where

$$A = \frac{\frac{201}{100}3^{3/2}\log^{\frac{701}{400}}\frac{1}{3n(1-a)}}{\log^2\frac{1}{n(1-a)}} + \frac{18\cdot 3^{2/3}\log^{\frac{601}{400}}\frac{1}{3\left(n(1-a)\right)}}{\log^2\frac{1}{n(1-a)}}$$

valid in the interval (1.1) and for T satisfying (1.2).

It is easy to realize that for all a-values from (1.1) we have $A < 1/10$ and the proof of the theorem follows.

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On the length of continued fractions representing a rational number with given denominator

by

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To the memory of my teacher and friend P. Turán

Let N be a given natural number. Denote by $l(a) = l(a, N)$ the length of the finite continued fraction

$$\frac{a}{N} = [0; b_1, \ldots, b_m], \quad (a, N) = 1, b_m \geqslant 2. \tag{1}$$

A few years ago Heilbronn [2] showed that

$$\sum_{\substack{1\leqslant a<N\\(a,N)=1}} l(a) = \frac{12\log 2}{\pi^2}\varphi(N)\log N + O\left(N\sigma_{-1}^3(N)\right), \tag{2}$$

where $\sigma_{-1}(N)$ denotes the sum $\sum_{d|N} 1/d$.

A classical result of A. Khintchine [3] and P. Lévy [4] states that, putting

$$t = [0; b_1, b_2, \ldots] \quad (t \text{ real}, \ t\in(0, 1)) \tag{3}$$

for almost all t we have

$$\sqrt[k]{B_k} \to e^{\frac{\pi^2}{12\log 2}} \quad (k\to\infty). \tag{4}$$

B_k being the denominator of the kth convergent of the continued fraction (3). Because of (4) Heilbronn's result (2) is not very surprising. The subtleness of his result is that we can make a statement for any given N, not only for "most N's" in some sense. In 1970 J. Dixon [1] proved a theorem about the length of the continued fractions of "most" rational numbers a/ν where $a < \nu$, $(a, \nu) = 1$ and $\nu \leqslant N$; the exceptional set was not for a given ν in the a's, but in pairs a, ν. One can ask whether a statement about "most a's" for any given N is true. In the present paper I prove a theorem in this direction. As a special case, it contains (2) even with a stronger remainder term than $O(N\sigma^3_{-1}(N))$. The proof is based on probability theory. To the question, whether convergence to the Gaussian distribution or the law of iterated logarithm holds, I hope to be able to return later.

THEOREM. *We have with at most $N^{1-\varepsilon}$ exceptions in the a's,*

$$\frac{\log N}{\log\gamma}(1-\varepsilon) < l(a) < \frac{\log N}{\log\gamma}(1+\varepsilon) \tag{5}$$

with $\gamma = e^{\frac{\pi^2}{12\log 2}}$.

The proof is contained in the next two sections.

1. Lemmas. Let t be a real number in $(0,1)$ and use the notation (3). Further, denote by $P(A)$ the Lebesgue measure of the set A in t. Then we have

LEMMA 1.1.

$$\begin{aligned} &P\big(B_k > (1+\varepsilon)^k\gamma^k\big) < e^{-c\varepsilon^2 k},\\ &P\big(B_k < (1-\varepsilon)^k\gamma^k\big) < e^{-c\varepsilon^2 k}. \end{aligned} \tag{1.1}$$

Proof. Let

$$\zeta_k = [b_k; b_{k+1}, \ldots]. \tag{1.2}$$

Then, because of $\zeta_k = b_k + \dfrac{1}{\zeta_{k+1}}$ we have

$$\frac{B_{k+1}\zeta_{k+2}+B_k}{B_k\zeta_{k+1}+B_{k-1}} = \zeta_{k+2},$$

or

$$\sum_{j=2}^{k+1}\log\zeta_j = \log(B_k\zeta_{k+1}+B_{k-1}) - \log\zeta_1.$$

Therefore

$$\log B_k + \log\left(\zeta_{k+1}+\frac{B_{k-1}}{B_k}\right) = \sum_{j=1}^{k+1}\log\zeta_j. \tag{1.3}$$

Consider t in $(0,1)$ as a uniformly distributed random variable. Since (because of (3)) the ζ_k's are functions of t, the ζ_k's are random variables also.

By my refinement of the Gauss–Kuzmin theorem [5] we have

$$E(\log\zeta_k) = \frac{1}{\log 2}\int_1^\infty \log x\,\frac{1}{x(x+1)}\,dx\big(1+O(q^k)\big), \tag{1.4}$$

where $E(\cdot)$ is the expectation and q a positive constant $< \frac{1}{2}$. The integral on the right-hand side of (1.4) is equal to $\pi^2/12$.

Since $P'(\zeta_k < x) = \dfrac{1}{\log 2}\dfrac{1}{x(x+1)}\big(1+O(q^k)\big)$, $P'(\cdot)$ denoting differentiation with respect to x, $E(\log^2\zeta_k)$ exists also. Further, by [5] it follows that

$$E(\log\zeta_k\log\zeta_l) = E(\log\zeta_k)E(\log\zeta_l)\big(1+O(q^{k-l})\big).$$

Now for mutually independent bounded random variables

$$\log\zeta_1^*,\ \log\zeta_2^*, \ldots$$

we have (see for instance Rényi [7], p. 323)

$$P\left(\left|\sum_{k=1}^{n}\big(\log\zeta_k - E(\log\zeta_k)\big)\right| > \varepsilon c_1 n\right) < e^{-nc_2\varepsilon^2}, \tag{1.5}$$

c_1 and c_2 being constants depending only on the bounds of the ζ_k^*'s and the variances $D^2(\log\zeta_k^*)$.

Now if $\zeta_1\ldots$ have the meaning (1.1) then they are neither mutually independent nor bounded. But their dependence and increase are so weak that the proof given in [7] works without essential change. I omit the details.

(1.4) and (1.5) give

$$P\left(\left|\sum_{k=1}^{n}\log\zeta_k - n\frac{\pi^2}{12\log 2}\right| > \varepsilon c_1 n\right) < e^{-c_2\varepsilon^2 n}; \tag{1.6}$$

because of (1.3), $\log B_k = \sum\limits_{j=1}^{k}\log\zeta_j + O(1)$ and so (1.1) follows immediately from (1.6).

LEMMA 1.2. *Let N be given to each a $(1 \leqslant a < N,\ (a,N)=1)$; define $m_0 = m_0(a,N)$ as the index of the greatest denominator of the convergents in*

$$\frac{a}{N} = [0; b_1, \ldots, b_m] \tag{1.7}$$

such that

$$B_{m_0} \leqslant N^{1/2},$$

that is,

$$B_{m_0} \leqslant N^{1/2}, \quad B_{m_0+1} > N^{1/2}. \tag{1.8}$$

Then we have, with at most $N^{1-c\varepsilon^2}$ exceptions in a

$$\frac{\log N}{2\log\gamma(1+\varepsilon)} < m_0 < \frac{\log N}{2\log\gamma(1-\varepsilon)}, \tag{1.9}$$

c being a positive constant.

Proof. Because of (1.7) we have (see Perron [6], p. 32)

$$\frac{a}{N} = \frac{A_{m_0}\zeta_{m_0+1}+A_{m_0-1}}{B_{m_0}\zeta_{m_0+1}+B_{m_0-1}},$$

A_k/B_k denoting the convergents of the continued fraction (1.7). Because of $B_{m_0} < N^{1/2}$ with $1 < t < \infty$

$$\frac{A_{m_0}t+A_{m_0-1}}{B_{m_0}t+B_{m_0-1}}$$

runs over an interval of length

$$\frac{1}{B_{m_0}(B_{m_0}+B_{m_0-1})} > \frac{1}{2N}.$$

Therefore if $m_0 \leqslant \dfrac{\log N}{2\log\gamma(1+\varepsilon)}$ for more than $N^{1-c\varepsilon^2}$ a's satisfying $1 \leqslant a < N$, $(a, N) = 1$, then

$$P(B_{\nu+1} > N^{1/2}) > N^{-c\varepsilon^2} \tag{1.10}$$

where $\nu \leqslant \dfrac{\log N}{2\log\gamma(1+\varepsilon)}$.

From (1.10) follows that

$$P(B_{m'} > N^{1/2}) > N^{-c\varepsilon^2} \tag{1.11}$$

where $m' = \dfrac{\log N}{2\log\gamma(1+\varepsilon)}$.

On the other hand, we have by Lemma 1.1

$$P(B_{m'} > N^{1/2}) = P\left(B_{m'} > \gamma^{m'}\exp\left(\frac{\log(1+\varepsilon)\log N}{2\log(1+\varepsilon)\gamma}\right)\right) < e^{\frac{-c\varepsilon^2}{2}m'}$$

which contradicts (1.11). Therefore the lower estimation of (1.9) is proved. The upper estimation follows similarly.

2. Conclusion of the proof. In the previous section we saw that with at most $N^{1-c\varepsilon^2}$ exceptions in a we have (1.9). On the other hand

$$\frac{a}{N} = [0; b_1, \ldots, b_{m_0}, \ldots, b_m] = \frac{A_{m_0}\zeta_{m_0+1}+A_{m_0-1}}{B_m\zeta_{m_0+1}+B_{m_0-1}}; \tag{2.1}$$

here ζ_{m_0+1} is a rational number with the continued fraction expansion

$$\zeta_{m_0+1} = [b_{m_0+1}; b_{m_0+2}, \ldots, b_m].$$

We only have to show that with at most $N^{1-c\varepsilon^2}$ exceptions in a,

$$\frac{\log N}{2\log\gamma(1+\varepsilon)} < m - m_0 < \frac{\log N}{2\log\gamma(1-\varepsilon)}. \tag{2.2}$$

Suppose that the number of a's for which (2.2) does not hold is greater than $N^{1-c\varepsilon^2}$. To each such "exceptional number" corresponds the continued fraction

$$\frac{a'}{N} = [0; b_m, b_{m-1}, \ldots, b_{m_0}, \ldots, b_1]. \tag{2.3}$$

It follows that the continued fractions

$$[0; b_m, \ldots, b_{m_0}]$$

have for more than $N^{1-c\varepsilon}$ a's more than $\dfrac{\log N}{2\log\gamma(1-\varepsilon)}$ or less than $\dfrac{\log N}{2\log\gamma(1+\varepsilon)}$ terms, which contradicts Lemma 1.2. This finishes our proof.

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