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On sets characterizing additive arithmetical functions, II

by

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To the memory of Professor Paul Turán

As in [1], f denotes an additive arithmetical function, A and B are subsequences of the natural numbers, consisting of the elements $a_1 < a_2 < a_3 < \dots$ and $b_1 < b_2 < b_3 < \dots$, respectively. A is called a *U-set*, if $f(a_k) = 0$, $k = 1, 2, \dots$, imply $f = 0$.

In [1] we proved the following assertions:

I. Let A be a U-set. Then

$$\liminf \frac{a_{k+1}}{a_k^2} \leqslant 1,$$

moreover, if we put $\frac{a_{k+1}}{a_k^2} = e_k$, then

$$\liminf(e_1 \dots e_k) = 0 \quad \text{(Theorem 2/I)}. \tag{1}$$

In fact, if A does not satisfy (1), then we can construct an additive f, which is "arbitrarily strongly" unbounded, though $f(a_k) = 0$ for all k (Theorem 4).

II. Let a_k be an arbitrary sequence of positive numbers satisfying

$$\liminf(\alpha_1 \dots \alpha_k) = 0 \quad \text{and} \quad \alpha_k \geqslant 2^{-k}.$$

Then there exists an A, for which

$$\frac{a_{k+1}}{a_k^2} \geqslant \alpha_k$$

holds, and A is a U-set, moreover, if

$$\sum_{k=1}^{\infty} f(a_k) \text{ is convergent,} \tag{2}$$

then $f = 0$ (Theorem 2/II).

A has also the following characterizing property:
If

(2a) the $\sum_{r=1}^{k} f(a_r)$ sums are bounded,

then f is bounded (Remark 4, after the proof of Theorem 2/II).

Now we examine the problem of replacing the characterizing conditions (2) and (2a) resp. with even weaker ones, namely, with the following:

(3) $f(a_k)$ is convergent,

(3a) $f(a_k)$ is bounded,

(4) $f(a_{k+1})-f(a_k)$ is convergent,

(4a) $f(a_{k+1})-f(a_k)$ is bounded.

We have the following results:

THEOREM 1. *Let $\alpha < 1$ be an arbitrary real number. We can construct an A, which satisfies*

(5)
$$\frac{a_{k+1}}{a_k^2} > \alpha$$

and (3) *implies $f = 0$.*

THEOREM 2. *Let α_k be an arbitrary sequence of real numbers tending to* 0. *We can construct an A, which satisfies*

(6)
$$\frac{a_{k+1}}{a_k^2} > \alpha_k$$

and (3a) *implies that f is bounded.*

Moreover, we can guarantee even

$$\sup_n |f(n)| = \sup_k |f(a_k)|.$$

THEOREM 3. *Let $\varepsilon > 0$ be arbitrary. We can construct an A, which satisfies*

(7)
$$a_{k+1} > a_k^{2-\varepsilon},$$

and (4) *and* (4a), *resp., imply $f = 0$ and the boundedness of f, resp.*

Remark. The proofs have some common features with the proof of Theorem 2/II in [1] (but involve several new ideas too).

Proof of Theorem 1. Let $t_1, t_2, \ldots$ be a sequence (of natural numers), which contains each natural number infinitely often.

The required set A will be the union of successive "blocks".

The ith block has the following elements:

$$u_{i1}, \ldots, u_{i,r-1}, t_i\cdot u_{ir}, u_{i,r+1}, \ldots, t_i\cdot u_{i,2r}, \ldots,$$
$$u_{i,N_i\cdot r-1}, t_i\cdot u_{i,N_i\cdot r}, u_{i1}\cdot u_{i2}\cdot \ldots \cdot u_{i,N_i\cdot r},$$

i.e. every rth u_{ij} is multiplied by t_i, and on the end we take the product of all the u_{ij}. The block has altogether $N_i\cdot r+1$ elements.

Now we put several prescriptions for the u_{ij}, r, and N_i.

The u_{ij} are primes and $(u_{ij}, t_i) = 1$.

We take a $\vartheta > 1$, for which $\gamma = \alpha\cdot\vartheta^2 < 1$ holds (ϑ does not depend on i).

r should be chosen in the following way:

(8)
$$\gamma^{-r} > t_i,$$

further, if we consider all the t_i having the same value, the corresponding numbers r take exactly two different values, alternately. (Thus r depends on t_i, and somewhat on i too.)

Let us suppose that we have already constructed the $(i-1)$-st block and denote its last element by a_s.

Having (5) in mind, we intend to construct the ith block in the following way:

$$a_{s+1} = u_{i1} = u > \alpha\cdot a_s^2,$$
$$a_{s+2} = u_{i2} \sim \alpha\cdot(a_{s+1})^2 = \alpha\cdot u^2,$$
$$a_{s+3} = u_{i3} \sim \alpha\cdot(a_{s+2})^2 \sim \alpha^3\cdot u^4,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{s+r} = t_i\cdot u_{ir} \sim \alpha\cdot(a_{s+r-1})^2 \sim \alpha^{2^{r-1}-1}\cdot u^{2^{r-1}}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{s+j} \sim \alpha^{2^{j-1}-1}\cdot u^{2^{j-1}} = \frac{(\alpha\cdot u)^{2^{j-1}}}{\alpha},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{s+N_i\cdot r+1} = u_{ij}\cdot \ldots \cdot u_{i,N_i\cdot r}.$$

Precisely, we act as follows: Let $u_{i1} = u$ be a "large" prime:

(9)
$$u > \alpha\cdot a_s^2,$$
$$u > t_i^2,$$

further, if $m > u/t_i$, then there is at least one prime between m and $m\cdot\vartheta$.

Put

$$\frac{(\gamma\cdot u)^{2^{j-1}}}{\gamma} < a_{s+j} < \frac{(\gamma\cdot u)^{2^{j-1}}}{\gamma}\cdot\vartheta,$$

$j = 2, 3, \ldots, N_i\cdot r$.

According to the choice of u we are able to select the a_{s+j} so that the u_{ij} should be primes. Also $u_{ij} > t_i$, and ths $(u_{ij}, t_i) = 1$ holds.

Now, for $j = 1, 2, \ldots, N_i \cdot r - 1$, we have

$$\frac{a_{s+j+1}}{(a_{s+j})^2} > \frac{(\gamma \cdot u)^{2^j}/\gamma}{\frac{(\gamma \cdot u)^{2^j}}{\gamma^2} \cdot \vartheta^2} = \frac{\gamma}{\vartheta^2} = a.$$

Further

$$\frac{a_{s+N_i \cdot r+1}}{(a_{s+N_i \cdot r})^2} = \frac{a_{s+1} \cdot \ldots \cdot a_{s+N_i \cdot r}}{t_i^{N_i} \cdot (a_{s+N_i \cdot r})^2}$$

$$> \frac{(\gamma \cdot u)^{1+2+\ldots+2^{N_i \cdot r-1}}}{\gamma^{N_i \cdot r} \cdot t_i^{N_i} \cdot (\gamma \cdot u)^{2^{N_i \cdot r}} \cdot \gamma^{-2} \cdot \vartheta^2} = \frac{a}{(\gamma^r \cdot t_i)^{N_i} \cdot u}.$$

By (8) $\gamma^r \cdot t_i < 1$. Hence, if N_i is large enough, then

$$\frac{a_{s+N_i \cdot r+1}}{(a_{s+N_i \cdot r})^2} > a.$$

Herewith we proved (5) for all k.

For later purposes we choose N_i so that

(10) $$\lim_i N_i = \infty$$

should hold.

Let now f be additive satisfying (3), i.e. $f(a_k) \to c$, and take an arbitrary natural number, say h.

We consider those blocks, where $t_i = h$, and denote by r_1 and r_2 the two values of r corresponding to h.

In any of these blocks we have by the additivity

(11) $$f(a_{s+N_i \cdot r+1}) = f(a_{s+1}) + f(a_{s+2}) + \ldots + f(a_{s+N_i \cdot r}) - N_i f(h).$$

Let $\varepsilon > 0$ be arbitrary. We can find an M such that, for $m > M$, we have $|f(a_m) - c| < \varepsilon$.

We consider only the blocks with $s > M$. Then by (11) we obtain

$$|r \cdot N_i \cdot c - N_i \cdot f(h)| < |c| + (N_i \cdot r + 1) \cdot \varepsilon,$$

i.e.

$$\left| c - \frac{f(h)}{r} \right| < \frac{|c|}{r \cdot N_i} + \varepsilon \cdot \left(1 + \frac{1}{r \cdot N_i}\right)$$

and hence, using (10),

(12) $$\left| c - \frac{f(h)}{r} \right| < 2\varepsilon, \quad \text{if } i \text{ is large enough.}$$

We consider first only those i, for which $r = r_1$. By (12) we obtain

$$c = \frac{f(h)}{r_1}.$$

Repeating the argument with r_2, we infer $c = 0, f(h) = 0$. Thus we proved $f = 0$.

Proof of Theorem 2. Now the ith block will be the following:

$$v_{i1}, \ldots, v_{iK_i}, \; t_i \cdot u_{i1}, \; t_i \cdot u_{i2}, \ldots, t_i \cdot u_{iN_i}, \; u_{i_1} \cdot \ldots \cdot u_{iN_i}.$$

The v_{ij} have only the role of "stuffing" elements, till a_k becomes "small enough".

We take an m_i such that for $m > m_i$ $a_m < 1/2t_i$.

After the $(i-1)$-st block we insert arbitrary v_{ij} satisfying the "prescribed rate of growth", and we stop at an $a_s = v_{iK_i}$, where $s > m_i$.

Now we choose the $a_{s+j} = t_i \cdot u_{ij}$ elements as in the previous proof ($r = 1$, and we put $1/2t_i$ instead of a), and obtain the validity of (6) for all k by the same arguments.

Let now f be additive, and $\sup_k |f(a_k)| = L$. Then

$$f(u_{i1} \cdot \ldots \cdot u_{iN_i}) = f(t_i \cdot u_{i1}) + \ldots + f(t_i \cdot u_{iN_i}) - N_i \cdot f(t_i)$$

and thus

$$N_i \cdot |f(t_i)| \leqslant (N_i + 1) \cdot L,$$

i.e.

(13) $$|f(t_i)| \leqslant \frac{N_i + 1}{N_i} \cdot L.$$

Let h be an arbitrary natural number, and consider those i, for which $t_i = h$. $N_i \to \infty$ for these i too, hence

$$\frac{N_i + 1}{N_i} \to 1, \text{ and so by (13) } |f(h)| \leqslant L. \;\blacksquare$$

Proof of Theorem 3. Let N be so large that $\varepsilon > 1/2^{N-1}$ should hold.

Let $t_1, t_2, \ldots$ be the usual sequence, and we form the ith block in the following way:

$$u_{i1}, \ldots, u_{iN}, \; t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN}, \; v_{i1}, \ldots, v_{iN},$$

$$t_i \cdot v_{i1} \cdot \ldots \cdot v_{iN}, \; w_{i1}, \ldots, w_{i,N+1}, \; t_i \cdot w_{i1} \cdot \ldots \cdot w_{i,N+1}.$$

Here N is fixed (does not depend on i), u_{ij}, v_{ij} and w_{ij} are primes, not dividing t_i.

Suppose that the $(i-1)$-st block has already been constructed, and its last element is a_s $(s = (i-1)\cdot(3N+4))$.

Put

$$u_{i1} = u > a_s^2,$$
$$u_{i1}^2 < u_{i2} < 2\cdot u_{i1}^2,$$
$$u_{i2}^2 < u_{i3} < 2\cdot u_{i2}^2,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$v_{i1} > (t_i\cdot u_{i1}\cdot\ldots\cdot u_{iN})^2,$$
$$v_{i1}^2 < v_{i2} < 2\cdot v_{i1}^2,$$
$$\ldots\ldots\ldots\ldots\ldots$$

To prove (7) we have to verify only

$$t_i\cdot u_{i1}\cdot\ldots\cdot u_{iN} > u_{iN}^{2-\varepsilon}$$

(and the two similar assertions with the v_{ij} and the w_{ij}). We have obviously

$$u^{2^{j-1}} \leqslant u_{ij} \leqslant 2^{2^{j-1}-1}\cdot u^{2^{j-1}} < (2u)^{2^{j-1}}$$

and hence

$$t_i\cdot u_{i1}\cdot\ldots\cdot u_{iN} \geqslant u^{1+2+4+2^{N-1}} = u^{2^N-1}$$
$$= [(2u)^{2^{N-1}}]^{\frac{2^N-1}{2^{N-1}}\cdot\frac{\log u}{\log 2u}} > u_{iN}^{\frac{2^N-1}{2^{N-1}}\cdot\frac{\log u}{\log 2u}} > u_{iN}^{2-\varepsilon},$$

if u is large enough.

Thus we proved (7) for all k.

Let now f be additive, $f(a_{k+1})-f(a_k)\to c$. Using the additivity we obtain:

$$f(t_i\cdot v_{i1}\cdot\ldots\cdot v_{iN})-f(t_i\cdot u_{i1}\cdot\ldots\cdot u_{iN})$$
$$= f(v_{i1})-f(u_{i1}) + \ldots + f(v_{iN})-f(u_{iN}).$$

If $i\to\infty$, then the left-hand side tends to $(N+1)\cdot c$, while the right-hand side tends to $N\cdot(N+1)\cdot c$. Hence $c = 0$.

Again, by the additivity

$$f(t_i\cdot w_{i1}\cdot\ldots\cdot w_{i,N+1})-f(t_i\cdot v_{i1}\cdot\ldots\cdot v_{iN})$$
$$= f(w_{i1})-f(v_{i1}) + \ldots + f(w_{iN})-f(v_{iN})+f(w_{i,N+1}).$$

Here the left-hand side tends to 0, and so does the right-hand side too with the exception of the last term, and thus $f(w_{i,N+1})\to 0$ necessarily (when $i\to\infty$).

But then e.g. for any fixed j $\lim_{i\to\infty} f(w_{ij}) = 0$, and also

$$\lim_{i\to\infty} f(t_i\cdot w_{i1}\cdot\ldots\cdot w_{i,N+1}) = 0.$$

By the additivity

$$f(t_i) = f(t_i\cdot w_{i1}\cdot\ldots\cdot w_{i,N+1})-f(w_{i1})-\ldots-f(w_{i,N+1}),$$

and thus $\lim_{i\to\infty} f(t_i) = 0$. But in the sequence $t_1, t_2, \ldots$ every natural number occurs infinitely often, i.e. only $f = 0$ is possible.

Finally, assuming (4a) we obtain the boundedness of f by similar arguments. This completes the proof.

Remarks. 1. We mention that our theorems can be generalized analogously to Theorem 5 in [1].

2. In [1] and in this paper we have constructed several sets, for which (2), (3) or (4) implied $f = 0$. Nearly all of these sets had the property that (2a), (3a) or (4a) resp., implied the boundedness of f (see Theorem 3 in this paper, and Remark 2 after the proof of Theorem 1, Remark 4 after the proof of Theorem 2/II in [1]). There was just one exception: we had no evidence, whether the set A constructed in the proof of Theorem 1 in this paper possessed this property too or not, and so we had to construct a different set for the corresponding characterization of the bounded functions.

Thus it is natural to ask the following question:

What is the relation between the conditions (2), (3) and (4) characterizing the $f = 0$ function and the corresponding conditions (2a), (3a) and (4a) characterizing the set of the bounded functions?

We give the answer in [2]: we obtain that, roughly speaking, there is no connection between the two types of characterization.

We mention that by a slight modification of the set constructed in the proof of Theorem 1, we can also obtain a set A, for which even (4) implies $f = 0$, but we can find an additive f satisfying (3a) and in the meantime being "very strongly" unbounded.

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For other references see [1].

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