The metrical theory of continued fractions
to the nearer integer

by

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Let \( I \) be \([ -\frac{1}{2}, \frac{1}{2} ]\) and let \( x_0 \in I \) be nonzero. Let \( a_1 = a_1(x_0) \) be the sign of \( x_0 \) and let \( b_1 = b_1(x_0) \) be \( \lfloor a_1/x_0 \rfloor \) if \( \{a_1/x_0\} \leq \frac{1}{2} \) and \( \lfloor a_1/x_0 \rfloor + 1 \) otherwise. Then \( x_1 = (a_1/x_0) - b_1 \) is in \( I \). This process may now be applied to \( x_1 \) to yield \( a_2, b_2 \), and \( x_2 \in I \). It is clear that for any irrational \( x_0 \in I \) this process may be continued indefinitely and thus a unique sequence of digits \( a_n, b_n \) is formed for each such \( x_0 \). The resulting expression

\[
x_0 = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots
\]

will be called the (semiregular) continued fraction to the nearer integer for \( x_0 \). The \( b_1, b_2, \ldots \) are the partial denominators and the \( a_1, a_2, \ldots \) are the partial numerators. If this continued fraction is terminated at the \( n \)-th term \((n \geq 1)\), the resulting rational number, the \( n \)-th convergent, may be expressed in the form

\[
\frac{A_n}{B_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}
\]

where the \( A_n \) and \( B_n \) satisfy the recursion formulas

\[
A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2}
\]

and where \( A_{-1} = 1, A_0 = 0, B_{-1} = 0, \) and \( B_0 = 1 \).

Let \( N(x) \) be the nearer integer function given by \( N(x) = \lfloor x \rfloor \) if \( \{x\} \leq \frac{1}{2} \) and \( \lfloor x \rfloor + 1 \) if \( \{x\} > \frac{1}{2} \). Then the transformation \( T : I \to I \) given by

\[
Tx = \text{sign}(x) \left( \frac{1}{x} - N\left( \frac{1}{x} \right) \right)
\]

generates the continued fraction to the nearer integer algorithm in the sense that \( T^n x_0 = x_n \) for \( n \geq 0 \) where \( T^n \) is the identity transformation of \( I \) and \( T^n \) is the \( n \)-fold composition of \( T \) with
itself for \( n \geq 1 \). That \( T \) is indecomposable with respect to the Lebesgue measure follows by translating Knopp’s Theorem (2)2 for regular continued fractions. Let \( \Delta \) be \( \frac{1}{2}(1+\sqrt{5}) \) and let \( p(t) \) be \( \frac{1}{1+(\Delta+1+t)} \) if \(-\frac{1}{2} < t < 0\) and \( \frac{1}{1+(\Delta+t)} \) if \( 0 \leq t \leq \frac{1}{2} \). Then if \( E \) is any Lebesgue measurable set, the measure

\[
\nu(E) = \frac{1}{\log \Delta} \int_E p(t) \, dt
\]

is invariant under \( T \), as may be checked by direct calculation. These results are included in [6] and also are announced by Rieger ([5]).

Let \( x \in I \) and let \( \nu_n(x) \) for \( n \geq 0 \) denote the Lebesgue measure of the set of \( x_0 \in I \) such that \(-\frac{1}{2} \leq x_0 < x \). The estimate \( \nu_n(x) = \nu'(x)(1+O(a^n)) \), where \( 0 < q < \frac{1}{2} \) is a constant, follows from the general

**Theorem.** Let \( f_0(x) \) be any twice differentiable function on \( I \) such that \( f_0(-\frac{1}{2}) = 0 \) and \( f_0(\frac{1}{2}) = 1 \). Let the sequence of functions \( f_0(x) \), \( f_1(x) \), … be defined recursively by

\[
\sum_{k=2}^{\infty} \left( f_0 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right) \left( f_0 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right) + \left( f_1 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right) \left( f_1 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right)
\]

\[
\text{if } -\frac{1}{2} \leq x < 0;
\]

\[
f_{n+1}(x) = f_0 \left( \frac{1}{2-x} \right) + f_1 \left( \frac{1}{2-x} \right) - f_n \left( \frac{1}{2-x} \right) + \left( f_0 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right) \left( f_0 \left( \frac{1}{k-x} \right) - f_n \left( \frac{1}{k-x} \right) \right)
\]

\[
\text{if } 0 \leq x \leq \frac{1}{2};
\]

and let \( g_n(x) \) for \( n \geq 0 \) be defined by \( f_n(x) = g_n(x) \nu_n(x) \). If \( g_0(x) = g_0(\frac{1}{2}) \) and \( g_n(x) \leq M_n \) on \( I \) for some positive constant \( M_n \) then \( g_{n+1}(x) \leq M_n q_0 \) on \( I \) where \( M_n \) is the maximum of \( |g_n(x)| \) on \( I \) and \( 0 < q < \frac{1}{2} \) is a constant independent of \( n \).

This result is proven in [6] using the methods of Szücs ([7]) and several numerical estimations are required. Using this Gauss–Kuzmin type theorem, I show in the following sections the results for continued fractions to the nearer integer corresponding to Khintchine’s Theorem (§ 1) and Levy’s Theorem (§ 2) for regular continued fractions. I will use these results to calculate the relative frequency of digits in the continued fraction to the nearer integer expansion of almost all numbers and will calculate the geometric mean of the partial denominators. In the course of these demonstrations, I will indicate how to extend these results to include central limit theorems and the law of the iterated logarithm.

I would like to express my thanks to Professor Peter Szücs for his inspiration and guidance throughout the preparation of this paper and my thesis ([5]).

1. Let \( t = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots \) be the continued fraction to the nearer integer representation of the number \( t \). Let \( E(t; P) \) be the set of numbers \( t \in I \) with the property \( \nu \) and let \( m(E) \) be the Lebesgue measure of the set \( E \). The phrase “almost all” will mean “except for a set of Lebesgue measure zero.”

**Theorem.** Let \( f(k) \) be any number theoretic function such that \( f(k) = O(k^{-\varepsilon}) \) for some \( \varepsilon > 0 \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(b_k) = \frac{1}{\log \Delta} \int f(t) \log \frac{A(t-\frac{1}{2})+1}{A(t+\frac{1}{2})+1} \frac{(A+1)(t-\frac{1}{2})-1}{(A+1)(t+\frac{1}{2})-1} dt
\]

for almost all \( t \).

This follows directly from the next two lemmas.

**Lemma 1.**

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x) \nu(x)}{\log \Delta} dt = \frac{1}{\log \Delta} \int f(t) \log \frac{A(t-\frac{1}{2})+1}{A(t+\frac{1}{2})+1} \frac{(A+1)(t-\frac{1}{2})-1}{(A+1)(t+\frac{1}{2})-1} dt + O(1).
\]

**Proof.**

\[
m(E(t; b_k = 2)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(t) \nu(b_k) \frac{1}{\log \Delta} \log \frac{A(t-\frac{1}{2})+1}{A(t+\frac{1}{2})+1} + O(1),
\]

and for \( i > 3 \),

\[
m(E(t; b_k = i)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(t) \nu(b_k) \frac{1}{\log \Delta} \log \frac{A(t-\frac{1}{2})+1}{A(t+\frac{1}{2})+1} + O\left(\frac{q}{i^2}\right).
\]
Thus
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=1}^{n} \frac{f(b_k)}{\log A} \, dt = \sum_{k=1}^{n} \left( \frac{f(2)}{\log A} \log \frac{5A + 3}{5A + 2} + O\left(q^k\right) \right) + \\
+ \sum_{k=1}^{n} \sum_{t=3}^{\infty} \left( \frac{f(t)}{\log A} \log \frac{A(i - \frac{1}{2} + 1)}{A(i + \frac{1}{2} + 1)} \cdot \frac{(A + 1)(i + \frac{1}{2}) - 1}{(A + 1)(i - \frac{1}{2}) - 1} + O\left(q^k\right) \right)
\]
and the result follows.

Lemma 1.2. Let \( E_n \) denote the integral of Lemma 1.1. Then
\[
m\left( \left[ \sum_{k=1}^{n} f(b_k) - E_n \right] > \delta n \right) < a/\delta n
\]
where \( a \) is a constant and \( \delta > 0 \).

Proof. This follows from Chebyshev's Inequality provided that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=1}^{n} f(b_k) - E_n \right)^2 \, dt = O(n).
\]
But the integral is just
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\log A} \left( \sum_{k=1}^{n} f(b_k) - E_n \right) \, dt = E_n^2 + \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{1}{\log A} m(\{ t : b_k = r, b_l = s \}),
\]
where \( m \) is the total sum is \( E_n^2 + O(n) \) and the whole expression becomes
\[
\frac{n}{\log A} [1 + O(1)] \left( \frac{f(2)}{\log A} \log \frac{5A + 3}{5A + 2} + \\
+ \sum_{t=3}^{\infty} \frac{f(t)}{\log A} \log \frac{A(i - \frac{1}{2} + 1)}{A(i + \frac{1}{2} + 1)} \cdot \frac{(A + 1)(i + \frac{1}{2}) - 1}{(A + 1)(i - \frac{1}{2}) - 1} + O(n) \right).
\]
The remainder of the proof proceeds as in Lemmas 1.1 and 1.2.

The critical step in all these results is (*) in the proof of Lemma 1.2. In the language of probability, this is the statement that the \( b_k \)'s determine a mixing sequence of random variables. Thus the results of Ibragimov [1] and of Rényi [4] together with the Corollaries to Theorem 1 give the obvious central limit theorems and law of iterated logarithm immediately (see also Philipp [3]).

2. Theorem 2 (Denominator of nth convergent). For almost all \( t \) the limit as \( n \to \infty \) of \( \sqrt{A_n} \) is \( \exp(K) \) where
\[
K = \frac{1}{2} \log \frac{A + \frac{1}{2}}{A} + O(q^l).
\]
Proof. Let \( t = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots \) be the continued fraction to the nearest integer expansion of \( t \). Let
\[
\varphi_n = \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \ldots
\]
so that
\[
\varphi_n = \frac{b_n + \varphi_{n+1}}{b_n + a_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \ldots
\]
Recall that \( B_n = b_n B_{n-1} + a_n B_n \) and so
\[
\frac{B_{n+1} \varphi_{n+1} + a_{n+1} B_n}{B_n \varphi_{n+1} + a_{n+1} B_n} = \frac{B_{n+1} + \frac{a_{n+1} B_n}{\varphi_{n+1}}}{B_n \left( \frac{b_n + \frac{a_{n+1}}{\varphi_{n+1}}}{b_n + a_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \ldots \right)}
\]

Since \( B_{-1} = 0 \) and \( B_0 = 1 \),
\[
\varphi_1 \cdots \varphi_{n+2} = \frac{1}{B_0 \varphi_1 + a_n B_{-1}} (B_{n+1} \varphi_{n+2} + a_{n+1} B_n)
\]
and so
\[
\varphi_1 \cdots \varphi_{n+1} = B_{n+1} \left( 1 + \frac{a_{n+1} B_n}{\varphi_{n+1} B_{n+1}} \right).
\]
or
\[
\frac{1}{n+1} \sum_{k=1}^{n+1} \log \varphi_k = \frac{1}{n+1} \left( \log B_{n+1} + \log \left( 1 + \frac{a_{n+1} B_n}{\varphi_{n+1} B_{n+1}} \right) \right).
\]

Now,
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \log \varphi_k \, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log \left( 1 + O(q^k) \right) \, dt(1)
\]
so that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \log \varphi_k \, dt = K + O(q^k).
\]

Let \( E_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=1}^{n+1} \log \varphi_k \, dt \). From the above, \( E_{n+1} = (n+1)K + O(1) \).

If it can be shown that \( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=1}^{n+1} \log \varphi_k - E_{n+1} \right) \, dt \) is \( O(n+1) \), then Chebyshev's Inequality will give the required result. But the integral in question is just
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=1}^{n+1} \log^2 \varphi_k \right) \, dt - E_{n+1}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{k=1}^{n+1} \log \varphi_k \right) \, dt
\]
so it suffices to show that the last integral is \( E_{n+1}^2 + O(n+1) \). Each term
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \log \varphi_k \log \varphi_i \, dt
\]
is \( \log x \log y \, dF(x, y) \) where the function \( F(x, y) \) is \( F(x, y) = m \{ E(t: 1/\varphi_i < x \text{ and } 1/\varphi_i < y) \} \). If it is true that
\[
F'(x, y) = m' \{ E(t: 1/\varphi_i < x) \} m' \{ E(t: 1/\varphi_i < y) \} (1 + O(q^{k-i}))
\]
then the proof will be done. Suppose that \( k < i \). Then
\[
\varphi_k = b_k + \frac{a_{k+1}}{b_{k+1}} + \cdots + \frac{a_i}{\varphi_i}
\]
so that the desired result is not immediate. However, let
\[
\varphi_k = b_k + \frac{a_{k+1}}{b_{k+1}} + \cdots + \frac{a_{k+N}}{b_{k+N}}
\]
for some \( N \), then \( \varphi_k \) and \( \varphi_i \) have the relation required for \( \varphi_k \) and \( \varphi_i \). But \( |\varphi_n - \varphi_k| < \varepsilon \) for any \( \varepsilon > 0 \) provided \( N \) is large and so the proof is complete.

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References


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