An exponential polynomial formed with the Legendre symbol  

by

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Dedicated to the memory of Paul Turán

We investigate the behavior of the sum

\[ S(\alpha) = S_p(\alpha) = \sum_{n=1}^{p-1} \left( \frac{n}{\alpha} \right) e(n\alpha), \]

which is a particular example of an exponential polynomial of the sort

\[ S(\alpha, \epsilon) = \sum_{n=1}^{N} \epsilon_n(n\alpha) \]

with \( \epsilon_n = \pm 1 \). Among such polynomials, \( S(\alpha) \) has the unusual property that

\[ |S(\alpha/p)| = \sqrt{p} \quad (1 \leq \alpha < p - 1), \]

and \( S(0) = 0 \). It is difficult to exhibit a choice of \( \epsilon \) for which \( |S(\alpha, \epsilon)| \leq C \sqrt{N} \) for all \( \alpha \). The example known was found by H. S. Shapiro [5] and W. Rudin [4]; a nice account of this and related problems was given by Littlewood [3], pp. 28–32. In view of (3), we ask whether the sum \( S(\alpha) \) also satisfies the bound \( S(\alpha) \leq \sqrt{p} \). Indeed, from Bernstein’s inequality it follows that if \( K > \frac{\pi}{2} N \) then

\[ \max_{\alpha} |S(\alpha, \epsilon)| \leq \left( 1 - \frac{\pi N - 1}{2K} \right) \max_{\alpha} |S(\alpha/K, \epsilon)|; \]

thus the points \( \alpha/p \) are nearly dense enough for us to deduce from (3) that \( S(\alpha) \leq p^{12} \). Hence it is surprising that this estimate is false for all large primes \( p \).

**Theorem.** For \( p > 2 \), \( S(\alpha) \leq p^{12} \log p \), and for all large \( p \),

\[ \max_{\alpha} |S(\alpha)| > \frac{2}{\pi} p^{12} \log p. \]
Later we discuss the distribution of values of $S(n)$, and we conjecture that there is a constant $C$ so that

$$\max_a |S(a)| \leq Cp^{12}\log \log p;$$

thus our lower bound (4) is expected to be sharp.

The proof of our theorem is based on the following two lemmas.

**Lemma 1.** Let $\chi$ be a primitive character (mod $q$), $q > 1$. Then

$$\sum_{n=1}^{q-1} \chi(n) e(na) = \tau(\chi) q^{-1} e(\frac{qa}{q}(\sin \pi a) T(a, \chi),$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e(nq/q),$$

and

$$T(a, \chi) = \sum_{a=1}^{q} \chi(a) \cot \pi (a-a/q).$$

**Proof.** Since $\chi$ is primitive, we have

$$\chi(n) = \tau(\chi)^{-1} \sum_{a=1}^{q} \chi(a) e(a n q/q)$$

for all $n$. Thus the left hand side of (6) is

$$= \tau(\chi)^{-1} \sum_{a=1}^{q} \chi(a) \sum_{n=1}^{q-1} e(n(a-a/q))$$

$$= \tau(\chi)^{-1} |1 - e(qa)| \sum_{a=1}^{q} \chi(a) |1 - e(a + a/q)|^{-1}. $$

We write $-a$ for $a$, and recall that $\tau(\chi) \tau(\chi) = \chi(-1) q$ for primitive $\chi$; thus the above is

$$= \tau(\chi)^{-1} |1 - e(qa)| \sum_{a=1}^{q} \chi(a) |1 - e(a + a/q)|^{-1}$$

$$= \tau(\chi)^{-1} |1 - e(qa)| \sum_{a=1}^{q} \chi(a) e\left(-\frac{1}{2}(a-a/q) \sin\pi(a-a/q)\right)^{-1}. $$

We now write $e\left(-\frac{1}{2}(a-a/q)\right) = \cos\pi(a-a/q) - i\sin\pi(a-a/q)$, and recall that $\sum_{a=1}^{q} \chi(a) = 0$ to see that the sum above is $= T(a, \chi)$; this gives (6).

On letting $a$ tend to $a/p$, we see that (6) includes the familiar relation

$$\sum_{n=0}^{q-1} \chi(n) e(\alpha n/q) = \tau(\chi) \chi(a).$$

In fact this is (9) with $\chi$ replaced by $\chi$. Since $|\tau(\chi)| = q^{12}$ for primitive $\chi$ we obtain (3) by taking $\chi(n) = \left(\frac{n}{p}\right)$. By the partial fraction formula for cot $\pi x$ we see that

$$\sum_{a=1}^{q} \chi(a) \cot \pi (a-a/q) = \frac{q}{\pi} \sum_{k=1}^{\infty} \frac{\chi(k)}{qk - k},$$

where the sum over $k$ is to be interpreted as $\lim_{K \to \infty} \sum_{k=1}^{K}$. Thus (8) can be considered to be a special case of the functional equation for a generalized zeta function.

**Lemma 2.** For $k \geq 1$ let $a_{1}, \ldots, a_{k}$ be integers, distinct (mod $p$), and put $f(x) = \prod_{a_{i}} (x-a_{i})$. Then

$$\left| \sum_{n=1}^{q} \left(\frac{f(n)}{p}\right) \right| \leq (k-1)p^{12}.$$  

This is a consequence of Weil's Riemann Hypothesis for the zeta function of a curve over a finite field; see Weil [6], [7]. The derivation of the particular bound above is given by Burgess ([1], § 2).

To obtain the theorem, we put $\chi(n) = \left(\frac{n}{p}\right)$ in Lemma 1. Since then $|\tau(\chi)| = p^{12}$, the first assertion is immediate on noting that

$$|\tau(\chi)\chi(a)| \leq \sum_{a=1}^{p-1} |\cot \pi (a-a/p)| \leq \frac{1}{\pi} \sum_{a=1}^{p-1} |a-a/p|^{-1} \leq p(||p||^{-1} + \log p).$$

Again from (6), the second assertion is a consequence of the inequality

$$\max \left| T'\left(\frac{2n+1}{2p}\right) \right| \geq \frac{\pi}{p} p \log p,$$

where $T(a) = T\left(\left(\frac{a}{p}\right), a\right)$. If $\left(\frac{a}{p}\right) = 1$ for $a = n, n-1, \ldots, n-H$, and $\left(\frac{a}{p}\right) = -1$ for $a = n+1, n+2, \ldots, n+H$, then we may expect that
\[ T\left(\frac{2n+1}{2p}\right) \text{ is approximately} \]
\[ \sum_{n=1}^{\frac{H}{p}} \left( \frac{2n-2h+1}{2p} \right) \approx \frac{2}{\pi} p \log H. \]

With this in mind we put

\[ (11) \quad W(n) = \prod_{h=1}^{H} \left( 1 - \left( \frac{n-h}{p} \right) \right) \prod_{h=0}^{H} \left( 1 + \left( \frac{n-h}{p} \right) \right), \]

and compute the size of the weighted sum

\[ (12) \quad \sum_{n=1}^{p} T\left(\frac{2n+1}{p}\right) W(n). \]

On multiplying out the product (11), we see that

\[ W(n) = 1 + \sum \pm \left( \frac{f(n)}{p} \right), \]

where \( f \) runs through \( 2^{2H+1} - 1 \) polynomials of the sort considered in Lemma 2. Hence by this lemma,

\[ (13) \quad \sum_{n=1}^{p} W(n) = p + O(H2^{2H}p^{1/3}). \]

Similarly,

\[ \sum_{n=1}^{p} W(n) \left( \frac{n-\alpha}{p} \right) = c(\alpha)p + O(H2^{2H}p^{1/3}) \]

where \( c(\alpha) = 1 \) if \( 0 \leq \alpha < H \), \( c(\alpha) = 0 \) if \( H < \alpha < p - H \), and \( c(\alpha) = -1 \) if \( p - H \leq \alpha < p \). Hence the expression (12) is

\[ = \frac{4p}{\pi} \sum_{n=1}^{H} \frac{1}{2a-1} + O(p/H) + O(H^2/p) \]

\[ = \frac{2}{\pi} p (\log H + C) + O(p/H) + O(H^2/p), \]

where \( C = 2\log 2 + \gamma = 1.9635... \) Taking \( H = \frac{1}{4} \log p \), we find that

\[ \sum_{n=1}^{p} W(n) T\left(\frac{2n+1}{2p}\right) = \frac{2}{\pi} p^{4}(\log \log p + \gamma) + O(p^{3}\log p). \]

But \( W(n) \) is non-negative, so

\[ \max_{n} T\left(\frac{2n+1}{2p}\right) \geq \left( \sum_{n} W(n) T\left(\frac{2n+1}{2p}\right) \right) \left( \sum_{n} W(n) \right)^{-1} \]

\[ = \frac{2}{\pi} p (\log \log p + \gamma) (1 + O\left(\frac{1}{\log p}\right)) \]

\[ > \frac{2}{\pi} p \log \log p \]

for large \( p \).

By the Pólya–Vinogradov inequality we see that if \( p \equiv 3 \pmod{4} \) then

\[ T\left(\frac{1}{2p}\right) = -\frac{2}{\pi} p L\left(1,\left(\frac{\ast}{p}\right)\right) + O(p). \]

Joshi [2] has shown that there are infinitely many primes \( p \equiv 3 \pmod{4} \) for which \( L\left(1,\left(\frac{\ast}{p}\right)\right) > (e^\gamma - \varepsilon) \log \log p \). Thus for infinitely many \( p \),

\[ \max_{n} |S(n)| \geq \left( \frac{2}{\pi} e^\gamma - \varepsilon \right) p^{1/2} \log \log p. \]

It seems likely that for \( N > (\log p)^{3}, \)

\[ \sum_{n=1}^{N} \left( \frac{n}{p} \right) \leq N \left( \log \frac{2N^{2}p^{-1}}{\log p} \right)^{-2}; \]

this would give \( T(a) \ll p\left(\|pa\|^{-1} + \log \log p\right), \) and consequently (5).

We have seen that the behavior of \( S(\alpha) \) is atypical when compared with that of \( S(\alpha, \varepsilon) \) for most \( \varepsilon \). However, it can be shown that the distribution of the values of \( S(\alpha)/r\left(\left(\frac{\ast}{p}\right)\right) \) is approaching a limiting distribution which is the same as the limit of the distribution of the values of \( S^*(\alpha, \varepsilon) \) for almost all \( \alpha \), where

\[ S^*(\alpha, \varepsilon) = p^{-1} c(\alpha/pa) \sin \pi a \sum_{d=1}^{p} \epsilon_{d} \cot \pi (\alpha - a/p). \]

References

Об интегралах, содержащих остаточный член проблемы делителей

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Памяти П. Турана посвящается

Пусть \( \tau_k(n) \) обозначает число решений уравнения \( n = m_1 \cdots m_k \) в целых положительных числах \( m_1, \ldots, m_k \). Положим

\[
\sum_{n \leq x} \tau_k(n) = xP_k(\log x) + A_k(x),
\]

где \( xP_k(\log x) \) — главный член роста суммы (входит в тело \( e = 1 \) функции \( \zeta(s) \zeta(s)/s \), \( \zeta \) — дзета-функция Римана).

В работе [1] были вписаны коэффициенты \( a_{ij}^{(k)} \) полинома:

\[
P_k(\log x) = a_{k-1}^{(0)} \log^{k-1} x + \cdots + a_{ij}^{(0)} \log^i x + a_{ij}^{(k)},
\]

которые выписываются через \( \gamma_0, \gamma_1, \ldots, \gamma_k \) определяемыми соотношением:

\[
\gamma_n = \frac{(-1)^n}{n!} \lim_{M \to \infty} \left[ \sum_{1 \leq m \leq M} \frac{\log^m n}{m} - \frac{\log^{m+1} M}{m+1} \right].
\]

Опираясь на этот результат здесь в отношении остаточного члена \( A_k(x) \) доказывается следующий

Теорема. Для любого целого числа \( k \geq 1 \) имеет

\[
\int_1^x \frac{A_k(u)}{u^2} \, du = a_0^{(k+1)} - \sum_{m=1}^{k-1} m! \gamma_m a_{m}^{(k)},
\]

Вывод теоремы основывается на двух леммах.

Лемма 1. Для любого целого \( n \geq 0 \) и любое вещественного \( x \geq 1 \) имеет место следующее равенство

\[
\sum_{1 \leq m \leq x} \frac{\log^m n}{m} = \frac{\log^{n+1} x}{n+1} + (-1)^n \gamma_n + O \left( \frac{\log^a x}{x} \right),
\]

\[
\sum_{1 \leq m \leq x} \frac{\log^m (x/m)}{m} = \frac{\log^{n+1} x}{n+1} + \sum_{k=0}^{n} k C_k \gamma_k \log^{n-k} x + O \left( \frac{\log^a x}{x} \right).
\]