ACTA ARITHMETICA
XXXVII (1980)

# On a constant of Turán and Erdös

by

D. H. LEHMER (Berkeley, Calif.)

*Dedicated to the memory of Paul Turán*

In a paper entitled *Combinatorics, partitions, group theory*, Turán reported on some joint research with Erdös on the asymptotic structure of the symmetric group $S_n$ on $n$ letters. One of the questions raised had to do with the $p(n)$ conjugacy classes of $S_n$. Since all elements of any such class $C$ have the same order, one can speak of the order $O(C)$ of the conjugacy class $C$. They found that, as $n\to\infty$,

$$O(C) = \exp\left[\sqrt{n}\,(A+o(1))\right]$$

holds for almost all classes $C$, that is, for all but $o(p(n))$ classes at most. The constant $A$ was determined to be [1]

$$A = \frac{2\sqrt{6}}{\pi}\sum_{j\neq 0}\frac{(-1)^{j+1}}{(3j^2+j)}, \tag{1}$$

that is $1/\sqrt{\zeta(2)}$ times the alternating sum of the reciprocals of the non-zero pentagonal numbers.

It is the purpose of this note to show that in fact

$$A = 4\sqrt{2}-6\sqrt{6}/\pi = .97867344209\ldots$$

Use will be made of the general harmonic sum associated with the arithmetic progression $r,\ r+k,\ r+2k,\ldots,$ namely

$$H(x, r, k) = \sum_{\substack{0<n\leqslant x\\ n\equiv r(\mathrm{mod}\,k)}} 1/n$$

whose properties were considered in a recent paper [2].

We begin by defining

$$S(N) = \sum_{-N\leqslant j\leqslant N}{}' (-1)^{j+1}/(3j^2+j).$$

Since
$$1/(3j^2+j) = 1/j - 3/(3j+1)$$
and because
$$\sum_{-N\leqslant j\leqslant N}' 1/j = 0,$$
we can write

(2) $$S(N) = 3\big(T_1(N) - T_2(N)\big) + o(N),$$

where
$$T_1(N) = \sum_{1\leqslant j\leqslant N} (-1)^j/(3j+1) = \sum_{1\leqslant j\leqslant N/2} 1/(6j+1) - \sum_{0\leqslant i\leqslant (N-1)/2} 1/(6i+4)$$
and
$$T_2(N) = \sum_{1\leqslant j\leqslant N} (-1)^j/(3j-1) = \sum_{1\leqslant i\leqslant N/2} 1/(6j-1) - \sum_{0\leqslant i\leqslant (N-1)/2} 1/(6i+2).$$
That is,
$$T_1(N) = H(3N+1, 1, 6) - 1 - H\big((3N+1)/2, 2, 3\big)/2,$$
$$T_2(N) = H(3N-1, 5, 6) - H\big((3N-1)/2, 1, 3\big)/2.$$

By [2], (1), we have
$$H(x, r, k) = k^{-1}\log x + \gamma(r, k) + o(x)$$
where $\gamma(r, k)$ is the associated Euler constant. It follows that
$$T_1(N) - T_2(N) = -1 + \gamma(1, 6) - \gamma(5, 6) + [\gamma(1, 3) - \gamma(2, 3)]/2 + o(N).$$
By [2], p. 133,
$$\gamma(1, 6) - \gamma(5, 6) = 3[\gamma(1, 3) - \gamma(2, 3)]/2.$$
Hence

(3) $$T_1(N) - T_2(N) = -1 + 2[\gamma(1, 3) - \gamma(2, 3)] + o(N).$$

Finally by [2], (12), we have
$$\gamma(1, 3) - \gamma(2, 3) = \frac{\pi}{3}\cot\frac{\pi}{3} = \frac{\sqrt{3}\pi}{9}.$$
Now (3) and (2) give
$$S(N) = \frac{2\pi\sqrt{3}}{3} - 3 + o(N).$$
Letting $N\to\infty$ we obtain from (1)
$$A = \frac{2\sqrt{6}}{\pi}\left(\frac{2\pi\sqrt{3}}{3} - 3\right) = 4\sqrt{2} - \frac{6\sqrt{6}}{\pi}.$$



### References

[1] P. Turán, *Combinatorics, partitions, group theory*, Colloq. Internat. sulle Teorie Combinatorie, Accad. Naz. Lincei, Rome 1976, vol. 2, p. 183.

[2] D. H. Lehmer, *Euler constants for arithmetic progressions*, Acta Arith. 27 (1976), pp. 125–142.

*Received on 15. 2. 1978* (1043)