *Let $i\geqslant 2$ be an integer then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{k\leqslant n\\ b_k=i}} 1 = \begin{cases}\dfrac{1}{\log A}\log\dfrac{5A+3}{5A+2} & \text{if}\quad i=2,\\[2ex] \dfrac{1}{\log A}\log\left(\dfrac{A(i-\frac12)+1}{A(i+\frac12)+1}\cdot\dfrac{(A+1)(i+\frac12)-1}{(A+1)(i-\frac12)-1}\right) & \text{if}\quad i\geqslant 3\end{cases}$$

for almost all t.

This follows from the theorem by letting $f(k)$ be 1 if $k=i$ and 0 otherwise. Letting $f(k)=\log k$ gives

COROLLARY 2 (Geometric mean of partial denominators). *For almost all t the limit as $n\to\infty$ of $\sqrt[n]{b_1\ldots b_n}$ is*

$$\left(\frac{5A+3}{5A+2}\right)^{\frac{\log 2}{\log A}}\prod_{k=3}^{\infty}\left(\frac{A(k-\frac12)+1}{A(k+\frac12)+1}\cdot\frac{(A+1)(k+\frac12)-1}{A+1)(k-\frac12)-1)}\right)^{\frac{\log k}{\log A}}.$$

COROLLARY 3 (Frequency of plus signs). *For almost all t,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{k\leqslant n\\ a_k=1}} 1 = \frac{1}{\log A}\log\left(1+\frac{1}{2A}\right).$$

Proof.

$$m\big(E(t:\ a_k=1)\big)=\int_0^{1/2} d\nu_k = \frac{1}{\log A}\log\frac{A+\frac12}{A}+O(q^k).$$

The remainder of the proof proceeds as in Lemmas 1.1 and 1.2.

The critical step in all these results is $(*)$ in the proof of Lemma 1.2. In the language of probability, this is the statement that the b_i's determine a mixing sequence of random variables. Thus the results of Ibragimov [1] and of Reznik [4] together with the Corollaries to Theorem 1 give the obvious central limit theorems and law of iterated logarithm immediately (see also Philipp [3]).

2. THEOREM 2 (Denominator of nth convergent). *For almost all t the limit as $n\to\infty$ of $\sqrt[n]{B_n}$ is* $\exp(K)$ *where* $K=\int_{-1/2}^{1/2}\log|1/t|\,d\nu(t)$.

Proof. Let $t=\dfrac{a_1}{b_1}\,\dfrac{}{+}\,\dfrac{a_2}{b_2}\,\dfrac{}{+}\cdots$ be the continued fraction to the nearer integer expansion of t. Let

$$\varphi_n = b_n\,\frac{}{+}\,\frac{a_{n+1}}{b_{n+1}}\,\frac{}{+}\,\frac{a_{n+2}}{b_{n+2}}\,\frac{}{+}\cdots$$

so that

$$\varphi_n = b_n\,\frac{}{+}\,\frac{a_{n+1}}{\varphi_{n+1}}\quad\text{and}\quad \varphi_{n+1}=\frac{1}{z_n}.$$

Recall that $B_n=b_nB_{n-1}+a_nB_{n-2}$ and so

$$\frac{B_{n+1}\varphi_{n+2}+a_{n+2}B_n}{B_n\varphi_{n+1}+a_{n+1}B_{n-1}}=\frac{\left(B_{n+1}+\dfrac{a_{n+2}B_n}{\varphi_{n+2}}\right)\varphi_{n+2}}{B_n\left(b_{n+1}+\dfrac{a_{n+2}}{\varphi_{n+2}}\right)+a_{n+1}B_{n-1}}=\varphi_{n+2}.$$

Since $B_{-1} = 0$ and $B_0 = 1$,

$$\varphi_2 \cdots \varphi_{n+2} = \frac{1}{B_0\varphi_1 + a_1 B_{-1}} (B_{n+1}\varphi_{n+2} + a_{n+2}B_n)$$

and so

$$\varphi_1 \cdots \varphi_{n+1} = B_{n+1}\left(1 + \frac{a_{n+2}B_n}{\varphi_{n+2}B_{n+1}}\right),$$

or

$$\frac{1}{n+1}\sum_{k=1}^{n+1} \log\varphi_k = \frac{1}{n+1}\left(\log B_{n+1} + \log\left(1 + \frac{a_{n+2}B_n}{\varphi_{n+2}B_{n+1}}\right)\right).$$

Now,

$$\int_{-1/2}^{1/2} \log\varphi_{k+1}\,dt = \int_{-1/2}^{1/2} \log\left|\frac{1}{t}\right|(1+O(q^k))\,d\nu(t)$$

so that

$$\int_{-1/2}^{1/2} \log\varphi_{k+1}\,dt = K + O(q^k).$$

Let $E_n = \int\limits_{-1/2}^{1/2} \big(\sum\limits_{k=1}^{n}\log\varphi_k\big)\,dt$. From the above, $E_{n+1} = (n+1)K + O(1)$. If it can be shown that $\int\limits_{-1/2}^{1/2}\big(\sum\limits_{k=1}^{n+1}\log\varphi_k - E_{n+1}\big)^2 dt$ is $O(n+1)$, then Chebyshev's Inequality will give the required result. But the integral in question is just

$$\int_{-1/2}^{1/2}\Big(\sum_{k=1}^{n+1}\log^2\varphi_k\Big)\,dt - E_{n+1}^2 + \int_{-1/2}^{1/2}\Big(\sum_{k=1}^{n+1}\sum_{\substack{i=1\\ i\neq k}}^{n+1}\log\varphi_k\log\varphi_i\Big)\,dt$$

so it suffices to show that the last integral is $E_{n+1}^2 + O(n+1)$. Each term $\int\limits_{-1/2}^{1/2}\log\varphi_k\log\varphi_i\,dt$ is $\int\limits_{2}^{\infty}\log x\log y\,dF(x,y)$ where the function $F(x,y)$ is $F(x,y) = m\big(E(t\colon 1/\varphi_k \leqslant x \text{ and } 1/\varphi_i \leqslant y)\big)$. If it is true that

$$F'(x,y) = m'\big(E(t\colon 1/\varphi_c \leqslant x)\big)m'\big(E(t\colon 1/\varphi_i \leqslant y)\big)\big(1+O(q^{|k-i|})\big)$$

then the proof will be done. Suppose that $k < i$. Then

$$\varphi_k = b_k + \frac{a_{k+1}}{b_{k+1}} + \cdots + \frac{a_i}{\varphi_i}$$

so that the desired result is not immediate. However, let

$$\bar\varphi_k = b_k + \frac{a_{k+1}}{b_{k+1}} + \cdots + \frac{a_{k+N}}{b_{k+N}}$$

for some N, then $\bar\varphi_k$ and $\bar\varphi_i$ have the relation required for φ_k and φ_i. But $|\bar\varphi_k - \varphi_k| < \varepsilon$ for any $\varepsilon > 0$ provided N is large and so the proof is complete.

Again, this observation shows that the central limit theorem and the law of the iterated logarithm mentioned at the end of § 1 carry over to the denominator of the nth convergent.

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