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An explicit estimate for zeros of Dedekind zeta functions

by

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In memory of Professor P. Turán

Denote by K an algebraic number field, by n and Δ the degree and the discriminant of K respectively, and by $\zeta_K(s)$, $s = \sigma + it$ the Dedekind zeta-functions (see [4]).

Further, denote by $N(a, T, K)$ the number of zeros of $\zeta_K(s)$ for $(\frac{1}{2} \leqslant)\, a \leqslant \sigma \leqslant 1,\ 0 < t \leqslant T$.

Basing ourselves on the ideas of G. Halász and P. Turán (see [2]), we shall prove the following

THEOREM. *For all*

$$1-(3n)^{-1}e^{-10^9} \leqslant a \leqslant 1 \tag{1.1}$$

values and

$$T \geqslant e, \tag{1.2}$$

the following inequality holds:

$$N(a, T, K) < \exp\exp\left(c_1 n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}\right) T^{10^{-1} n^2 (n(1-a))^{3/2} \log^2 \frac{1}{n(1-a)}}, \tag{1.3}$$

where c_1 is a positive pure numerical constant.

For the Riemann zeta-function, the estimate of the form (1.1)–(1.3) is due to G. Halász and P. Turán (see [2]). A slightly stronger result for $\zeta(s)$ is due to H. L. Montgomery (see [5]).

The proof of Theorem (1.1)–(1.3) will rest on the following lemmas:

LEMMA 1 (Turán's second main theorem). *For arbitrary positive m, integer $n \leqslant N^*$ and complex $w_1, w_2, \ldots, w_n$ numbers, there is an integer ν_0 satisfying the condition*

$$m \leqslant \nu_0 \leqslant m + N^*,$$

so that

$$\left|\sum_{j=1}^{n} w_j^{\nu_0}\right| \geqslant \left(\frac{N^*}{8e(m+N^*)}\right)^{N^*} |w_k|^{\nu_0}, \tag{2.1}$$

where w_k stands for any of the w_j's (see [10], p. 52 and [7]).

Lemma 2. *If*

$$1-\frac{1}{n+1}\leqslant\sigma\leqslant 1,\quad t\geqslant e,$$

then

(2.2) $$|\zeta_K(\sigma+it)|\leqslant\exp(c_2 n^8|\Delta|^2)t^{6\cdot 10^2 n^2(n(1-\sigma))^{3/2}}\log^{2/3}t,$$

where c_2 is a positive pure numerical constant (see [9]).

From Lemma 2 in the way shown by Landau we immediately get

Lemma 3. *$\zeta_K(\sigma+it)$ has no zeros in the region*

(2.3) $$\sigma\geqslant 1-\frac{1}{c_3 n^{11}|\Delta|^2\log^{2/3}t(\log\log t)^{1/3}},\quad t\geqslant 4,$$

where c_3 is a pure numerical constant (compare [1]).

Lemma 4. *Denoting by $V(T)$ the number of roots of $\zeta_K(\sigma+it)$ in the region $\sqrt{\delta}\leqslant\sigma\leqslant 1$, $T\leqslant t\leqslant T+1$ where $0<\delta<1/100$, we have the estimate*

(2.4) $$V(T)\leqslant 4\delta^{-5/6}\log\big(c_4^n|\Delta|^{3/2}(|T|+3)^{3n/2+2}\big)$$

for each real T, and c_4 denotes a positive numerical constant (see [8]).

Lemma 4 easily implies

Lemma 5. *Denoting by $N(T)$ the number of roots of $\zeta_K(\sigma+it)$ in the region $\sqrt{\delta}\leqslant\sigma\leqslant 1$, $|t|\leqslant T$, $T>0$ where $0<\delta<1/100$, we have the estimate*

(2.5) $$N(T)\leqslant 8\delta^{-5/6}(T+1)\log\big(c_5^n|\Delta|^{3/2}(T+3)^{3n/2+2}\big),$$

and c_5 denotes a positive numerical constant.

Lemma 6. *Let $G(z)$ be regular for $|z|\leqslant R$, and let for $|z|<R$ the inequality*

(2.6) $$\left|\frac{G(z)}{G(0)}\right|\leqslant U.$$

hold. If $0<\vartheta<1$ and the zeros of $G(z)$ in the disc $|z|\leqslant\vartheta R$ are $z_1, z_2, \ldots$, then for all nonnegative integers μ's we have

(2.7) $$\left|\frac{1}{\mu!}\frac{G'}{G}(z)\Big|_{z=0}^{(\mu)}+(-1)^{\mu}\sum_{|z_j|\leqslant\vartheta R}\frac{1}{z_j^{\mu+1}}\right|\leqslant\frac{2(\mu+1)\log U}{(\vartheta R)^{\mu+1}}\left(1+\frac{1}{\log(1/\vartheta)}\right)$$

(see [3]).

3. Proof of the Theorem (compare [2]). From Lemma 3 it follows that $\zeta_K(\sigma+it)\neq 0$ for

$$\sigma\geqslant 1-\frac{1}{c_3 n^{11}|\Delta|^2\log^{2/3}T(\log\log T)^{1/3}},\quad \tfrac{1}{2}T\leqslant t\leqslant T,\ T\geqslant 4.$$

Let θ be such that

(3.1) $$\frac{e^{-10^9}}{n}\geqslant 1-\theta\geqslant\frac{2}{c_3 n^{11}|\Delta|^2\log^{2/3}T(\log\log T)^{1/3}},$$

for $T>c_6$ where c_6 is a sufficiently large pure numerical constant.

With such a θ we introduce the following abbreviations

(3.2) $$\sigma_0=2-\theta,$$

(3.3) $$\varepsilon=(1-\theta)^2,$$

(3.4) $$\lambda=n^2\big(n(1-\theta)\big)^{3/2}\log^{\frac{701}{400}}\frac{1}{n(1-\theta)},$$

(3.5) $$s_0'=2\sigma_0-1-2\varepsilon+it_0,\quad r_0'=2\sigma_0-1-2\varepsilon-\theta,\quad t_0\geqslant 4,$$

(3.6) $$s_0''=2\sigma_0-1-2\varepsilon,\quad r_0''=2\theta(1-\theta)\left(1-\frac{1}{\mu}\right),\quad \mu\geqslant 6.$$

Analogously to [2], p. 125, we get for the derivatives of the Riemann zeta-function $\zeta(s)$ the estimates

(3.7) $$|\zeta^{(\mu)}(s_0')|\leqslant c_7\frac{\mu!\,t_0^{100(1-\theta)^{3/2}}}{(2\sigma_0-1-2\varepsilon-\theta)^{\mu}}\log^{2/3}t_0,$$

(3.8) $$|\zeta^{(\mu)}(s_0'')|\leqslant\frac{3(\mu+1)!}{(s_0''-1)^{\mu+1}}.$$

In the proof of (3.7) we used Richert's estimate (see [6]) instead of (2.2).

It is easy to notice that the function

$$x^{3/2}\log^{\frac{601}{400}}\frac{1}{x}$$

is monotonically increasing in the interval $(0,\exp(-10^9))$.

Hence, owing to (3.1), (3.4), we get for $T>c_8$

(3.9) $$\frac{\lambda\log T}{\log^{1/4}\dfrac{1}{n(1-\theta)}}\geqslant\frac{1}{c_9 n^{13}|\Delta^3|}(\log\log T)^{\frac{401}{400}}.$$

4. Let us consider on the segment

(4.1) $$I:\ \sigma=\sigma_0,\quad \tfrac{1}{2}T\leqslant t\leqslant T,$$

with a fixed natural ν, the set $H = H(\nu)$ on which the following inequality holds:

$$\left|\frac{\zeta_K'}{\zeta_K}(s)^{(\nu)}\right| \geqslant \frac{\nu!}{(\sigma_0-1)^\nu} T^{-\lambda}. \tag{4.2}$$

Then, denoting the measure of the set H by $|H|$, we assert

LEMMA 7. *For λ determined by* (3.4) *and θ determined by* (3.1) *and*

$$T \geqslant \exp\exp(c_{10} n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}), \tag{4.3}$$

$$\frac{\lambda \log T}{\log(\frac{1}{2}+\theta)}\left(1+\frac{51}{\log^{1/4}\frac{1}{n(1-\theta)}}\right) \leqslant \nu \leqslant \frac{\lambda\log T}{\log(\frac{1}{2}+\theta)}\left(1+\frac{53}{\log^{1/4}\frac{1}{n(1-\theta)}}\right), \tag{4.4}$$

the following inequality holds:

$$|H| < c_{11} n^{123}|\Delta|^{22} T^{\frac{201}{100}\lambda}\log^9 T. \tag{4.5}$$

Proof (compare [2], pp. 126–128). Let τ_1 be the smallest t-value in H and, $\tau_1, \tau_2, \ldots, \tau_l$ being defined, let τ_{l+1} be the smallest t-value in H satisfying

$$\tau_{l+1} \geqslant \tau_l + 6 \tag{4.6}$$

(if there is any). If $\tau_1, \ldots, \tau_M$ are all points so defined, then

$$H \subset \bigcup_{l=1}^{M} [\tau_l, \tau_l+6],$$

and hence

$$|H| \leqslant 6M. \tag{4.7}$$

Denote

$$s_j = \sigma_0 + i\tau_j, \quad j = 1, 2, \ldots, M, \tag{4.8}$$

and

$$\frac{\zeta_K'}{\zeta_K}(s_j)^{(\nu)} = F(s_j, K). \tag{4.9}$$

Hence, owing to (4.2), we have

$$\frac{M\nu!}{(\sigma_0-1)^\nu} T^{-\lambda} \leqslant \sum_{j=1}^{M} |F(s_j, K)|. \tag{4.10}$$

Putting

$$\eta_j = e^{-i\arg F(s_j, K)}, \tag{4.11}$$

we have

$$|\eta_j| = 1, \quad |F(s_j, K)| = F(s_j, K)\eta_j.$$

But

$$F(s_j, K) = \left(-\sum_{m=2}^{\infty}\frac{G(m)}{m^{s_j}}\right)^{(\nu)} = (-1)^{\nu+1}\sum_{m=2}^{\infty}\frac{G(m)\log^\nu m}{m^{s_j}} \tag{4.12}$$

where

$$G(m) = \sum_{(N\mathfrak{p})^l = m} \log N\mathfrak{p} \leqslant \frac{n}{\log 2}\log^2 m$$

(see [8], p. 183).

Therefore, owing to (4.10), we get

$$\frac{M\nu!}{(\sigma_0-1)^\nu} T^{-\lambda} \leqslant \frac{n}{\log 2}\sum_{m=2}^{\infty}\frac{\log^2 m}{m^{1/2+\varepsilon}}\,\frac{\log^\nu m}{m^{\sigma_0-1/2-\varepsilon}}\left|\sum_{j=1}^{M}\frac{\eta_j}{m^{i\tau_j}}\right|. \tag{4.13}$$

Applying Cauchy's inequality, we obtain

$$M^2\nu!^2\frac{T^{-2\nu}}{(\sigma_0-1)^{2\nu}} \leqslant \frac{c_{12}n^2}{\varepsilon^5}\sum_{1\leqslant j_1, j_2\leqslant M}\left|\zeta^{(2\nu)}\big(2\sigma_0-1-2\varepsilon+i(\tau_{j_1}-\tau_{j_2})\big)\right|. \tag{4.14}$$

Using estimates (3.7) and (3.8) with $j_1 \neq j_2$ and $j_1 = j_2$ respectively, we have

$$\frac{M\nu!^2}{(\sigma_0-1)^{2\nu}} T^{-2\lambda} \leqslant \frac{c_{13}n^2}{\varepsilon^5}\,\frac{(2\nu+1)!}{(2\sigma_0-2-2\varepsilon)^{2\nu+1}} + {} + \frac{c_{14}Mn^2}{\varepsilon^5}\,\frac{(2\nu)!\,T^{100(1-\theta)^{3/2}}}{(2\sigma_0-1-2\varepsilon-\theta)^{2\nu}}\log^{2/3}T. \tag{4.15}$$

Applying (4.4) and then (3.9), we get

$$(\tfrac{1}{2}+\theta)^{2\nu} \geqslant T^{2\lambda\left(1+\frac{50}{\log^{1/4}\frac{1}{n(1-\theta)}}\right)}\frac{1}{e^{c_9 n^{13}|\Delta|^3}}(\log\log T)^{\frac{401}{400}}. \tag{4.16}$$

Since

$$(2\sigma_0-1-2\varepsilon-\theta)^{2\nu} = 2^{2\nu}(\sigma_0-1)^{2\nu}(\theta+\tfrac{1}{2})^{2\nu},$$

owing to (3.3), (3.1) and by the use of the Stirling formula, we have by (4.3) the estimate

$$\frac{c_{14}Mn^2}{\varepsilon^5}\,\frac{(2\nu)!\,T^{100(1-\theta)^{3/2}}\log^{2/3}T}{(2\sigma_0-1-2\varepsilon-\theta)^{2\nu}} < \tfrac{1}{2}(\nu!)^2 M\,\frac{T^{-2\lambda}}{(\sigma_0-1)^{2\nu}}. \tag{4.17}$$

Hence, owing to (4.15), (4.3), we get

$$M \leqslant \frac{2c_{12}n^2(2\nu+1)!(\sigma_0-1)^{2\nu}T^{2\lambda}}{\varepsilon^5(\sigma_0-1-\varepsilon)^{2\nu+1}\nu!^2 2^{2\nu+1}}. \tag{4.18}$$

Using (4.4) and (3.9), we then have

$$\theta^{-2\nu} = \exp\left(2\nu\log\left(1+\frac{1-\theta}{\theta}\right)\right) \leqslant T^{\frac{1}{100}\lambda}. \tag{4.19}$$

Therefore, by (4.18), (4.3), (4.4), (3.1)–(3.3), we get

$$M \leqslant c_{15}n^{123}|\Delta|^{22}T^{\frac{201}{100}\lambda}\log^9 T, \tag{4.20}$$

and thus (4.5) follows.

In the following we shall only use a corollary to Lemma 7.

Let us consider on the segment I defined by (4.1) the set H^* of s-values for which the inequality

$$\left|\frac{\zeta_K'}{\zeta_K}(s)^{(\nu)}\right| < \frac{\nu!T^{-\lambda}}{(\sigma_0-1)^\nu} \tag{4.21}$$

holds for all ν-values permitted by (4.4).

Its complementary set $\overline{H^*}$ with respect to $[\frac{1}{2}T, T]$ is certainly covered by the union of the above $H = H(\nu)$ sets

$$\overline{H^*} \subset \bigcup H(\nu). \tag{4.22}$$

Hence, owing to (4.4), we have from Lemma 7 the estimate

$$|\overline{H^*}| < c_{11}n^{125}|\Delta|^{22}T^{\frac{201}{100}\lambda}\log^{10}T, \tag{4.23}$$

and we can formulate the following

COROLLARY. *If we drop from the segment I a suitable set $\overline{H^*}$ (consisting of finitely many closed intervals) of measure*

$$|\overline{H^*}| \leqslant c_{11}n^{125}|\Delta|^{22}T^{\frac{201}{100}\lambda}\log^{10}T \tag{4.24}$$

at the remaining points of I, inequality (4.21) *holds simultaneously for all ν-values given by* (4.4) *with* (4.3).

5. Let us consider the horizontal strips l_j defined by

$$\frac{T}{2}+\frac{j}{\log^3 T} \leqslant t \leqslant \frac{T}{2}+\frac{j+1}{\log^3 T}; \quad j = 0, 1, \ldots, \left[\frac{T}{2}\log^3 T\right]. \tag{5.1}$$

We call a strip l_j a "good" one, if its intersection with I contains at least one point of the set H^*; otherwise we call it a "bad" one. The number of "bad" strips is, by (4.24),

$$\leqslant \frac{|\overline{H^*}|}{1/\log^3 T} < T^{\frac{201}{100}}(\log T)^{14}. \tag{5.2}$$

Owing to Lemma 4, each "bad" strip contains at most

$$\leqslant c_{16}\log\left(c_4^n|\Delta|^{3/2}(|T|+3)^{\frac{3}{2}n+2}\right)$$

zeros of $\zeta_K(s)$. Hence the number of roots in all "bad" strips of the rectangle

$$\frac{e^{-10^9}}{n} \geqslant 1-\theta \geqslant \frac{2}{c_3 n^{11}|\Delta|^2\log^{2/3}T(\log\log T)^{1/3}}, \quad \tfrac{1}{2}T \leqslant t \leqslant T, \tag{5.3}$$

$$T > \exp\exp(c_{17}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2})$$

cannot exceed

$$T^{\frac{201}{100}\lambda}\log^{16}T. \tag{5.4}$$

Let us fix any "good" strip and let

$$z^* = \sigma_0 + it^*$$

be a point of H^* in such a strip.

Hence for all ν-values from the interval (4.4) we have

$$\left|\frac{\zeta_K'}{\zeta_K}(z^*)^{(\nu)}\right| < \nu!\frac{T^{-\lambda}}{(\sigma_0-1)^\nu}. \tag{5.5}$$

Applying Lemma 6 with

$$G(z) = \zeta_K(z+z^*), \quad R = e^2(\sigma_0-1), \quad \vartheta = 1/e, \quad \mu = \nu,$$

we obtain

$$\left|\frac{1}{\nu!}\frac{\zeta_K'}{\zeta_K}(z^*) - (-1)^{\nu+1}\sum_{|z^*-\varrho|\leqslant e(1-\theta)}\frac{1}{(z^*-\varrho)^{\nu+1}}\right| \leqslant \frac{4(\nu+1)\log U}{(e(1-\theta))^{\nu+1}}, \tag{5.6}$$

where

$$U = \sup_{|z|\leqslant e^2(1-\theta)}\left|\frac{\zeta_K(z^*+z)}{\zeta_K(z^*)}\right|. \tag{5.7}$$

Since for $\sigma > 1$

$$\frac{1}{|\zeta_K(\sigma+it)|} \leqslant \left(\frac{\sigma}{\sigma-1}\right)^n, \tag{5.8}$$

we have by (3.1), (3.2)

$$\frac{1}{|\zeta_K(z^*)|} \leqslant c_{18}^n n^{11n}|\Delta|^{2n}\log^{\frac{2}{3}n}T(\log\log T)^{n/3}. \tag{5.9}$$

Using Lemma 2, we get in the circle

$$|s-z^*| \leqslant e^2(\sigma_0-1)$$

the estimate

$$|\zeta_K(s)| \leqslant e^{c_2 n^8 |\Delta|^2}(T+1)^{6\cdot 10^2 n^2 (n(e^2-1)(1-\theta))^{3/2}} \log^{2/3} T. \tag{5.10}$$

Hence we can put

$$U = T^{6\cdot 10^2 n^2 (n(e^2-1)(1-\theta))^{3/2}} \log^{\frac{2}{3}n+1} T \tag{5.11}$$

with T determined by (4.3).

Owing to (5.5) we can write formula (5.6) in the form

$$\Big|\sum_{|z^*-\varrho|\leqslant e(1-\theta)} \Big(\frac{\sigma_0-1}{z^*-\varrho}\Big)^{\nu+1}\Big| \leqslant \frac{4(\nu+1)\log U}{e^{\nu+1}} + (\sigma_0-1)T^{-\lambda}. \tag{5.12}$$

Using (4.4) and (3.9), we get

$$e^{\nu} \geqslant T^{\lambda} \exp\Big(\frac{1}{c_{18} n^{13}|\Delta|^3}(\log\log T)^{\frac{401}{400}}\Big), \tag{5.13}$$

$$\frac{4(\nu+1)}{e} \leqslant \log T (\log\log T)^2 \tag{5.14}$$

and (5.11) implies

$$\log U \leqslant \log^2 T. \tag{5.15}$$

Therefore for T determined by (4.3) and for all ν-values from interval (4.4) we have

$$\Big|\sum_{|z^*-\varrho|\leqslant e(1-\theta)} \Big(\frac{\sigma_0-1}{z^*-\varrho}\Big)^{\nu+1}\Big| \leqslant \frac{1}{2} T^{-\lambda}. \tag{5.16}$$

6. In order to estimate the sum of (5.16) from below we shall apply Lemma 1.

We choose

$$m = \frac{\lambda \log T}{\log(\frac{1}{2}+\theta)} \Bigg(1 + \frac{51}{\log^{1/4}\dfrac{1}{n(1-\theta)}}\Bigg) + 1. \tag{6.1}$$

To estimate the number of terms in (5.16) we apply Jensen's inequality, which gives for the number of zeros of the function $f(s)$ in the circle $|s-s_0| \leqslant \vartheta R$ the estimate

$$\frac{1}{\log(1/\vartheta)} \log \max_{|s-s_0|\leqslant R} \Big|\frac{f(s)}{f(s_0)}\Big|. \tag{6.2}$$

Owing to (5.11) and by the use of inequality (3.9), we get for $T > \exp\exp(c_{19} n^{56\cdot 10^2} |\Delta|^{16\cdot 10^2})$

$$\log U < \frac{\lambda \log T}{\log(\frac{1}{2}+\theta)\log^{1/4}\dfrac{1}{n(1-\theta)}}. \tag{6.3}$$

Hence, owing to (6.2), we can choose

$$N^* = \frac{\lambda \log T}{\log(\frac{1}{2}+\theta)\log^{1/4}\dfrac{1}{n(1-\theta)}}, \tag{6.4}$$

with the values for T determined above.

It is easy to verify that for θ from interval (3.1) and owing to (3.9) we have

$$\frac{N^*}{8e(m+N^*)} > \Big(\log\frac{1}{n(1-\theta)}\Big)^{-1/2}. \tag{6.5}$$

Hence

$$\Big(\frac{N^*}{8e(m+N^*)}\Big)^{N^*} > \exp\left(\frac{-\lambda\log T \log\log\dfrac{1}{n(1-\theta)}}{2\log(\frac{1}{2}+\theta)\log^{1/4}\dfrac{1}{n(1-\theta)}}\right). \tag{6.6}$$

From Lemma 1 it follows that there exists such a ν_0 in interval (4.4) that then

$$\Big|\sum_{|\varrho-z^*|\leqslant e(\sigma_0-1)} \Big(\frac{\sigma_0-1}{z^*-\varrho}\Big)^{\nu_0+1}\Big| \geqslant \Big(\frac{\sigma_0-\sigma_\varrho^*}{|z^*-\varrho^*|}\Big)^{\nu_0+1} \times$$
$$\times \exp\left(-\frac{\lambda\log T\log\log\dfrac{1}{n(1-\theta)}}{2\log(\frac{1}{2}+\theta)\log^{1/4}\dfrac{1}{n(1-\theta)}} - (\nu_0+1)\log\frac{\sigma_0-\sigma_\varrho^*}{\sigma_0-1}\right), \tag{6.7}$$

where for w_k we choose the term belonging to $\varrho^* = \sigma_\varrho^* + it_\varrho^*$ with maximal σ_ϱ^*.

Since

$$|t^*-t_\varrho^*| \leqslant \frac{1}{\log^3 T}, \qquad \sigma_0-\sigma_\varrho^* > \sigma_0-1 \geqslant \frac{1}{\log T}, \tag{6.8}$$

we have

$$\Big|\frac{\sigma_0-\sigma_\varrho^*}{z^*-\varrho^*}\Big|^{\nu_0+1} \geqslant \Big(1+\frac{1}{\log^2 T}\Big)^{-\nu_0-1}. \tag{6.9}$$

Owing to (4.3), (4.4), we have

$$(\nu_0+1)\log\left(1+\frac{1}{\log^2 T}\right)<\log 2. \tag{6.10}$$

Hence

$$\left|\frac{\sigma_0-\sigma_\varrho^*}{z^*-\varrho^*}\right|^{\nu_0+1}>\frac{1}{2}. \tag{6.11}$$

From (5.16), (6.7) and (6.11) we get

$$-\lambda\log T\geqslant-\frac{\lambda\log T\log\log\dfrac{1}{n(1-\theta)}}{2\log(\frac{1}{2}+\theta)\log^{1/4}\dfrac{1}{n(1-\theta)}}-(\nu_0+1)\log\frac{\sigma_0-\sigma_\varrho^*}{\sigma_0-1}. \tag{6.12}$$

Since

$$\nu_0+1\leqslant\frac{\lambda\log T}{\log(\frac{1}{2}+\theta)}\left(1+\frac{53}{\log^{1/4}\dfrac{1}{n(1-\theta)}}\right)+1<\frac{\lambda\log T}{\log(\frac{1}{2}+\theta)}\left(1+\frac{54}{\log^{1/4}\dfrac{1}{n(1-\theta)}}\right),$$

we have from (6.12) the inequality

$$\left(1+\frac{54}{\log^{1/4}\dfrac{1}{n(1-\theta)}}\right)\log\frac{\sigma_0-\sigma_\varrho^*}{\sigma_0-1}\geqslant\log(\tfrac{1}{2}+\theta)-\frac{\log\log\dfrac{1}{n(1-\theta)}}{2\log^{1/4}\dfrac{1}{n(1-\theta)}}. \tag{6.13}$$

Hence, owing to (3.1), we get

$$\log\frac{\sigma_0-\sigma_\varrho^*}{\sigma_0-1}>\log\frac{1335}{1000} \tag{6.14}$$

and

$$1-\sigma_\varrho^*>\tfrac{1}{3}(1-\theta). \tag{6.15}$$

7. We have succeeded in proving that the $\varrho=\sigma_\varrho+it_\varrho$ zeros of $\zeta_K(s)$ in "good" strips satisfy the inequality

$$\sigma_\varrho<1-\tfrac{1}{3}(1-\theta). \tag{7.1}$$

Putting

$$1-\tfrac{1}{3}(1-\theta)=\alpha, \tag{7.2}$$

we get by (5.3), (5.4), in the region

$$\frac{2}{3c_3n^{11}|\Delta|^2\log^{2/3}T(\log\log T)^{1/3}}\leqslant 1-\alpha\leqslant\frac{e^{-10^9}}{3n}, \tag{7.3}$$

$$T>\exp\exp(c_{20}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2})$$

the estimate

$$N(\alpha,T,K)-N(\alpha,T/2,K)<T^{\frac{201}{100}\lambda}\log^{16}T. \tag{7.4}$$

From Lemma 3 it follows that (7.4) holds also in the interval (1.1). Then, replacing T by

$$\frac{T}{2},\frac{T}{2^2},\ldots,\frac{T}{2^i}\geqslant\exp\exp(c_{20}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2})\geqslant\frac{T}{2^{i+1}},$$

we get after summation

$$N(\alpha,T,K)-N\left(\alpha,\frac{T}{2^{i+1}},K\right)\leqslant(i+2)T^{\frac{201}{100}\lambda}\log^{16}T.$$

Hence, in view of Lemma 5, we get in (1.1) with

$$T>\exp\exp(c_{21}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}) \tag{7.5}$$

the estimate

$$N(\alpha,T,K)<3T^{\frac{201}{100}\lambda}\log^{17}T+\exp(c_{22}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}). \tag{7.6}$$

Applying inequality (3.9), we get from (7.6), (7.5)

$$N(\alpha,T,K)<3T^{\left(\frac{201}{100}+\frac{17}{\log^{1/4}\frac{1}{n(1-\theta)}}\right)\lambda}+\exp\exp(c_{22}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}). \tag{7.7}$$

Using Lemma 5 again, we get for

$$e\leqslant T\leqslant\exp\exp(c_{22}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2})$$

the estimate

$$N(\alpha,T,K)\leqslant\exp\exp(c_{22}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2}). \tag{7.8}$$

Owing to (3.4), (7.2), we get from (7.7), (7.8) the inequality

$$N(\alpha,T,K)\leqslant\exp\exp(c_{23}n^{56\cdot 10^2}|\Delta|^{16\cdot 10^2})T^{An^2(n(1-\alpha))^{3/2}\log^2\frac{1}{n(1-\alpha)}}, \tag{7.9}$$

where

$$A = \frac{\frac{201}{100} 3^{3/2} \log^{\frac{701}{400}} \frac{1}{3n(1-a)}}{\log^2 \frac{1}{n(1-a)}} + \frac{18 \cdot 3^{2/3} \log^{\frac{601}{400}} \frac{1}{3(n(1-a))}}{\log^2 \frac{1}{n(1-a)}}$$

valid in the interval (1.1) and for T satisfying (1.2).

It is easy to realize that for all a-values from (1.1) we have $A < 1/10$ and the proof of the theorem follows.

References

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On the length of continued fractions representing a rational number with given denominator

by

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To the memory of my teacher and friend P. Turán

Let N be a given natural number. Denote by $l(a) = l(a, N)$ the length of the finite continued fraction

(1) $$\frac{a}{N} = [0; b_1, \ldots, b_m], \quad (a, N) = 1,\ b_m \geq 2.$$

A few years ago Heilbronn [2] showed that

(2) $$\sum_{\substack{1 \leq a < N \\ (a,N)=1}} l(a) = \frac{12 \log 2}{\pi^2} \varphi(N) \log N + O\big(N\sigma_{-1}^3(N)\big),$$

where $\sigma_{-1}(N)$ denotes the sum $\sum_{d|N} 1/d$.

A classical result of A. Khintchine [3] and P. Lévy [4] states that, putting

(3) $$t = [0; b_1, b_2, \ldots] \quad (t \text{ real},\ t \in (0, 1))$$

for almost all t we have

(4) $$\sqrt[k]{B_k} \to e^{\frac{\pi^2}{12 \log 2}} \quad (k \to \infty).$$

B_k being the denominator of the kth convergent of the continued fraction (3). Because of (4) Heilbronn's result (2) is not very surprising. The subtleness of his result is that we can make a statement for any given N, not only for "most N's" in some sense. In 1970 J. Dixon [1] proved a theorem about the length of the continued fractions of "most" rational numbers a/ν where $a < \nu$, $(a, \nu) = 1$ and $\nu \leq N$; the exceptional set was not for a given ν in the a's, but in pairs a, ν. One can ask whether a statement about "most a's" for any given N is true. In the present paper I