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On the one-sided boundedness of discrepancy-function of the sequence $\{n\alpha\}$

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Let

$$\Delta_N([a, b); \alpha) = \sum_{n=1}^{N} \chi_{[a,b)}(\{n\alpha\}) - (b-a)N$$

where $\{n\alpha\}$ is the fractional part of $n\alpha$, $0 \leqslant a < b < 1$ and $\chi_{[a,b)}$ is the characteristic function of $[a, b)$.

It was proved by Hecke [7] (part of sufficiency) and by Kesten [8] (the more difficult part of necessity), that $\Delta_N([a, b); \alpha)$ is bounded in N if and only if

$$b-a \in R(\alpha) \doteq \{\beta: \beta = \{k\alpha\} \text{ for some nonzero integer } k\}.$$

Very elegant proofs of this theorem in the framework of ergodic theory are due to Furstenberg, Keynes and Shapiro [5], Halász [6] and Petersen [14]. It is remarkable, that on the other side this theorem — and further properties of Δ_N — have consequences for ergodic theory. (See e.g. Herman [9], [10], Deligne [1].)

Here we consider the question of one-sided boundedness. Some previous results already show the phenomenon, that the irregularity of the sequence $\{n\alpha\}$ is not necessarily a two-sided irregularity: e.g. though

$$\sum_{n=1}^{N} \{n\alpha\} - \tfrac{1}{2}N = \Omega(\log N)$$

(Ostrowski [13]) it can be one-sidedly bounded (T. Sós [15]).

It was observed (T. Sós [16], [17], Monteferrante — unpublished) that analogously, though $\Delta_N([a, b); \alpha)$ is unbounded when $b-a \notin R(\alpha)$ yet it can be one-sidedly bounded. More detailed and specified results have been proved by Dupain [3], [4], e.g. the following ones:

1. Suppose the partial quotients (a_n) of the irrational a satisfy:

$$a_{2n-1} = 2 \quad \text{for } n \in N \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_{2n}} < \infty.$$

Then $\Delta_N([0, \frac{1}{2}); a)$ is bounded from below.

2. Suppose a has unbounded partial quotients. Then there exist $\beta \notin R(a)$ for which $\Delta_N([0, \beta))$ is one-sidedly bounded.

3. For $a = (\sqrt{5}-1)/2$ and $\beta \notin R(a)$ $\Delta_N([0, \beta))$ cannot be bounded from above.

The following theorem gives a necessary and sufficient condition for the one-sided boundedness of $\Delta_N([0, \beta); a)$ in the case when a has bounded partial quotients. The "sufficient" part remains true for arbitrary irrational a but not the "necessary" part. This result includes 1 and 3.

THEOREM A. *Suppose the irrational a has bounded partial quotients.*

Then $\Delta_N([0, \beta); a)$ is bounded from above if and only if for some nonnegative integers k, n (with the usual notation (7)–(9))

$$\beta \equiv \{ka\} - r\{q_{2n+1}a\} \bmod 1, \tag{1}$$

where

$$0 \leqslant k < q_{2n+2}, \tag{2}$$

(3) $0 \leqslant r \leqslant 1$ *and* $ra_{2\nu}$ *is nonnegative integer for* $\nu > n$. [1]

Analogously, $\Delta_N([0, \beta); a)$ is bounded from below if and only if for some nonnegative integers k, n

$$\beta \equiv \{ka\} + r\{-q_{2n}a\} \quad (\{-q_0 a\} =: 1) \tag{4}$$

where

$$0 \leqslant k < q_{2n+1}, \tag{5}$$

(6) $0 \leqslant r \leqslant 1$ *and* $ra_{2\nu+1}$ *is nonnegative integer for* $\nu \geqslant n$.

THEOREM B. *Let a be an arbitrary irrational number and β be a number given by* (1)–(3), *resp.* (4)–(6). *Then $\Delta_N([0, \beta); a)$ is bounded from above, resp. from below.*

Corollaries. Let a be an irrational number, (a_n) be the sequence of the partial quotients of a.

[1] i.e. for $r = p/q$, $(p, q) = 1$, $q | a_{2\nu}$ for $\nu > n$.

I. *Suppose*

$$a_{2n+1} \text{ even for } n = 0, 1, \ldots,$$

resp.

$$a_{2n} \text{ even for } n = 1, 2, \ldots$$

Then

$$\Delta_N([0, \tfrac{1}{2}); a), \quad \textit{resp.} \quad \Delta_N([0, a/2); a) \quad \textit{and} \quad \Delta_N([0, 1-a/2); a)$$

is bounded from below, resp. from above. (See 1.)

II. *Suppose*

$$\text{g.c.d.}\ (a_{2n}, a_{2n+2}, \ldots, a_{2n+2k}, \ldots) = 1, \text{ [2]}$$

resp.

$$\text{g.c.d.}\ (a_{2n+1}, \ldots, a_{2n+2k+1}, \ldots) = 1 \quad \textit{for} \quad n = 1, 2, \ldots$$

Then $\Delta_N([0, \beta); a)$ is bounded from above, resp. from below if and only if it is bounded; if and only if $\beta = \{ka\}$ with some integer k. (See 2.)

III. *Suppose a has bounded partial quotients. Then the set of all the β's with one-sidedly bounded $\Delta_N([0, \beta); a)$ is a countable set.* [3]

Notations and some previous results. Let $a = [0, a_1, a_2, \ldots]$ be the continued fraction expansion of a. We shall use the notations and consequences:

$$\frac{p_n}{q_n} = [0, a_1, \ldots, a_{n-1}], \quad q_{n+1} = a_n q_n + q_{n-1}, \quad p_{n+1} = a_n p_n + p_{n-1}, \tag{7}$$

$$\theta_n = q_n a - p_n, \quad \theta_{n+1} = a_n \theta_n + \theta_{n-1},$$

$$\lambda_n = |\theta_n| = (-1)^{n+1}\theta_n, \tag{8}$$

$$\sum_{\nu=0}^{\infty} a_{k+2\nu}\theta_{k+2\nu} = -\theta_{k-1}, \quad k = 1, \ldots \ (\theta_0 = -1),$$

(9)

$$\sum_{\nu=0}^{n} a_{k+2\nu} q_{k+2\nu} = q_{k+2n+1} - q_{k-1}, \quad k = 1, \ldots \ (q_0 = 0).$$

It is well known that each positive integer N can be uniquely represented in the form

$$N = \sum_{k=1}^{m} d_k q_k, \tag{10}$$

where

$$0 \leqslant d_1 \leqslant a_1 - 1, \quad 0 \leqslant d_k \leqslant a_k \quad \text{for} \quad k \geqslant 2, \tag{11}$$

$$d_k = 0 \quad \text{if} \quad d_{k+1} = a_{k+1}. \tag{12}$$

[2] g.c.d. $(c_1, \ldots)$ denotes the greatest common divisor of $(c_1, \ldots)$.

[3] However, it could be shown that for a with unbounded partial quotients it can be a set of power of continuum.

It also is known (Descombes [2], Sós [18], Lesca [11], [12]) that each β with $-\alpha \leqslant \beta < 1-\alpha$ can be uniquely represented in the form

$$\beta = \sum_{k=1}^{\infty} b_k\theta_k, \tag{13}$$

where

$$0 \leqslant b_1 \leqslant a_1-1, \quad 0 \leqslant b_k \leqslant a_k \quad \text{for} \quad k \geqslant 2, \tag{14}$$

$$b_k = 0 \quad \text{if} \quad b_{k+1} = a_{k+1}, \tag{15}$$

$$b_{2k+1} \neq a_{2k+1} \quad \text{for infinitely many positive integer } k. \tag{16}$$

Conversely, every sequence which satisfies (14)–(16) by (13) determines a $\beta \in [-\alpha, 1-\alpha)$. The following simple properties of the above expansions hold:

$$N = \sum_\nu d_\nu q_\nu > N' = \sum_\nu d'_\nu q_\nu \tag{17}$$

iff for some k

$$b_\nu = b'_\nu \quad \text{if } \nu > k \quad \text{and} \quad b_k > b'_k,$$

$$\Big|\sum_{\nu=n}^{\infty} b_\nu\theta_\nu\Big| < \lambda_{n-1}, \quad n = 1, \ldots \; (\lambda_0 = 1), \tag{18}$$

$$\Big|\sum_{\nu=1}^{n} b_\nu q_\nu\Big| < q_{n+1}, \quad n = 1, \ldots, \tag{19}$$

(20) $b_k = 0$ for $k > k_0(\beta)$ iff $\beta \equiv \{k\alpha\} \bmod 1$ with some nonnegative integer k,
$b_{2k} = a_{2k}$ for $k > k_0(\beta)$ iff $\beta \equiv \{-k\alpha\} \bmod 1$ with some positive integer k.

From the above propositions it follows easily that β satisfies (1), (2) and (3), resp. (4), (5) and (6) iff (using the representation (13)–(16))

$$\beta \equiv \sum_{\nu=1}^{\infty} b_\nu\theta_\nu \pmod 1,$$

where

$$b_{2\nu+1} = 0 \quad \text{and} \quad b_{2\nu} = ra_{2\nu} \quad \text{for} \quad \nu > n(\beta), \tag{21}$$

resp.

$$\begin{cases} b_{2\nu} = 0 & \text{and} \quad b_{2\nu+1} = ra_{2\nu+1} \quad \text{for} \quad \nu > n(\beta) \quad \text{or} \\ b_{2\nu} = a_{2\nu} & \text{for} \quad \nu > n(\beta). \end{cases} \tag{22}$$

In the proofs of our theorems we shall consider instead of (3)–(6) (resp. (6)–(8)) the equivalent (21), resp. (22).

The expansion above turned out to be useful for different type of investigations in diophantine approximation. Our proof will be based on the observation that it is possible to handle the discrepancy-function Δ_N by this.

Let

$$\Delta_N(\beta) = \Delta_N([0, \beta); \alpha)$$

and also

$$\Delta_N(\beta) = \Delta_N([0, 1+\beta); \alpha) \quad \text{for} \quad -\alpha < \beta < 0.$$

We shall use with the notation of (10) and (13) the "explicit" formula for $\Delta_N(\beta)$ (see V. T. Sós [17]):

$$\Delta_N(\beta) = \sum_{k=1}^{m}(-1)^{k+1}\min(b_k, d_k) - d_k\Big(q_k\sum_{\nu=k+1}^{\infty} b_\nu\theta_\nu + \theta_k\sum_{\nu=1}^{k} b_\nu q_\nu\Big) + \sum_{k=1}^{m}\delta_k, \tag{23}$$

where

$$\delta_k = \begin{cases} 1 & \text{if } k \text{ odd}, \; b_k > d_k \text{ and } \sum_{\nu=1}^{k-1} d_\nu q_\nu > \sum_{\nu=1}^{k-1} b_\nu q_\nu, \\ -1 & \text{if } k \text{ even}, \; b_k < d_k \text{ and } \sum_{\nu=1}^{k-1} d_\nu q_\nu \leqslant \sum_{\nu=1}^{k-1} b_\nu q_\nu, \\ 0 & \text{otherwise}. \end{cases} \tag{24}$$

We shall also use the following corollaries of (23):

$$\delta_{2k+1} = 0 \text{ if } b_{2k+1} = 0, \qquad \delta_{2k} = 0 \text{ if } d_{2k} = 0. \tag{25}$$

(26) Suppose for the expansions (10) and (13) that

$$\beta = \sum_{\nu=1}^{\infty} b_{2\nu}\theta_{2\nu}, \qquad N = \sum_{\nu=0}^{n} d_{2\nu+1}q_{2\nu+1}. \text{ (4)}$$

Then

$$\Delta_N(\beta) = \sum_{k=0}^{n} d_{2k+1}\Delta_{q_{2k+1}}(\beta).$$

(4) When we write $\beta = \sum b_k\theta_k$, $N = \sum d_k q_k$ it means the expansions (10), (13) under the conditions (11)–(12), resp. (14)–(16).

Put

$$D_k = (-1)^{k+1}\min(b_k, d_k) - d_k\Big(q_k \sum_{\nu>k} b_\nu\theta_\nu + \theta_k \sum_{\nu=1}^{k} b_\nu q_\nu\Big),$$

(27)

$$B_k = (-1)^{k+1}\min(b_k, d_k) - b_k\Big(q_k \sum_{\nu>k} d_\nu\theta_\nu + \theta_k \sum_{\nu=1}^{k} d_\nu q_\nu\Big).$$

Then (by (18), (19) and (7)–(9)) $|D_k| \leqslant a_k$, $|B_k| \leqslant a_k$

(28) $$|\Delta_{q_k}(\beta)| \leqslant 2.$$

Now we give the proof of the theorems for the case of boundedness from above. For the other one the proof runs analogously.

Proof of Theorem A is based on the following lemmata.

LEMMA 1. *Let*

$$\beta = \sum_{\nu=1}^{\infty} b_\nu\theta_\nu, \quad \beta' = \sum_{\nu=1}^{\infty} b'_\nu\theta_\nu,$$

where

$$b_\nu = b'_\nu \quad \textit{for} \quad \nu > \nu_0.$$

Then $\Delta_N(\beta) - \Delta_N(\beta')$ is bounded, more exactly

$$|\Delta_N(\beta) - \Delta_N(\beta')| \leqslant \sum_{b_\nu \neq b'_\nu} 2(a_\nu + 2).$$

Proof. It is enough to prove that in case

$$b_\nu = b'_\nu \quad \text{if} \quad \nu \neq n$$

we have

(29) $$|\Delta_N(\beta) - \Delta_N(\beta')| \leqslant 2a_n + 4.$$

By (23)

$$\Delta_N(\beta) - \Delta_N(\beta') = \sum_{k=1}^{\infty}(D_k - D'_k) + \sum_{k=1}^{\infty}(\delta_k - \delta'_k)$$
$$= \sum_{k=1}^{\infty}(B_k - B'_k) + \sum_{k=1}^{\infty}(\delta_k - \delta'_k) = B_n - B'_n + \sum_{k=1}^{\infty}(\delta_k - \delta'_k).$$

By (24) it is easy to see that

$$\delta_\nu = \delta'_\nu \quad \text{if} \quad \nu \neq n \quad \text{and} \quad \nu \neq l,$$

where l is defined by

$$b_\nu = b'_\nu = d_\nu \quad \text{if} \quad n < \nu < l,$$
$$b_l = b'_l \neq d_l.$$

Thus by this and by (25) we have (29).

LEMMA 2. *Let α be an irrational number with bounded partial quotients; $a_n \leqslant A$ and*

$$\beta = \sum_{\nu=1}^{\infty} b_\nu\theta_\nu$$

for which $b_{2k+1} \neq 0$ holds for infinitely many nonnegative integer k. Then $\Delta_N(\beta)$ is unbounded from above.

Proof. From the assumption and (16) it follows that at least one of the following two conditions

(a) $$b_{2k+1} = a_{2k+1}, \quad b_{2k+3} = 0,$$

(b) $$b_{2k+1} < a_{2k+1}$$

holds for infinitely many k.

Suppose (a). Then

$$\Delta_{a_{2k+1}q_{2k+1}}(\beta) = a_{2k+1}\Big(1 - \theta_{2k+1}\sum_{\nu=1}^{2k+1} b_\nu q_\nu - q_{2k+1}\sum_{\nu=2k+2}^{\infty} b_\nu\theta_\nu\Big)$$
$$\geqslant a_{2k+1}(1 - \lambda_{2k+2}q_{2k+2} - \lambda_{2k+4}q_{2k+1})$$
$$= a_{2k+1}(\lambda_{2k+2} - \lambda_{2k+4})q_{2k+1} \geqslant \frac{1}{4(A+1)}.$$

Secondly suppose (b). Then

$$\Delta_{b_{2k+1}q_{2k+1}}(\beta) = b_{2k+1}\Big(1 - \theta_{2k+1}\sum_{\nu=1}^{2k+1} b_\nu q_\nu - q_{2k+1}\sum_{\nu=2k+2}^{\infty} b_\nu\theta_\nu\Big)$$
$$\geqslant b_{2k+1}\big(1 - \lambda_{2k+1}(b_{2k+1}+1)q_{2k+1} - q_{2k+1}\lambda_{2k+2}\big)$$
$$\geqslant b_{2k+1}\lambda_{2k+1}q_{2k} \geqslant \frac{1}{(A+1)^2}.$$

Let $c = \frac{1}{4}\cdot\frac{1}{(A+1)^2}$ and $2k_1+1 < 2k_2+1 < \ldots$ be all the indices for which

$$\Delta_{b_{2k_i+1}q_{2k_i+1}}(\beta) > c$$

holds. Put

$$N = \sum_{i=1}^{n} b_{2k_i+1}q_{2k_i+1}.$$

By (23) and (24) we have

$$\Delta_N(\beta) \geqslant \sum_{i=1}^{n} \Delta_{b_{2k_i+1}q_{2k_i+1}}(\beta) > nc.$$

By Lemma 1 and Lemma 2 to finish the proof of the necessity of the conditions (1)–(3) ((21)) we may suppose that $b_{2k+1} = 0$ for $k = 0, 1, \ldots$

LEMMA 3. *Let*

$$\beta = \sum_{\nu=1}^{\infty} b_{2\nu}\theta_{2\nu}.$$

If with a positive constant c we have

$$|\Delta_{q_{2k+1}}(\beta)| > c$$

for infinitely many k, then $\Delta_N(\beta)$ is unbounded from above.

Proof. When

$$\Delta_{q_{2k+1}}(\beta) > c$$

holds for infinitely many k, then by the same argument as in Lemma 2 it is obvious that $\sup_N \Delta_N(\beta) = +\infty$.

Now suppose that

$$\Delta_{q_{2k+1}}(\beta) < -c$$

holds and consequently

$$\Delta_{a_{2k+1}q_{2k+1}}(\beta) < -c$$

holds for infinitely many k; for $0 < k_1 < k_2 < \ldots$ let $K = \{k_1, k_2, \ldots\}$. Put

$$N = \sum_{i=1}^{n} a_{2k_i+1}q_{2k_i+1} \qquad (k_i \neq 0).$$

Then

$$\Delta_N(\beta) < -nc.$$

Now let

$$N' = \sum_{\substack{k\leqslant k_n\\ k\notin K}} a_{2k+1}q_{2k+1} - 1.$$

Since

$$N+N' = q_{2k_n+2} - 1 = (a_1-1)q_1 + \sum_{k=1}^{n} a_{2k+1}q_{2k+1}$$

by (25), (26) and (28) we have

$$|\Delta_{q_{2k_n+2}-1}| = |\Delta_N(\beta)+\Delta_{N'}(\beta)| \leqslant |\Delta_{q_{2k_n+2}}|+1 \leqslant 3.$$

Hence

$$\Delta_{N'}(\beta) \geqslant nc-3.$$

LEMMA 4. *Let α be an irrational number with bounded partial quotients; $a_n \leqslant A$ and*

$$\beta = \sum_{\nu=1}^{\infty} b_{2\nu}\theta_{2\nu}.$$

If

$$r = \varliminf_{n\to\infty} \frac{b_{2n}}{a_{2n}} \neq \varlimsup_{n\to\infty} \frac{b_{2n}}{a_{2n}}, \tag{30}$$

$\Delta_N(\beta)$ is unbounded from above.

Proof. Since by the assumption the range of $r_{2k} = b_{2k}/a_{2k}$ is finite, by Lemma 1 we may suppose, that

$$r = \varliminf_{n\to\infty} \frac{b_{2n}}{a_{2n}} = \min \frac{b_{2n}}{a_{2n}}.$$

(30) implies that for infinitely many k we have with some $m = m(k) > k$

$$r_{2k} > r, \quad r_{2\nu} = r \quad \text{for} \quad k+1 \leqslant \nu \leqslant m, \quad r_{2m+2} > r. \tag{31}$$

By (23) we have, using $b_{2\nu} = r_{2\nu}a_{2\nu}$ and (31)

$$\Delta_{q_{2m+1}} = q_{2m+1}\sum_{\nu>m} b_{2\nu}\lambda_{2\nu} - \lambda_{2m+1}\sum_{\nu=1}^{m} b_{2\nu}q_{2\nu}$$
$$= r\lambda_{2m+1} + q_{2m+1}\sum_{\nu>m}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu} - \lambda_{2m+1}\sum_{\nu=1}^{k}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu}$$

and

$$\Delta_{q_{2k+1}} = q_{2k+1}\sum_{\nu>m} b_{2\nu}\lambda_{2\nu} - \lambda_{2k+1}\sum_{\nu=1}^{k}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu}$$
$$= r\lambda_{2k+1} + q_{2k+1}\sum_{\nu>m}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu} - \lambda_{2k+1}\sum_{\nu=1}^{k}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu}.$$

Thus

$$\Delta_{q_{2m+1}} - \Delta_{q_{2k+1}} = r(\lambda_{2m+1}-\lambda_{2k+1}) + (q_{2m+1}-q_{2k+1})\sum_{\nu>m}(r_{2\nu}-r)a_{2\nu}\lambda_{2\nu} +$$
$$+ (\lambda_{2k+1}-\lambda_{2m+1})\sum_{\nu=1}^{k}(r_{2\nu}-r)a_{2\nu}q_{2\nu}$$
$$\geqslant (q_{2m+1}-q_{2k+1})(r_{2m+2}-r)a_{2m+2}\lambda_{2m+2} +$$
$$+ (\lambda_{2k+1}-\lambda_{2m+1})(r_{2k}-r)a_{2k}\lambda_{2k} - \lambda_{2k+1}.$$

Since

$$q_{2m+1}-q_{2k+1} \geqslant a_{2m}q_{2m}, \qquad \lambda_{2k+1}-\lambda_{2m+1} \geqslant a_{2k+2}\lambda_{2k+2}$$

and

$$r_{2m+2}-r \geqslant \frac{1}{a_{2m+2}a_{2m}}, \qquad r_{2k}-r > \frac{1}{a_{2k}a_{2m}},$$

we easily get

$$\Delta_{q_{2m+1}} - \Delta_{q_{2k+1}} \geqslant \frac{1}{4}\frac{1}{(A+1)^3} - \lambda_{2k+1}.$$

Hence, with $c = \frac{1}{8}\frac{1}{(A+1)^3}$

$$\max\left(|\Delta_{q_{2m+1}}(\beta)|,\ |\Delta_{q_{2k+1}}(\beta)|\right) > c \quad \text{if} \quad k > k_0.$$

Thus by Lemma 3 we get that under the condition (30)

$$\sup_N \Delta_N(\beta) = +\infty.$$

Now by the above lemmata we know that when $\Delta_N(\beta)$ is bounded from above, then in the expansion

$$\beta = \sum_{k=1}^{\infty} b_k\theta_k$$

we have

$$b_{2k+1} = 0 \quad \text{for} \quad k > k_0$$

and

$$b_{2k} = ra_{2k} \quad \text{for} \quad k > k_0^*.$$

But this means that, with some n_0

$$\beta = \sum_{k=1}^{n_0-1} b_k\theta_k + r\sum_{2k>n_0} a_{2k}\theta_{2k}.$$

This proves the "necessity" part of Theorem A.

Proof of Theorem A will be finished, when we prove the "sufficient" part by the more general Theorem B.

Proof of Theorem B. The proof will follow by the following simple

LEMMA 5. *Let with* $0 \leqslant r \leqslant 1$

$$\beta = \sum_{\nu=1}^{\infty} b_{2\nu}\theta_{2\nu} \quad \textit{and} \quad b_{2\nu} = ra_{2\nu} \quad \textit{for} \quad \nu = 1, \ldots,$$

$$N = \sum_{\nu=1}^{m} d_\nu q_\nu \quad \textit{and} \quad N^+ = N - \sum_{\nu} d_{2\nu}q_{2\nu}.$$

Then

(32) $$\Delta_{N^+}(\beta) \geqslant \Delta_N(\beta).$$

Proof. By (23) and (25)

$$\Delta_{N^+}(\beta) - \Delta_N(\beta) = -\sum_k D_{2k}(N).$$

But

$$\begin{aligned} -D_{2k} &= \min(b_{2k}, d_{2k}) - d_{2k}\left[q_{2k}\sum_{\nu=k+1}^{\infty} b_{2\nu}\lambda_{2\nu} + \lambda_{2k}\sum_{\nu=1}^{k} b_{2\nu}q_{2\nu}\right] \\ &= \min(ra_{2k}, d_{2k}) - rd_{2k}[q_{2k}\lambda_{2k+1} + \lambda_{2k}(q_{2k+1}-1)] \\ &\geqslant \min(ra_{2k}, d_{2k}) - rd_{2k} \geqslant 0 \end{aligned}$$

which proves (32).

Now let N^+ be the set of integers N, for which in their expansion (with respect to α)

$$N = \sum_{\nu=1}^{m} d_\nu q_\nu, \quad d_{2\nu} = 0 \quad \text{for all } \nu.$$

To finish the proof of the theorem by Lemma 1 and Lemma 5 it is enough to prove that if

$$\beta = \sum b_{2\nu}\theta_{2\nu}, \quad b_{2\nu} = ra_{2\nu}, \quad \nu = 1, 2, \ldots$$

then

$$\sup_{N\in N^+} \Delta_N(\beta) < \infty.$$

Using (9) and (26) we get

$$\begin{aligned} \Delta_N(\beta) &= \sum d_{2k+1}\Delta_{2k+1}(\beta) \\ &= -\sum_{k=1}^{m} d_{2k+1}\left[q_{2k+1}\sum_{\nu>k} b_{2\nu}\theta_{2\nu} + \theta_{2k+1}\sum_{\nu=1}^{k} b_{2\nu}q_{2\nu}\right] \\ &= \sum d_{2k+1}[q_{2k+1}r\lambda_{2k+1} - \lambda_{2k+1}r(q_{2k+1}-1)] \\ &= \sum d_{2k+1}\lambda_{2k+1} \leqslant 1. \end{aligned}$$

Remark. It could be shown, that the conclusion in Lemma 3 remains true without the assumption α has bounded partial quotients.

Also the conclusion in Lemma 5 remains true with the weaker assumption (a_{2n}) is bounded. This means, that the conclusion of Theorem A remains true when we suppose only the boundedness of (a_{2n}) (for the boundedness of $\Delta_N(\beta)$ from above) resp. the boundedness of a_{2n+1} (for the boundedness of $\Delta_N(\beta)$ from below).

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An exponential polynomial formed with the Legendre symbol

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Dedicated to the memory of Paul Turán

We investigate the behavior of the sum

$$S(\alpha) = S_p(\alpha) = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right)e(n\alpha), \tag{1}$$

which is a particular example of an exponential polynomial of the sort

$$S(\alpha, \varepsilon) = \sum_{n=1}^{N}\varepsilon_n(n\alpha) \tag{2}$$

with $\varepsilon_n = \pm 1$. Among such polynomials, $S(\alpha)$ has the unusual property that

$$|S(a/p)| = \sqrt{p} \quad (1 \leqslant a \leqslant p-1), \tag{3}$$

and $S(0) = 0$. It is difficult to exhibit a choice of ε for which $|S(\alpha, \varepsilon)| \leqslant C\sqrt{N}$ for all α. The example known was found by H. S. Shapiro [5] and W. Rudin [4]; a nice account of this and related problems was given by Littlewood [3], pp. 25–32. In view of (3), we ask whether the sum $S(\alpha)$ also satisfies the bound $S(\alpha) \ll \sqrt{p}$. Indeed, from Bernstein's inequality it follows that if $K > \frac{\pi}{2}N$ then

$$\max_\alpha |S(\alpha, \varepsilon)| \leqslant \left(1 - \frac{\pi N}{2K}\right)^{-1} \max_a |S(a/K, \varepsilon)|;$$

thus the points a/p are nearly dense enough for us to deduce from (3) that $S(\alpha) \ll p^{1/2}$. Hence it is surprising that this estimate is false for *all* large primes p.

THEOREM. *For $p > 2$, $S(\alpha) \ll p^{1/2}\log p$, and for all large p,*

$$\max_\alpha |S(\alpha)| > \frac{2}{\pi}p^{1/2}\log\log p. \tag{4}$$