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On the coprimality of certain multiplicative functions

by

E. J. SCOURFIELD (London)

1. Introduction. An integer-valued multiplicative function f is said to be *polynomial-like* if there exists a polynomial W with coefficients in $\mathbf{Z}$ (the set of all integers) such that

(1) $$f(p) = W(p) \quad \text{for all primes } p;$$

it will not be necessary for us to impose any corresponding condition on $f(p^a)$ for $a \geqslant 2$. Obvious examples of functions in this class are Euler's function φ and the divisor functions

(2) $$\sigma_\nu(n) = \sum_{d|n} d^\nu$$

for ν a non-negative integer.

In an earlier paper [8] we investigated the sum

(3) $$\Sigma_f(x) = \sum_{\substack{1\leqslant n\leqslant x\\ (n,f(n))=1}} 1,$$

and for f a polynomial-like multiplicative function such that the polynomial W in (1) has degree l and satisfies $W(0) \neq 0$, we obtained the asymptotic formula

(4) $$\Sigma_f(x) \sim \begin{cases} Cx(\log\log\log x)^{-\lambda} & \text{if} \quad l > 0,\\ Cx & \text{if} \quad l = 0\end{cases}$$

as $x \to \infty$, where C, λ are positive constants with λ rational and $\lambda \leqslant 1$ When $W(0) = 0$, we deduced easily that

$$\Sigma_f(x) = O(x^{1/2}),$$

and indeed for some f in this category, one can obtain, by a minor adaptation of the argument in [8], an asymptotic formula of the type in (4) but with x replaced by x^a for some rational a with $0 < a \leqslant \frac{1}{2}$. The proof of (4) is elementary, although complicated, and depends in part on a double application of the Sieve of Eratosthenes. The result of (4) is a generalization of one obtained earlier by Erdös [2] for the case $f = \varphi$, when $\lambda = 1$ and $C = e^{-\gamma}$. In contrast to (4), several authors have established the expected result

$$\Sigma_f(x) \sim 6\pi^{-2}x$$

as $x \to \infty$ for certain classes of non-multiplicative functions f for which the arithmetic structure of n and $f(n)$ are largely independent, and some references to this work are given in [8].

Our motivation for writing this paper is provided by the observation that n is itself a polynomial-like multiplicative function, and this suggests replacing the first n in the condition $(n, f(n)) = 1$ in (3) by a general integer-valued polynomial-like multiplicative function $g(n)$. An obvious example arises when we take $f = \varphi$ and $g = \sigma = \sigma_1$ in the definition (2) above, and look at the sum

$$\sum_{\substack{n \leqslant x \\ (\varphi(n), \sigma(n)) = 1}} 1;$$

however a snag immediately presents itself, for $2 | (\varphi(p), \sigma(p)) = (p-1, p+1)$ for all odd primes p, whence $2 | (\varphi(n), \sigma(n))$ whenever there is an odd prime p with $p \| n$. Thus our sum is rather small, for it is bounded above by a constant multiple of the number of squarefull integers not exceeding x (see (23) below and [3]). Fortunately the prime 2 is exceptional in this respect, for no odd prime divides $(\varphi(p), \sigma(p))$ for all but a finite number of primes p, and this suggests that we modify our original sum and consider instead

$$\Sigma_{\varphi,\sigma}(x) = \sum_{\substack{n \leqslant x \\ p \nmid (\varphi(n), \sigma(n)) \forall p \geqslant 3}} 1$$

where throughout p will denote a prime; the case $\nu = 1$ of Corollary 1 below gives quite close upper and lower bounds for this sum. In general, for (f, g) a pair of polynomial-like multiplicative functions of a suitable type, we shall estimate from above and below the sum

$$\Sigma_{f,g}(x) = \sum_{\substack{1 \leqslant n \leqslant x \\ p \nmid (f(n), g(n)) \forall p \notin S_0}} 1, \tag{5}$$

where S_0 is a certain finite set of exceptional primes defined in (6).

Before we can state the results obtained, we need some notation. Let f, g be integer-valued polynomial-like multiplicative functions, so that there exist polynomials W_1, $W_2 \in \mathbf{Z}[x]$, of degree l_1, l_2, respectively, and with positive leading coefficients, such that

$$f(p) = W_1(p), \quad g(p) = W_2(p) \quad \text{for all primes } p.$$

In order that our problem is non-trivial and does not reduce to one already considered, we impose the following conditions on W_1, W_2:

(i) $l_1 > 0$, $l_2 > 0$;

(ii) $W_2(0) \neq 0$ and $W_1(x) = x^{l'} W_1^*(x)$ where $0 \leqslant l' < l_1$, $W_1^*(0) \neq 0$, $W_1^* \in \mathbf{Z}[x]$;

(iii) W_1, W_2 are coprime polynomials.

We observe that if $W_1(0) \neq 0$, then $W_1^* = W_1$ and in any case W_*^1 has positive degree $l_1^* = l_1 - l'$, and that (iii) would be violated if $W_1(0) = 0 = W_2(0)$. Condition (iii) implies that there is a least positive integer e such that there exist polynomials U, $V \in \mathbf{Z}[x]$ satisfying

$$U(x)W_1(x) + V(x)W_2(x) = e$$

identically. Hence if p is a prime and there exists a prime q such that $p | (f(q), g(q)) = (W_1(q), W_2(q))$, then $p | e$ and so p belongs to a finite or empty set. We define

$$S_0 = \{p\colon\ p | (W_1(q), W_2(q)) \,\forall q \neq p\} \tag{6}$$

where, as usual, p, q denote primes and where, if $p \in S_0$, the condition $p | (W_1(p), W_2(p))$, i.e. $p | (W_1(0), W_2(0))$, may, but need not, hold; then S_0 is a finite or empty set such that $p | e$ for all $p \in S_0$. When $f = \varphi$, $g = \sigma$, we have $S_0 = \{2\}$. Clearly if $p \in S_0$, then $p | (f(n), g(n))$ whenever there is a prime $q \neq p$ with $q \| n$, whence $(f(n), g(n)) \neq 1$ for all but $O(x^{1/2})$ values of $n \leqslant x$; however if $p \notin S_0$, there are infinitely many primes q such that $p \nmid (W_1(q), W_2(q))$, namely all q in at least one residue class $(\bmod\, p)$ coprime to p. Thus the primes in S_0 are exceptional in some sense; we take this set S_0 to be the one appearing in (5), so $\Sigma_{f,g}(x)$ is now properly defined.

We can now state the main result of this paper, weaker forms of which were stated without proof in [9]:

THEOREM 1. *Let f, g be integer-valued polynomial-like multiplicative functions such that* (i), (ii), (iii) *above hold, put $l = l_1 + l_2 \geqslant 2$, and define $\Sigma_{f,g}(x)$ by* (5) *and* (6). *Then there exist positive constants A, B, λ with λ rational and $\lambda \leqslant 1$ such that*

$$\frac{x}{L_1(x)} \exp\left(A \frac{L_2(x)}{(L_3(x))^l}\right) \ll \Sigma_{f,g}(x) \ll \frac{x}{L_1(x)} \exp\left(B \frac{L_2(x)}{(L_3(x))^{\lambda}}\right)$$

where $L_k(x)$ $(k = 1, 2, 3, \ldots)$ stands for the iterated logarithm given by

$$L_1(x) = \max(1, \log x), \quad L_k(x) = \max(1, \log L_{k-1}(x)) \quad (k \geqslant 2). \tag{7}$$

We note that for each $\varepsilon > 0$,

$$\exp\left(BL_2(x)(L_3(x))^{-\lambda}\right) < (L_1(x))^{\varepsilon}$$

for all x sufficiently large, and thus the two bounds for $\Sigma_{f,g}(x)$ are not too far apart. Both bounds are obtained by using a sieve argument, and the left bound is derived by considering squarefree integers with j prime factors where j is chosen optimally in terms of x. The upper bound is probably nearer the true order of magnitude than the lower bound, but it would seem that, to obtain an asymptotic formula, additional or different techniques may be required.

When S_0 is empty, Theorem 1 estimates the number of positive integers $n \leqslant x$ such that $(f(n), g(n)) = 1$.

If condition (i) is dropped and W_2 (say) is a constant (non-zero) polynomial, then the problem of estimating $\Sigma_{f,g}(x)$ is closely allied to that of estimating

$$\sum_{\substack{n \leqslant x \\ (f(n),c)=1}} 1,$$

where c is a given integer, and this problem was one investigated by Narkiewicz in [7]. Similarly if we relax condition (ii) to allow $l' = l_1$, then the sum $\Sigma_{f,g}(x)$ is closely related to a sum of the type (3) for which the estimate (4) holds. Finally if condition (iii) fails to hold, $(W_1(q), W_2(q))$ has a prime divisor $p \notin S_0$ for "most" primes q and, whenever $q \| n$ for such a prime q, n does not contribute to $\Sigma_{f,g}(x)$, which is therefore rather small.

We have already referred to the special case $f = \varphi$, $g = \sigma$, when $S_0 = \{2\}$, and here we find that $l = 2$, $\lambda = 1$ in Theorem 1. This example is included in the first of the following corollaries to Theorem 1:

COROLLARY 1. *Let ν be a positive integer with $2^\beta \| \nu$, and put $\lambda = 2^{-\beta}$; then there exist positive constants A, B such that*

$$\frac{x}{L_1(x)} \exp\left(A \frac{L_2(x)}{(L_3(x))^{\nu+1}}\right) \ll \Sigma_{\varphi,\sigma_\nu}(x) \ll \frac{x}{L_1(x)} \exp\left(B \frac{L_2(x)}{(L_3(x))^{\lambda}}\right)$$

where $\Sigma_{\varphi,\sigma_\nu}(x)$ is defined by (5) *with $S_0 = \{2\}$.*

COROLLARY 2. *Let $\nu, \varkappa$ be positive integers such that exactly one of the integers $\nu/(\nu,\varkappa)$, $\varkappa/(\nu,\varkappa)$ is even, suppose that $2^\beta \| \nu$, $2^\gamma \| \varkappa$ where $\beta > \gamma$, and put $\lambda = 2^{-\beta}$; then there exist positive constants A, B such that*

$$\frac{x}{L_1(x)} \exp\left(A \frac{L_2(x)}{(L_3(x))^{\nu+\varkappa}}\right) \ll \Sigma_{\sigma_\nu,\sigma_\varkappa}(x) \ll \frac{x}{L_1(x)} \exp\left(B \frac{L_2(x)}{(L_3(x))^{\lambda}}\right)$$

where $\Sigma_{\sigma_\nu,\sigma_\varkappa}(x)$ is defined by (5) *with $S_0 = \{2\}$.*

It is easy to check that conditions (i), (ii), (iii) of Theorem 1 hold in both cases and that $S_0 = \{2\}$; the value in these cases of the constant λ in Theorem 1 is given by the Corollary of Lemma 1 below, and so $\lambda = 2^{-\beta}$ in both corollaries above. We observe that in Corollary 2, the condition on $\nu, \varkappa$ ensures that $\beta \neq \gamma$; if both $\nu/(\nu,\varkappa)$ and $\varkappa/(\nu,\varkappa)$ are odd, then condition (iii) is violated, for $x^{(\nu,\varkappa)}+1$ is then a common factor of $W_1(x) = x^\nu + 1$ and $W_2(x) = x^\varkappa + 1$.

The method used to prove Theorem 1 can also be used to estimate the following interesting quantity: Let $\Sigma(x)$ denote the number of positive integers $n \leqslant x$ with the property that for no odd prime p does n have a prime divisor from each residue class $(\bmod\, p)$ coprime to p. Thus, in particular, if n contributes to $\Sigma(x)$, either n has no prime divisor $q \equiv 1 \pmod 3$ or n has no prime divisor $q \equiv 2 \pmod 3$, and more generally to each prime $p \geqslant 3$ there corresponds c_p (depending on n) with $1 \leqslant c_p < p$ such that $q \nmid n$ if $q \equiv c_p \pmod p$, q prime. We have:

THEOREM 2. *There exist positive constants A, B such that*

$$\frac{x}{L_1(x)} \exp\left(A \frac{L_2(x)}{L_3(x)}\right) \ll \Sigma(x) \ll \frac{x}{L_1(x)} \exp\left(B \frac{L_2(x)}{L_3(x)}\right).$$

Notice that the upper and lower bounds in Theorem 2 take the same form, differing only in the values of the constants A, B, and those implied by the $\ll$ sign, and so the result of Theorem 2 is tighter than that of Theorem 1. It follows that almost all positive integers n have a prime divisor from every non-zero residue class $(\bmod\, p)$ for some odd prime p.

Finally, I should like to thank Professor Erdös for his interest in the problem of improving my original lower bound for the sum $\Sigma_{\varphi,\sigma}(x)$ and for his helpful comments.

2. Preliminary lemmas. We need some additional notation to that given in § 1. Throughout p, q (with or without suffices) will denote primes.

With S_0 defined by (6), let

$$S_0' = \{p \notin S_0 \colon \exists q \text{ with } p \mid (W_1(q), W_2(q))\};$$

then by remarks in § 1, whenever $p \in S_0'$ we have $p \mid e$ and $p \nmid (W_1(q), W_2(q))$ for infinitely many primes q.

Furthermore for $j = 1, 2$, let

$S_j = \{p \colon \exists a \in \mathbf{Z} \text{ with } p \mid W_j(a)\}$,

$\varrho_j(p)$ be the number of solutions of the congruence

$$W_j(u) \equiv 0 \pmod p, \tag{8}$$

$\varrho_j'(p)$ be the number of solutions of (8) with $(p, u) = 1$, and let $\varrho(p)$, $\varrho'(p)$ be the corresponding functions for the polynomial

$$W = W_1 W_2$$

of degree $l = l_1+l_2$. Thus

$$\varrho'(p) = \begin{cases} \varrho(p) & \text{if} \quad p \nmid W(0), \\ \varrho(p)-1 & \text{if} \quad p \mid W(0), \end{cases}$$

$$0 \leqslant \varrho(p) \leqslant l \quad \text{or} \quad \varrho(p) = p$$

by Lagrange's Theorem, and moreover ϱ and ϱ' are multiplicative functions; similar remarks apply to ϱ_j, ϱ_j' $(j = 1, 2)$. If $p \in S_0$, then

$$\min\big(\varrho_1'(p), \varrho_2'(p)\big) = p-1$$

and the left side equals $\max\big(\varrho_1(p), \varrho_2(p)\big)$ if $p \nmid W(0)$.

If τ is the number of irreducible components of W (so $\tau \geqslant 2$ since $W = W_1W_2$), it is well known that

$$T(x) = \sum_{p \leqslant x} \varrho(p)p^{-1} = \tau L_2(x)+C+O\big((L_1(x))^{-1}\big) \tag{9}$$

using (7), where the constant C and the O-constant may depend on W and $T(x)$ is defined by (9).

Let $\nu(n)$ denote the number of distinct prime divisors of n.

The notation $L_k(x)$, $k \geqslant 1$ (see (7)), is used throughout this paper.

LEMMA 1. *There exist constants D and λ with $0 < \lambda \leqslant 1$ and λ rational such that*

$$\sum_{\substack{p \leqslant x \\ p \in S_1 \cap S_2}} p^{-1} = \lambda L_2(x)+D+O\big((L_1(x))^{-1}\big).$$

Hence in particular $S_1 \cap S_2$ is infinite.

Proof. Let

$$S_3 = S_1 \cup S_2 = \{p\colon \exists a \in \boldsymbol{Z} \text{ with } p \mid W_1(a)W_2(a)\}.$$

By a result due to Schinzel (see Theorem 2 of [8]), there exist rational numbers λ_j with $0 < \lambda_j \leqslant 1$ and constants D_j such that

$$\sum_{\substack{p \leqslant x \\ p \in S_j}} p^{-1} = \lambda_j L_2(x)+D_j+O\big((L_1(x))^{-1}\big) \quad (j = 1, 2, 3).$$

Hence

$$\sum_{\substack{p \leqslant x \\ p \in S_1 \cap S_2}} p^{-1} = \sum_{\substack{p \leqslant x \\ p \in S_1}} p^{-1}+\sum_{\substack{p \leqslant x \\ p \in S_2}} p^{-1}-\sum_{\substack{p \leqslant x \\ p \in S_3}} p^{-1} \tag{10}$$
$$= (\lambda_1+\lambda_2-\lambda_3)L_2(x)+(D_1+D_2-D_3)+O\big((L_1(x))^{-1}\big)$$

where clearly $\lambda_1+\lambda_2-\lambda_3 \geqslant 0$. Thus the result of the lemma holds provided that the sum on the left of (10) is divergent as $x \to \infty$, for then $\lambda = \lambda_1+\lambda_2-\lambda_3 > 0$. The result is obvious if S_1 or S_2 consists of all but a finite

number of primes, for then λ_1 or $\lambda_2 = 1$ and $\lambda_3 = 1$; this certainly holds if W_1 or W_2 has a linear factor, so we may assume below that neither has.

For $j = 1, 2$, let $\theta_j \notin \boldsymbol{Q}$ (the set of all rationals) be a zero of $W_j(u)$, and let $k = [\boldsymbol{Q}(\theta_1, \theta_2)\colon \boldsymbol{Q}]$. With a finite number of exceptions, every prime p splitting completely in $\boldsymbol{Q}(\theta_1, \theta_2)$ satisfies $p \in S_1 \cap S_2$. When p does split completely in $\boldsymbol{Q}(\theta_1, \theta_2)$, $p = \mathfrak{p}_1 \ldots \mathfrak{p}_k$ where the $\mathfrak{p}_i$ are prime ideals of the first degree and $N\mathfrak{p}_i = p$. Since the sum $\sum (N\mathfrak{p})^{-1}$ over all prime ideals of degree at least two is convergent, we have

$$\sum_{\substack{p \leqslant x \\ p \in S_1 \cap S_2}} p^{-1} \geqslant \sum_{p \leqslant x}' p^{-1}+O(1) \geqslant k^{-1} \sum_{N\mathfrak{p} \leqslant x}'' (N\mathfrak{p})^{-1}+O(1)$$
$$\geqslant k^{-1} \sum_{N\mathfrak{p} \leqslant x} (N\mathfrak{p})^{-1}+O(1) = k^{-1}L_2(x)+O(1)$$

as $x \to \infty$, by well known results, where $\sum'$ denotes the sum over primes p splitting completely in $\boldsymbol{Q}(\theta_1, \theta_2)$ and $\sum''$ denotes the sum over prime ideals $\mathfrak{p}$ of the first degree. Hence, as required, the sum on the left of (10) always diverges as $x \to \infty$.

COROLLARY. (i) *If $W_1(u) = u-1$, $W_2(u) = u^\nu+1$ where $2^\beta \| \nu$, then $\lambda_1 = \lambda_3 = 1$ and $\lambda = \lambda_2 = 2^{-\beta}$.*

(ii) *If $W_1(u) = u^\nu+1$, $W_2(u) = u^\varkappa+1$ where $2^\beta \| \nu$, $2^\gamma \| \varkappa$ and $\beta > \gamma$, then $\lambda_2 = \lambda_3 = 2^{-\gamma}$, $\lambda = \lambda_1 = 2^{-\beta}$.*

Proof. If p is an odd prime, $u^\nu+1 \equiv 0 \pmod{p}$ is solvable if and only if $(\nu, p-1) \mid \frac{1}{2}(p-1)$, whence $p \equiv 1 \pmod{2^{\beta+1}}$. The results now follow.

LEMMA 2. *If $(a, t) = 1$ and $1 \leqslant a < t$, then*

$$\sum_{\substack{t < q \leqslant x \\ q \equiv a \,(\mathrm{mod}\, t)}} q^{-1} \ll \big(\varphi(t)\big)^{-1}\big(L_1(t)+L_2(x)\big)$$

where the implied constant is independent of the choice of a.

This is proved in the same way as Lemma 2 in Erdös's paper [2], where the case $a = 1$, t prime is established.

LEMMA 3. *Let h be an arithmetic function satisfying $h(1) = 1$, $0 \leqslant h(n) \leqslant 1$ for all $n \geqslant 1$, $h(mn) \leqslant h(m)h(n)$ whenever $(m, n) = 1$. Then*

$$\sum_{1 \leqslant n \leqslant x} h(n) \leqslant x\big(L_1(x)\big)^{-1}\big(1+O\big(L_2(x)/L_1(x)\big)\big) \prod_{p \leqslant x} \Big\{\sum_{\alpha=0}^{\infty} h(p^\alpha)p^{-\alpha}\Big\}$$

where the O-constant does not depend on h.

This is proved by Hall in [5].

LEMMA 4. *Let $Q \geqslant 1$, Q squarefree, $(Q, P_0) = 1 = (a_0, P_0)$, $\varrho'(p) < p-1$ if $p \nmid P_0$, and $\mathscr{P} = \max_{p \mid Q} p \leqslant \exp\big(L_1(x)/L_3(x)\big)$, where a_0 and P_0 are indepen-*

dent of x but Q may depend on x. Then

$$|\{p\leqslant x:\ p\equiv a_0(\mathrm{mod}\ P_0),\ (W(p),Q)=1\}| = x\big(\varphi(P_0)L_1(x)\big)^{-1}\prod_{p|Q}\big(1-\varrho'(p)(p-1)^{-1}\big)\Big(1+O\big((L_2(x))^{-1}\big)\Big)$$

where the O-constant depends on the degree l of the polynomial W but not on any other property of W (such as the size of the coefficients, for example).

Proof. When $Q=1$, this follows from the Prime Number Theorem for primes in arithmetic progression. When $Q>1$, an application of Theorem 2.5′ in [4] establishes the lemma. For, in the notation of Halberstam and Richert in that theorem, take

$$\mathscr{A}=\{W(p):\ p\leqslant x,\ p\equiv a_0\ (\mathrm{mod}\ P_0)\},$$

$$\mathfrak{P}=\{p:\ p\nmid Q'P_0\}\quad\text{where}\quad Q'=\prod_{p\leqslant\mathscr{P},\,p\nmid Q}p,$$

$$X=\mathrm{li}\,x/\varphi(P_0),\qquad z=\mathscr{P}+1,$$

$$\omega(p)=\begin{cases}p\varrho'(p)/(p-1) & \text{if}\quad p\in\mathfrak{P},\\ 0 & \text{if}\quad p\notin\mathfrak{P}\end{cases}$$

where ω is multiplicative on the squarefree integers; then the conditions (Ω_1), $(\Omega_2(\varkappa))$, (R_0), $(\mathrm{R}_1(\varkappa,\alpha))$, with $\varkappa=l+1$ (for example), $\alpha=\frac{1}{2}$, follow by routine calculations, and consequently so does the lemma above.

Lemma 5.

$$(11)\qquad |\{p\leqslant x:\ W(p)>0\ \text{ and }\ \nu(W(p))>2T(x)\}|\ll x\big(L_1(x)L_2(x)\big)^{-1}$$

where the $\ll$ constant depends on W and $T(x)$ is defined in (9).

This is an immediate consequence of the result of (9) and of Theorem 3.4 in [6], for the condition $W(n)>0$ for all positive integers n, given in that theorem, is assumed only for convenience. The result of Lemma 5 is strong enough for our purposes; however sharper results hold. For example, in 1934 in [1], Erdös proved a result from which it follows that, when $W(p)=p-1$, the right side of (11) can be replaced by $x/(\log x)^{1+\delta}$ for a suitable $\delta>0$.

Lemma 6. *Let Q be a positive integer such that $\varrho'(p)<p-1$ for all $p|Q$ and $\nu(Q)\leqslant JL_2(x)$ where J is a positive integer satisfying $J\leqslant L_2(x)$. Then there exists a constant $C_1>0$ such that*

$$\prod_{p|Q}\big(1-\varrho'(p)(p-1)^{-1}\big)\geqslant C_1\big(L_3(x)\big)^{-l}.$$

Proof. Let $t=(L_2(x))^3$; then for all sufficiently large x, $\nu(Q)\leqslant(L_2(x))^2\leqslant\pi(t)$, and hence

$$\sum_{p|Q}p^{-1}\leqslant\sum_{p<t}p^{-1}=L_4(x)+O(1).$$

Thus, since $\varrho'(p)\leqslant\min(p-2,l)$ for all $p|Q$,

$$\log\prod_{p|Q}\big(1-\varrho'(p)(p-1)^{-1}\big)=-\sum_{p|Q}\varrho'(p)p^{-1}+O(1)\geqslant -lL_4(x)+O(1)$$

and the lemma follows.

3. Upper bounds in Theorems 1 and 2. Let $2\leqslant p_1<p_2<\ldots$ be a subsequence of the sequence of primes, and let $P_k=p_1p_2\ldots p_k$ $(k=1,2,\ldots)$. To each $i\geqslant 1$, associate $\psi(p_i)$ distinct residue classes $b_{ij}\ (\mathrm{mod}\ p_i)$ with $p_i\nmid b_{ij}$, given by $j=1,2,\ldots,\psi(p_i)$, where $0\leqslant\psi(p_i)<\varphi(p_i)$, and let

$$(12)\qquad \alpha_k=\prod_{i=1}^{k}\big(1-\psi(p_i)/\varphi(p_i)\big).$$

Let $N(x)$ denote the number of squarefree integers n with $1\leqslant n\leqslant x$ and with no prime divisor q satisfying $q\equiv b_{ij}(\mathrm{mod}\ p_i)$ for any j with $1\leqslant j\leqslant\psi(p_i)$ and any i with $1\leqslant i\leqslant k$.

Lemma 7. *There exists a positive constant C_2 such that, uniformly in the choice of the b_{ij} and for k with $P_k\leqslant\log x$,*

$$N(x)\leqslant x\big(L_1(x)\big)^{-1}\exp\Big(C_2\big(\alpha_kL_2(x)+L_3(x)\big)\Big).$$

Proof. If n contributes to $N(x)$ and $q|n$ but $q\nmid P_k$, then the prime q lies in one of $p_i-1-\psi(p_i)$ residue classes $(\mathrm{mod}\ p_i)$ coprime to p_i for $i=1,2,\ldots,k$, so q lies in one of

$$(13)\qquad \beta_k=\varphi(P_k)\alpha_k=\prod_{i=1}^{k}\big(p_i-1-\psi(p_i)\big)>0$$

residue classes $(\mathrm{mod}\ P_k)$, represented by a_j with $1\leqslant a_j<P_k$ and $(a_j,P_k)=1$ $(j=1,2,\ldots,\beta_k)$, say. Thus n contributing to $N(x)$ has no prime divisor in $\varphi(P_k)-\beta_k$ residue classes $(\mathrm{mod}\ P_k)$ coprime to P_k.

Let

$$h(n)=\begin{cases}1 & \text{if }\ \mu^2(n)=1\ \text{ and }\ q|n\Rightarrow q|P_k\ \text{ or }\ q\equiv a_j\ (\mathrm{mod}\ P_k)\ \text{ for }\ 1\leqslant j\leqslant\beta_k,\\ 0 & \text{otherwise.}\end{cases}$$

Then $h(p^\alpha)=0$ for all $\alpha\geqslant 2$, and h is multiplicative and satisfies the conditions of Lemma 3. Hence

$$(14)\quad N(x)=\sum_{n\leqslant x}h(n)\leqslant x\big(L_1(x)\big)^{-1}\Big(1+O\big(L_2(x)/L_1(x)\big)\Big)\prod_{p\leqslant x}\big(1+h(p)p^{-1}\big),$$

and

$$(15)\qquad \prod_{p\leqslant x}\big(1+h(p)p^{-1}\big) = \exp\Big(\sum_{p\leqslant x} h(p)p^{-1}+O(1)\Big)$$
$$= \exp\Big(\sum_{i=1}^{k} p_i^{-1} + \sum_{j=1}^{\beta_k}\ \sum_{\substack{p\leqslant x\\ p\equiv a_j (\mathrm{mod}\, P_k)}} p^{-1}+O(1)\Big)$$
$$\leqslant \exp\Big(\sum_{p\leqslant P_k} p^{-1} + \sum_{j=1}^{\beta_k}\ \sum_{\substack{P_k<p\leqslant x\\ p\equiv a_j (\mathrm{mod}\, P_k)}} p^{-1}+O(1)\Big)$$
$$\leqslant \exp\Big(L_2(P_k)+C_3\alpha_k\big(L_1(P_k)+L_2(x)\big)+O(1)\Big)$$

by Lemma 2 and (13), where C_3 is a positive constant. Since by hypothesis $P_k \leqslant L_1(x)$, the result of the lemma now follows from (14) and (15) provided $C_2 > 0$ is suitably chosen.

Proof of the upper bound in Theorem 1. Consider first square-free $n \leqslant x$ for which $p \nmid (f(n), g(n))$ for all primes $p \notin S_0$; then for no primes q, r dividing n can we have both $p|W_1(q)$ and $p|W_2(r)$ if $p \notin S_0$. Hence $q\nmid n$ if q belongs to one of the $\varrho_1'(p)$ residue classes $a_1 \pmod p$ with $p|W_1(a_1)$ and $p\nmid a_1$, or $r\nmid n$ if r belongs to one of the $\varrho_2'(p)$ residue classes $a_2 \pmod p$ with $p|W_2(a_2)$ and $p\nmid a_2$.

Let

$$S_4 = \{p\colon \exists\, a, b\in \mathbf{Z} \text{ with } p\nmid ab \text{ and } p|\big(W_1(a), W_2(b)\big)\}.$$

If $W_1(0)\neq 0$, $S_1\cap S_2 \setminus S_4$ is finite but, by Lemma 1, $S_1\cap S_2$ is infinite, and hence S_4 is infinite. If $W_1(0) = 0$, then by condition (ii) of Theorem 1,

$$W_1(u) = u^{l'}W_1^*(u) \quad \text{where} \quad 0<\deg W_1^* = l_1^* < \deg W_1 = l_1 \text{ and } W_1^*(0)\neq 0,$$

and corresponding to S_1 we have

$$S_1^* = \{p\colon \exists\, a\in \mathbf{Z} \text{ with } p|W_1^*(a)\},$$

but S_0 is unaltered when W_1 is replaced by W_1^*; in this case

$$S_4 = \{p\colon \exists\, a, b\in \mathbf{Z} \text{ with } p\nmid ab \text{ and } p|\big(W_1^*(a), W_2(b)\big)\}.$$

Hence, as in the case $W_1(0)\neq 0$, we have $S_1^*\cap S_2\setminus S_4$ is finite, $S_1^*\cap S_2$ is infinite by Lemma 1, and so S_4 is infinite. Thus in either case, $S_4\setminus S_0$ is infinite. Note that $2\notin S_4\setminus S_0$.

Let $p_1, p_2, \ldots, p_k, p_{k+1}$ be the first $k+1$ primes in $S_4\setminus S_0$ where k is such that $p_k \leqslant \frac{1}{2}L_2(x) < p_{k+1}$, so that for x sufficiently large,

$$P_k \leqslant \exp\big(\theta(p_k)\big) \leqslant \log x \quad \text{since} \quad \theta(p_k) = \sum_{p\leqslant p_k}\log p \sim p_k.$$

For $i = 1, \ldots, k$, let $\varepsilon(i) = 1$ or 2, so that there are 2^k possible k-tuples $\big(\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(k)\big)$, which we assume to be enumerated in some well defined way, and take $\psi(p_i) = \varrho'_{\varepsilon(i)}(p_i)$; then $0<\psi(p_i)<p_i-1$ since $p_i\in S_4$ but $p_i\notin S_0$. Now let b_{ij} $(1\leqslant j\leqslant \psi(p_i), 1\leqslant i\leqslant k)$ be those residue classes $(\bmod\, p_i)$ coprime to p_i satisfying $p_i|W_{\varepsilon(i)}(b_{ij})$, and let $N_u(x)$ be the number on the left of Lemma 7 associated with the uth k-tuple $\big(\varepsilon(1), \ldots, \varepsilon(k)\big)$ $(u = 1, 2, \ldots, 2^k)$. Then by Lemma 7 we have

$$(16)\qquad \sum_{\substack{n\leqslant x\\ p\nmid (f(n), g(n))\forall p\notin S_0}} \mu^2(n) \leqslant \sum_{\substack{n\leqslant x\\ p_i\nmid (f(n), g(n)), i=1,\ldots,k}} \mu^2(n) \leqslant \sum_{u=1}^{2^k} N_u(x)$$
$$\leqslant x\big(L_1(x)\big)^{-1}\sum_{u=1}^{2^k}\exp\Big(C_2\big(\alpha_{k,u}L_2(x)+L_3(x)\big)\Big)$$

where μ is the Möbius function and by (12)

$$(17)\qquad \alpha_{k,u} = \prod_{i=1}^{k}\big(1-\varrho'_{\varepsilon(i)}(p_i)/\varphi(p_i)\big) \leqslant \prod_{i=1}^{k}(1-p_i^{-1})$$
$$\leqslant \exp\Big(-\sum_{i=1}^{k}p_i^{-1}+O(1)\Big) \leqslant \exp\Big(-\sum_{\substack{p\leqslant p_k\\ p\in S_1^*\cap S_2}} p^{-1}+O(1)\Big)$$

since S_0 and $S_1^*\cap S_2\setminus S_4$ are finite, where $S_1^* = S_1$ if $W_1(0)\neq 0$. By Lemma 1,

$$(18)\qquad \sum_{\substack{p\leqslant p_k\\ p\in S_1^*\cap S_2}} p^{-1} = \lambda L_2(p_k)+D+O\Big(\big(L_1(p_k)\big)^{-1}\Big)$$

for some rational λ $(0<\lambda\leqslant 1)$ and some constant D. Hence there exists $C_4>0$ such that for each u

$$(19)\qquad \alpha_{k,u} \leqslant C_4\big(L_1(p_k)\big)^{-\lambda}.$$

From the definition of k it is easily seen using (18) that for x sufficiently large,

$$(20)\qquad p_k > \big(L_2(x)\big)^{1/2} \quad \text{and} \quad k\leqslant \pi(p_k) \leqslant 2p_k/L_1(2p_k) \leqslant L_2(x)/L_3(x).$$

Hence from (16) to (20),

$$(21)\qquad \sum_{\substack{n\leqslant x\\ p\nmid (f(n), g(n))\forall p\notin S_0}} \mu^2(n) \leqslant x\big(L_1(x)\big)^{-1}2^k\exp\Big(C_2\big(C_4L_2(x)\big(L_1(p_k)\big)^{-\lambda}+L_3(x)\big)\Big)$$
$$\leqslant x\big(L_1(x)\big)^{-1}\exp\Big(C_5L_2(x)/\big(L_3(x)\big)^{\lambda}\Big)$$

for a suitable positive constant C_5.

We must now extend the result of (21) to the sum in Theorem 1

Every positive integer n can be written uniquely in the form

$$n = n_1 n_2 \quad \text{where} \quad (n_1, n_2) = 1, \quad \mu^2(n_1) = 1, \quad \text{and} \quad p|n_2 \Rightarrow p^2|n_2.$$

If $p \nmid (f(n), g(n))$, then $p \nmid (f(n_1), g(n_1))$ but not vice versa. Hence, putting $v = (\log x)^2$ and using (21), we have

$$(22) \quad \sum_{\substack{n \leqslant x \\ p \nmid (f(n), g(n)) \forall p \notin S_0}} 1 \leqslant \sum_{n_2 \leqslant x} \sum_{\substack{n_1 \leqslant x/n_2 \\ p \nmid (f(n_1), g(n_1)) \forall p \notin S_0}} 1$$

$$\leqslant \sum_{n_2 \leqslant v} \frac{x}{n_2}\left(L_1\left(\frac{x}{n_2}\right)\right)^{-1} \exp\left(C_5 L_2\left(\frac{x}{n_2}\right) \Big/ \left(L_3\left(\frac{x}{n_2}\right)\right)^{\lambda}\right) + \sum_{v < n_2 \leqslant x} \frac{x}{n_2}$$

$$\leqslant x\left(L_1\left(\frac{x}{v}\right)\right)^{-1} \exp\left(C_5 L_2(x)/(L_3(x))^{\lambda}\right) \sum_{n_2 \leqslant v} n_2^{-1} + x \sum_{v < n_2 \leqslant x} n_2^{-1}$$

since $x/v \leqslant x/n_2 \leqslant x$ in the first sum. From a result of Erdös and Szekeres [3], it follows that

$$(23) \quad \sum_{n_2 \leqslant x} n_2^{-1} = C_6 - C_7 x^{-1/2} + o(x^{-1/2});$$

using this in (22) we have

$$(24) \quad \sum_{\substack{n \leqslant x \\ p \nmid (f(n), g(n)) \forall p \notin S_0}} 1$$

$$\leqslant C_6 x (L_1(x))^{-1} \exp\left(C_5 L_2(x)/(L_3(x))^{\lambda}\right)\left(1 + O(L_2(x)/L_1(x))\right) + O(x/L_1(x))$$

$$\leqslant C_8 x (L_1(x))^{-1} \exp\left(C_5 L_2(x)/(L_3(x))^{\lambda}\right)$$

for a suitable positive constant C_8. This establishes the upper bound in Theorem 1.

Proof of the upper bound in Theorem 2. We need only consider the squarefree n contributing to the sum $\Sigma(x)$ in Theorem 2, for then we can appeal to the argument used to deduce (24) from (21) above. Let $\Sigma^*(x)$ denote the number of squarefree n contributing to $\Sigma(x)$; for such an integer n, to each odd prime p there corresponds c_p such that $p \nmid c_p$ and $q \nmid n$ for all primes $q \equiv c_p \pmod{p}$. The number of possible values for c_p depends on both n and p, but is positive and does not exceed $\varphi(p)$. Let $p_1, \ldots, p_k$ be the first k odd primes and write c_i for c_{p_i}; the ordered k-tuple $(c_1, \ldots, c_k)$ can be chosen in $\varphi(P_k)$ ways, and any n contributing to $\Sigma^*(x)$ is associated with one or more such k-tuples. Let p_k be the largest prime not exceeding $\frac{1}{2}L_2(x)/L_3(x)$, so that for sufficiently large x,

$$p_k > (L_2(x)/L_3(x))^{1/2}, \quad P_k \leqslant \exp(2p_k) \leqslant \exp(L_2(x)/L_3(x)) < \log x,$$

and

$$\varphi(P_k) = P_k \prod_{i=1}^{k} (1 - p_i^{-1}) \sim 2e^{-\gamma} P_k / L_1(p_k)$$

whence

$$\varphi(P_k) \leqslant C_9 \exp\left(L_2(x)/L_3(x)\right)/L_3(x)$$

for a suitable positive constant C_9.

Now apply Lemma 7 with $\psi(p_i) = 1$ and $b_{i1} = c_i$ $(i = 1, \ldots, k)$, so that using (12)

$$a_k = \prod_{i=1}^{k} \left(1 - (p_i - 1)^{-1}\right) \leqslant C_{10}/L_3(x)$$

for some $C_{10} > 0$; then we have from our choice of p_k that

$$\Sigma^*(x) \leqslant \varphi(P_k) x (L_1(x))^{-1} \exp\left(C_2\left(C_{10} L_2(x)(L_3(x))^{-1} + L_3(x)\right)\right)$$

$$\leqslant C_9 x (L_1(x))^{-1} \exp\left(C_{11} L_2(x)/L_3(x)\right)$$

where C_{11} is a suitable positive constant. On using the arguments used to derive (24), the upper bound in Theorem 2 now follows.

4. Lower bounds in Theorems 1 and 2. We use Lemma 8 below to establish these lower bounds, and before stating it, we need some further notation.

Let $\mathcal{Q}_t$ be a set of primes, depending on a parameter t, such that $\mathcal{Q}_{t_1} \subset \mathcal{Q}_{t_2}$ whenever $t_1 < t_2$ and

$$(25) \quad |\{p \leqslant t \colon p \notin \mathcal{Q}_t\}| \ll t/L_1(t) L_2(t).$$

Suppose $(a_0, P_0) = 1$, where a_0, P_0 are fixed, and let

$$(26) \quad \mathcal{Q}_t^0 = \{p \in \mathcal{Q}_t \colon p \equiv a_0 \pmod{P_0}\}.$$

We shall define sets $\mathcal{S}_j(t)$ $(j = 1, 2, \ldots)$ of positive integers inductively. We shall assume that, once $\mathcal{S}_{j-1}(t)$ has been defined, then to each $m_{j-1} \in \mathcal{S}_{j-1}(t)$ there are assigned a unique positive squarefree integer M_{j-1} and a unique polynomial w_{j-1}, with integer coefficients (possibly depending on m_{j-1}) and of fixed positive degree l, which satisfy $(M_{j-1}, P_0) = 1$ and the following conditions:

(i) There exists a positive integer k such that

$$\nu(M_{j-1}) \leqslant kj L_2(t);$$

(ii) $\max_{p|M_{j-1}} p < \max\left((m_{j-1})^l, C_{12}\right)$ where C_{12} is an absolute constant;

(iii) $\varrho'_{j-1}(p) < p-1$ for all $p \nmid P_0$ (and so for all $p \mid M_{j-1}$ and for all $p \in \mathcal{Q}_l^0$), where $\varrho'_{j-1}(p)$ is the number of solutions of $w_{j-1}(u) \equiv 0 \pmod p$ with $p \nmid u$.

In our applications, (i) follows from our definition of $\mathcal{Q}_l$, and by well known results, $\varrho'_{j-1}(p) \leqslant l$ whenever (iii) holds.

Let $\mathcal{S}_1(t) = \mathcal{Q}_t^0$; suppose that $\mathcal{S}_{j-1}(t)$ has been defined for some $j \geqslant 2$, and suppose furthermore that $\mathcal{S}_{j-1}(t_1) \subset \mathcal{S}_{j-1}(t_2)$ whenever $t_1 < t_2$ and that $q \mid m_{j-1} \in \mathcal{S}_{j-1}(t) \Rightarrow q \in \mathcal{Q}_t^0$. Then we define $\mathcal{S}_j(t)$ to be the set of all positive integers $m_j = m_{j-1} q_j$ such that

$$(27) \qquad m_{j-1} \in \mathcal{S}_{j-1}(t), \qquad q_j \in \mathcal{Q}_t^0, \qquad q_j > \max_{q \mid m_{j-1}} q, \qquad \big(w_{j-1}(q_j), M_{j-1}\big) = 1.$$

It is easily seen that if $m_j \in \mathcal{S}_j(t)$, then $\nu(m_j) = j$, $\mu^2(m_j) = 1$, $q \in \mathcal{Q}_t^0$ $\forall q \mid m_j$, and $\mathcal{S}_j(t_1) \subset \mathcal{S}_j(t_2)$ whenever $t_1 < t_2$ (since $\mathcal{Q}_{t_1}^0 \subset \mathcal{Q}_{t_2}^0$ and $\mathcal{S}_{j-1}(t_1) \subset \mathcal{S}_{j-1}(t_2)$). Now define

$$S_j(x, t) = |\{m_j \in \mathcal{S}_j(t) \colon m_j \leqslant x\}|, \qquad S_j(x) = S_j(x, x).$$

Our next aim is to establish a lower bound for $S_j(x)$.

LEMMA 8. *There exist positive constants C_{13}, C_{14}, C_{15} such that for all $j \geqslant 1$ and all x satisfying*

$$(28) \qquad j(1+j^{-1})^j \leqslant C_{13} L_2(x)/L_4(x),$$

we have

$$(29) \qquad S_j(x) \geqslant C_{14} \frac{x}{L_1(x)} \frac{\big(C_{15} L_2(x)\big)^{j-1}}{(j-1)!\,\big(L_3(x)\big)^{l(j-1)}}.$$

Notes: (1) Clearly $S_j(x) = 0$ if $x < 2 \cdot 3 \cdot \ldots \cdot p_j = \exp\big(\theta(p_j)\big)$; since $\theta(p_j) \sim p_j \sim j \log j$, (28) certainly ensures that $x > \exp\big(\theta(p_j)\big)$ for each $j \geqslant 1$, provided C_{13} is suitably chosen.

(2) The right side of (29) starts decreasing when j increases from $C_{15} L_2(x)/\big(L_3(x)\big)^l$; we utilize this in our applications, but we prove Lemma 8 for the larger range (28).

(3) Since the sequence $\big((1+j^{-1})^j\big)$ increases and converges to e, we have

$$(30) \qquad 2 \leqslant 2j \leqslant j(1+j^{-1})^j \leqslant ej;$$

hence (28) implies that

$$(31) \qquad j \leqslant \tfrac{1}{2} C_{13} L_2(x)/L_4(x),$$

and conversely if

$$j \leqslant e^{-1} C_{13} L_2(x)/L_4(x),$$

then (28) follows.

(4) From (28) and (30),

$$(32) \qquad L_2(x)/L_4(x) \geqslant 2/C_{13};$$

hence we can always choose C_{13} small enough so that each of a finite number of statements true for all sufficiently large x holds. We shall assume in the proof that this has been done, and we note that such a condition on x below will always be independent of the value of j. We begin by assuming that $C_{13} < 1$ and that x is sufficiently large for $L_4(x) > 1$.

Proof of Lemma 8. Consider first the case $j = 1$, and suppose that (28) with $j = 1$, i.e. (32), holds. Now

$$\begin{aligned} S_1(x) &= |\{q \leqslant x \colon q \in \mathcal{Q}_x^0\}| \\ &= \pi(x; P_0, a_0) + O\big(x/L_1(x) L_2(x)\big) \\ &= x\Big(1 + O\big(\big(L_2(x)\big)^{-1}\big)\Big)/\varphi(P_0) L_1(x) \end{aligned}$$

on using (25), (26) and the Prime Number Theorem for primes in arithmetic progression; we recall that P_0 is fixed. If we take C_{14} to be a constant satisfying $0 < C_{14} < \big(\varphi(P_0)\big)^{-1} \leqslant 1$ and choose C_{13} small enough, we have

$$S_1(x) \geqslant C_{14} x/L_1(x) \qquad \text{whenever} \qquad 2 \leqslant C_{13} L_2(x)/L_4(x),$$

which establishes the case $j = 1$ of the lemma. From now on, C_{14} is fixed and we may not increase C_{13}; decreasing C_{13} will have the effect of increasing the lower bound for x in (32).

Suppose next that the lemma holds for $S_{j-1}(t)$ for some $j \geqslant 2$, and assume that (28) is satisfied. Define y and v_j by

$$(33) \qquad L_1(y) = L_1(x)/\big(L_3(x)\big)^2, \qquad L_1(v_j) = \big(L_1(x)\big)^{1-j^{-1}} \qquad (j \geqslant 2);$$

then for sufficiently large x, $1 < v_j < y < x$ and $L_4(v_j) > 1$. Furthermore from (28) and (33), we have for $j \geqslant 2$

$$\begin{aligned} (34) \qquad (j-1)\big(1+(j-1)^{-1}\big)^{j-1} &\leqslant j(1+j^{-1})^j (1-j^{-1}) \\ &\leqslant (1-j^{-1}) C_{13} L_2(x)/L_4(x) \leqslant C_{13} L_2(v_j)/L_4(v_j) \end{aligned}$$

which is (28) with j, x replaced by $j-1$, v_j respectively. Hence by our induction hypothesis, whenever $v_j \leqslant t \leqslant x$,

$$(35) \qquad S_{j-1}(t) \geqslant C_{14} \frac{t}{L_1(t)} \frac{\big(C_{15} L_2(t)\big)^{j-2}}{(j-2)!\,\big(L_3(t)\big)^{l(j-2)}}.$$

Now consider

$$S_j(x) = |\{m_j \in \mathcal{S}_j(x) \colon m_j \leqslant x\}|,$$

and recall that the elements $m_j = m_{j-1}q_j$ of $\mathscr{S}_j(x)$ satisfy (27). Clearly

$$(36)\qquad S_j(x) \geqslant \sum_{\substack{m_{j-1}\leqslant y \\ m_{j-1}\in\mathscr{S}_{j-1}(x)}} \sum_{\substack{y<q_j\leqslant x/m_{j-1} \\ q_j\equiv a_0 (\mathrm{mod}\, P_0) \\ q_j\in\mathscr{Q}_x \\ (w_{j-1}(q_j), M_{j-1})=1}} 1$$

$$\geqslant \sum_{\substack{m_{j-1}\leqslant y \\ m_{j-1}\in\mathscr{S}_{j-1}(x)}} \Big\{ \sum_{\substack{y<q_j\leqslant x/m_{j-1} \\ q_j\equiv a_0 (\mathrm{mod}\, P_0) \\ (w_{j-1}(q_j), M_{j-1})=1}} 1 + O\Big(\sum_{\substack{q_j\leqslant x/m_{j-1} \\ q_j\notin\mathscr{Q}_{x/m_{j-1}}}} 1\Big)\Big\}$$

since $q \notin \mathscr{Q}_x \Rightarrow q \notin \mathscr{Q}_{x/m_{j-1}}$ by the definition of $\mathscr{Q}_x$, and further by (25), the error term is for $1 \leqslant m_{j-1} \leqslant y$

$$(37)\qquad \ll (x/m_{j-1})/\big(L_1(x/m_{j-1})L_2(x/m_{j-1})\big) \ll (x/m_{j-1})/\big(L_1(x)L_2(x)\big).$$

To the main inner sum of (36) we apply Lemma 4; our conditions (ii) and (iii) ensure that $\varrho'_{j-1}(q) < q-1$ for $q \nmid P_0$, and that $\mathscr{P} = \max_{p|M_{j-1}} p < \max\big((m_{j-1})^l, C_{12}\big) < y^l$ for x sufficiently large, and so for $1 \leqslant m_{j-1} \leqslant y$,

$$L_1(\mathscr{P}) < lL_1(x)/\big(L_3(x)\big)^2 < L_1(x/y)/L_3(x/y) < L_1(x/m_{j-1})/L_3(x/m_{j-1})$$

since $L_1(x/y)/L_3(x/y) \sim L_1(x)/L_3(x)$ by (33). Hence by Lemma 4,

$$(38)\qquad \sum_{\substack{y<q_j\leqslant x/m_{j-1} \\ q_j\equiv a_0 (\mathrm{mod}\, P_0) \\ (w_{j-1}(q_j), M_{j-1})=1}} 1$$

$$= \frac{x/m_{j-1}}{\varphi(P_0)L_1(x/m_{j-1})} \prod_{p|M_{j-1}} \Big(1 - \frac{\varrho'_{j-1}(p)}{p-1}\Big)\Big\{1 + O\Big(\big(L_2(x/m_{j-1})\big)^{-1}\Big)\Big\} + O\big(\pi(y)\big)$$

where the first O-constant depends on the fixed degree l but not on the coefficients of w_{j-1}, and as in (37) for $m_{j-1} \leqslant y$ we have

$$O\Big(\big(L_2(x/m_{j-1})\big)^{-1}\Big) = O\Big(\big(L_2(x)\big)^{-1}\Big).$$

Since $j = o\big(L_2(x)\big)$ by (31), $\nu(M_{j-1}) \leqslant kjL_2(x)$ by condition (i), and $\varrho'_{j-1}(p) < p-1$ for all $p|M_{j-1}$ by (iii), we have by Lemma 6

$$\prod_{p|M_{j-1}} \big(1 - \varrho'_{j-1}(p)(p-1)^{-1}\big) \geqslant C_1/\big(L_3(x)\big)^l.$$

Substituting this in (38) and then using (36) and (37), we have

$$(39)\qquad S_j(x) \geqslant \frac{x}{L_1(x)} \Big\{ \frac{C_1}{\varphi(P_0)} \big(L_3(x)\big)^{-l}\Big(1 + O\Big(\big(L_2(x)\big)^{-1}\Big) +$$

$$+ O\Big(\big(L_2(x)\big)^{-1}\Big)\Big\} \sum_{\substack{m_{j-1}\leqslant y \\ m_{j-1}\in\mathscr{S}_{j-1}(x)}} (m_{j-1})^{-1} + O\big(y^2/L_1(y)\big)$$

for sufficiently large x, so that the expression $\{\ldots\}$ is positive, and by (33) the last error term is certainly $O(x^{1/2})$.

Finally we must consider the sum, and for that we use (34) and our induction hypothesis (35) and the fact that $\mathscr{S}_{j-1}(t) \subset \mathscr{S}_{j-1}(x)$ for $t < x$ whence $S_{j-1}(t, x) \geqslant S_{j-1}(t)$. By partial summation, we have since $1 < v_j < y < x$

$$(40)\qquad \sum_{\substack{m_{j-1}\leqslant y \\ m_{j-1}\in\mathscr{S}_{j-1}(x)}} (m_{j-1})^{-1}$$

$$= S_{j-1}(y, x)y^{-1} + \int_1^y S_{j-1}(t, x)t^{-2}\,dt > \int_{v_j}^y S_{j-1}(t)t^{-2}\,dt$$

$$> \frac{C_{14}(C_{15})^{j-2}}{(j-2)!\big(L_3(x)\big)^{l(j-2)}} \int_{v_j}^y \big(L_2(t)\big)^{j-2}\big(tL_1(t)\big)^{-1}\,dt$$

$$\geqslant \frac{C_{14}\big(C_{15}L_2(x)\big)^{j-1}}{(j-1)!\big(L_3(x)\big)^{l(j-2)}} (C_{15})^{-1}\Big\{\Big(\frac{L_2(y)}{L_2(x)}\Big)^{j-1} - \Big(\frac{L_2(v_j)}{L_2(x)}\Big)^{j-1}\Big\}.$$

By (33)

$$(41)\qquad \big(L_2(v_j)/L_2(x)\big)^{j-1} = (1-j^{-1})^{j-1} \leqslant \tfrac{1}{2} \quad \text{for all } j \geqslant 2$$

since $(1-j^{-1})^{j-1}$ decreases as j increases from 2 (and converges to e^{-1}). For $0 < t < \frac{1}{2}$, $(1-t)^{j-1} > \exp\big(-2t(j-1)\big)$; hence if x is sufficiently large for $0 < 2L_4(x)/L_2(x) < \frac{1}{2}$, we have by (31) and (33) that

$$(42)\qquad \big(L_2(y)/L_2(x)\big)^{j-1} = \big(1 - 2L_4(x)/L_2(x)\big)^{j-1}$$

$$> \exp\big(-4(j-1)L_4(x)/L_2(x)\big)$$

$$> \exp(-2C_{13}) > 3/4$$

if C_{13} is chosen small enough. Hence by (41) and (42),

$$(43)\qquad \big(L_2(y)/L_2(x)\big)^{j-1} - \big(L_2(v_j)/L_2(x)\big)^{j-1} > \exp(-2C_{13}) - \tfrac{1}{2} > \tfrac{1}{4}.$$

From (40) and (43), it follows that

$$\sum_{\substack{m_{j-1}\leqslant y \\ m_{j-1}\in\mathscr{S}_{j-1}(x)}} (m_{j-1})^{-1} > \frac{C_{14}\big(C_{15}L_2(x)\big)^{j-1}}{(j-1)!\big(L_3(x)\big)^{l(j-2)}} (4C_{15})^{-1},$$

and substituting this in (39) we have, when (28) and the result of the lemma for $j-1$ hold, that

$$S_j(x) \geqslant \frac{C_{14}x}{L_1(x)} \frac{(C_{15}L_2(x))^{j-1}}{(j-1)!(L_3(x))^{l(j-1)}} \times$$
$$\times \{C_1(4\varphi(P_0)C_{15})^{-1} + O((L_3(x))^l/L_2(x))\} + O(x^{1/2})$$
$$\geqslant C_{14}\frac{x}{L_1(x)} \frac{(C_{15}L_2(x))^{j-1}}{(j-1)!(L_3(x))^{l(j-1)}}$$

for a suitable choice of C_{15} and for all sufficiently large x. This completes the proof by induction of the lemma.

Proof of the lower bound in Theorem 1. We shall apply the previous lemma to suitable sets $\mathscr{S}_j(t)$, and to do so we need some definitions. First we define a_0, P_0. Let p_0 be the least prime exceeding

$$\max(l+1, W_2(0), \max_{p \in S_0 \cup S_0'} p, \max_{W_1(p) \leqslant 0} p, \max_{W_2(p) \leqslant 0} p)$$

(a valid definition by our assumptions), and put $P_0 = \prod_{p<p_0} p$. If $p \in S_0'$, let a_p be the least positive integer such that $p \nmid (W_1(a_p), W_2(a_p))$; since $p \notin S_0$, a_p with $1 \leqslant a_p < p$ exists. If $p|P_0$ but $p \notin S_0'$, let $a_p = 1$. Then we define a_0 $(1 \leqslant a_0 < P_0, (a_0, P_0) = 1)$ to be the simultaneous solution of all the congruences of the form

$$u \equiv a_p (\mathrm{mod}\, p), \quad p|P_0.$$

We note that

$$a_0 = 1 \quad \text{if} \quad S_0' = \emptyset \text{ (the empty set)};$$

$$p \nmid (W_1(a_0), W_2(a_0)) \quad \text{for all } p \text{ satisfying } p|P_0 \text{ and } p \notin S_0;$$

$$q \equiv a_0 (\mathrm{mod}\, P_0) \quad \text{and} \quad q \text{ prime} \Rightarrow q \geqslant p_0;$$

$$p \nmid P_0 \Rightarrow p \geqslant p_0, \quad \varrho'(p) \leqslant l < p-1, \quad p \nmid g(p), \quad W(p) > 0,$$
$$\text{and} \quad p \nmid (f(q), g(q)) \text{ for all primes } q,$$

where $\varrho'(p)$ was defined at the beginning of § 2.

Now let $\mathfrak{Q}_t = \{q: \nu(W(q)) \leqslant 2T(t)\}$; then (25) holds by Lemma 5, for $W(p) > 0$ for all but a finite number of primes p, and clearly $\mathfrak{Q}_{t_1} \subset \mathfrak{Q}_{t_2}$ if $t_1 < t_2$. We now define $\mathfrak{Q}_t^0$ by (26).

We must define M_{j-1} and w_{j-1} before we can construct inductively the sets $\mathscr{S}_j(t)$ by the method described earlier. For $m_{j-1} \in \mathscr{S}_{j-1}(t)$, let

$$M_{j-1} = \prod_{\substack{p \nmid P_0 \\ p | f(m_{j-1})g(m_{j-1})}} p$$

and let w_{j-1} always be the polynomial W; then we must check that our conditions (i), (ii), (iii) all hold. Since $q \in \mathfrak{Q}_t^0$ $\forall q|m_{j-1}$ and $\mu^2(m_{j-1}) = 1$, $\nu(m_{j-1}) = j-1$,

(i) $$\nu(M_{j-1}) \leqslant \nu\Big(\prod_{q|m_{j-1}} W(q)\Big) \leqslant 2(j-1)T(t) = 2(j-1)(\tau L_2(t) + O(1))$$
$$\leqslant kjL_2(t)$$

for a suitable positive integer k, since ν is additive and (9) holds;

(ii) $$\max_{p|M_{j-1}} p \leqslant \max_{p|W(q), q|m_{j-1}} p < (m_{j-1})^l \quad \text{for all but a finite number of } m_{j-1}$$

since W is reducible and of degree l;

(iii) $$\varrho'(p) < p-1 \quad \text{for} \quad p \nmid P_0$$

by a remark above. Thus we can define the sets $\mathscr{S}_j(t)$ and the quantities $S_j(x, t)$, $S_j(x)$ as above and estimate $S_j(x)$ by Lemma 8.

We show next that for this particular choice of the sets $\mathscr{S}_j(t)$ $(j = 1, 2, \ldots)$,

(44) $$p \nmid (f(m_j), g(m_j)) \,\forall p \notin S_0 \quad \text{whenever} \quad m_j \in \mathscr{S}_j(t).$$

This clearly holds when $j = 1$ since $m_1 = q \equiv a_0 (\mathrm{mod}\, P_0)$. Assume it holds for $j-1 \geqslant 1$ and consider $m_j \in \mathscr{S}_j(t)$; then $m_j = m_{j-1}q_j = m_{j-1}q$ where (27) holds and thus since $(W(q), M_{j-1}) = 1$

$$p \nmid (f(m_{j-1}), g(m_{j-1}))(f(q), g(q)) \,\forall p \notin S_0,$$
$$p \nmid (f(q)g(q), f(m_{j-1})g(m_{j-1})) \,\forall p \nmid P_0;$$

moreover $p \nmid (f(q), g(m_{j-1}))(f(m_{j-1}), g(q))$ for $p|P_0$ but $p \notin S_0$ since a prime $r|qm_{j-1}$ satisfies $r \equiv a_0 (\mathrm{mod}\, P_0)$ and we can then use a remark above. Since $q \nmid m_{j-1}$ and f, g are multiplicative, it is easily seen that these relations imply (44).

We can now deduce our result. For

(45) $$\Sigma_{f,g}(x)$$
$$\geqslant |\{n \leqslant x: \mu^2(n) = 1, \nu(n) = j, p|n \Rightarrow p \in \mathfrak{Q}_x^0, p \nmid (f(n), g(n)) \,\forall p \notin S_0\}|$$
$$\geqslant S_j(x) \geqslant C_{14}\frac{x}{L_1(x)} \frac{(C_{15}L_2(x))^{j-1}}{(j-1)!(L_3(x))^{l(j-1)}} = C_{14}\frac{x}{L_1(x)} E(x),$$

say, provided that (28) holds. Our aim now is to choose j in terms of x to make the magnitude of the right side as large as possible, and we utilize the remark in note (2) that the right side starts decreasing when j increases beyond $C_{15}L_2(x)/(L_3(x))^l$. Hence we take

$$j = [C_{15}L_2(x)/(L_3(x))^l].$$

Then by Stirling's formula

$$(46)\quad E(x) = \exp\{(j-1)(L_3(x)+\log C_{15}-lL_4(x))-(j-\tfrac{1}{2})\log j+j+O(1)\}$$
$$= \exp\left\{\frac{C_{15}L_2(x)}{(L_3(x))^l}+O(L_3(x))\right\} \geqslant \exp\left(C_{16}\frac{L_2(x)}{(L_3(x))^l}\right)$$

for $C_{16} < C_{15}$ and x sufficiently large. Hence the lower bound in Theorem 1 follows from (45) and (46).

Proof of the lower bound in Theorem 2. We again reduce the problem to an application of Lemma 8. Let $P_0 = 2$, $a_0 = 1$, and take $\mathcal{Q}_j$ to be the set of all primes (so that the parameter t is redundant here). Then $S_1(x) = \pi(x)-1$.

Let p_i denote the ith odd prime. We show that we can define the sets $\mathcal{S}_j = \mathcal{S}_j(t)$ $(j \geqslant 1)$ in such a way that if $m_j \in \mathcal{S}_j$, then to each odd prime p_i there corresponds c_i coprime to p_i such that $q \not\equiv c_i \pmod{p_i}$ for all primes $q|m_j$; this is certainly true when $j = 1$. Note that if $i \geqslant j \geqslant 1$, then since $p_i \geqslant 2i+1$, $\varphi(p_i) \geqslant 2i \geqslant 2j > j$, and hence there are more residue classes (mod p_i) coprime to p_i than primes dividing m_j, so in this case c_i exists trivially. Suppose that $\mathcal{S}_{j-1}$ has been defined for some $j \geqslant 2$ to satisfy the above property. Then to each $m_{j-1} \in \mathcal{S}_{j-1}$, we can find a_{j-1} such that $(a_{j-1}, p_1 \dots p_{j-1}) = 1$, $1 \leqslant a_{j-1} < p_1 \dots p_{j-1}$, and $q \not\equiv a_{j-1}$ (mod $p_1 \dots p_{j-1}$) for all primes $q|m_{j-1}$, for let $a_{j-1} \equiv c_i \pmod{p_i}$ $(i = 1, \dots, j-1)$. We define

$$M_{j-1} = p_1 \dots p_{j-1}, \qquad w_{j-1}(u) = u - a_{j-1},$$

and then conditions (i), (ii), (iii) at the beginning of § 4 are all satisfied, for since $\nu(M_{j-1}) = j-1$ and $\varrho'_{j-1}(p) \leqslant 1 = l < p-1$ for all odd primes p, (i) and (iii) hold trivially and (ii) holds since

$$\max_{p|M_{j-1}} p = p_{j-1} \leqslant \max_{p|m_{j-1}} p \leqslant m_{j-1}$$

with strict inequality except when $j = 2$ and $m_1 = 3$. Hence if we define $\mathcal{S}$ to be the set of all positive integers $m_j = m_{j-1}q_j$ satisfying (27), we see that since $(q_j - a_{j-1}, p_1 \dots p_{j-1}) = 1$, q_j and all the primes dividing m_{j-1} avoid the residue class $c_i \pmod{p_i}$ for $i = 1, \dots, j-1$ and moreover, by a remark above, for each $i \geqslant j$ there is a residue class (mod p_i) coprime to p_i containing no prime dividing $m_{j-1}q_j$. Hence $\mathcal{S}_j$ has been defined so that it has the required property as well as conforming to the general definition given earlier.

It is now clear that by Lemma 8 (since $l = 1$ here)

$$\Sigma(x) \geqslant S_j(x) \geqslant C_{14}\frac{x}{L_1(x)}\,\frac{(C_{15}L_2(x))^{j-1}}{(j-1)!\,(L_3(x))^{j-1}}$$

provided (28) holds. Now choose

$$j = [C_{15}L_2(x)/L_3(x)]$$

and then it follows, as in the proof of Theorem 1, that

$$\Sigma(x) \geqslant C_{14}\frac{x}{L_1(x)}\exp\left(C_{17}L_2(x)/L_3(x)\right)$$

for $C_{17} < C_{15}$ and x sufficiently large.

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WESTFIELD COLLEGE
London NW3 7ST

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