Some consequences of the Riemann hypothesis

by

P. X. Gallagher (New York, N.Y.)

In memory of Paul Turán

The main object of this note is to show that the remainder term in the prime number theorem, assuming the Riemann hypothesis, can be reduced from

\[ \psi(x) = x + O(x^{1/2} \log^2 x) \]

(1)

to

\[ \psi(x) = x + O(x^{1/2} (\log \log x)^2) \]

(2)

except on a set of finite logarithmic measure.

We also give short proofs of Cramér's conditional estimates ([1], [2]), of the mean value of the remainder term

\[ \int_1^x (\psi(x) - x) \frac{dx}{x} \ll X, \]

(3)

and

\[ \int_1^x (\psi(x) - x)^2 \frac{dx}{x^2} \sim \log X. \]

(4)

It follows from (3) that for each function \( \varphi = \varphi_x \) for which \( \varphi_x \to \infty \) as \( x \to \infty \), we have

\[ \psi(x) = x + O(x^{1/2} \varphi_x) \]

for almost all \( x \), i.e. except on a set whose intersection with the interval \([1, X]\) has measure \( o(X) \). The proof that for \( \varphi_x = (\log \log x)^2 \) the exceptional set has finite logarithmic measure is a combination of the arguments which prove (1) and (3). A similar method yields a short proof of Selberg's
conditional result [6] on the normal density of primes in short intervals:
\[
\psi(x + h) - \psi(x) - h \sim o(h^2 x),
\]
for increasing functions \(h = h_x\) which satisfy
\[
h_x \leq a, \quad h_x / \log x \to 0.
\]

J. H. Mueller and the author have recently shown [4] that the Riemann hypothesis together with Montgomery's conjecture [5] on the pair density of zeros implies that \(O\) may be replaced by \(o\) in (1).

\[
\psi(x) = x - \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} + O(\log^2 x),
\]
valid for \(X < x \leq X\). Here the sum is over the complex zeros \(\gamma = \frac{1}{2} + iy\) of the Riemann zeta function. We will use the fact that the number of terms with \(T \eta \leq t + 1\) is \(\log t\) for \(T \gg 1\). It follows from this that the contribution of the terms with \(|y| \leq \log x\) is \(\ll X^{1/3}(\log x)^3\).

For \(T < x\), we have
\[
\int_{T}^{x} \left| \sum_{|\gamma| \leq \theta} \frac{\alpha^\gamma}{\gamma} \right|^2 \frac{dx}{x^3} \ll \frac{\log^3 T}{T}.
\]
In fact, Lemma 1 of [3] shows that the integral is
\[
\frac{1}{\log X} \int \left| \sum_{|\gamma| \leq \theta} \frac{\alpha^\gamma}{\gamma} \right|^2 \frac{d\gamma}{\gamma} \ll \frac{1}{T} \int \left( \sum_{|\gamma| \leq \theta} \frac{1}{|\gamma|^2} \right) dt,
\]
and the bound in (7) now follows from the fact mentioned above.

It follows from (7) that the logarithmic measure of the set of \(\alpha\) in \([x, e^x]\) for which
\[
\left| \sum_{|\gamma| \leq \theta} \frac{\alpha^\gamma}{\gamma} \right| \geq \alpha^{1/2}(\log x)^3
\]
is
\[
< \frac{\log^3 T}{T(\log x)^3} \leq \frac{1}{T \log^2 T} \quad \text{for} \quad T = \log x,
\]
Choosing \(X = e^x\) with \(T = 2, 3, \ldots\), we see that (2) holds except on a set whose total logarithmic measure is finite.

2. From (6) and (7) with \(T = 2\), we get
\[
\int_{X}^{x} \left| \psi(x) - x \right|^2 \frac{dx}{x^3} \ll 1,
\]
from which (3) follows easily by a splitting-up argument.

To get (4), we observe first that for each fixed \(T\), and \(X \to \infty\),
\[
\int_{x}^{X} \left| \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} \right|^2 \frac{dx}{x^3} = \int_{x}^{X} \left( \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} \right)^2 \frac{du}{u^3} \sim \left( \sum_{|\gamma| \leq \theta} \frac{m^2(\gamma)}{|\gamma|^2} \right) \log x,
\]
where the dash indicates that the sum is over distinct zeros, and \(m(\gamma)\) is the multiplicity of the zero at the point \(\gamma\). It follows that
\[
\int_{x}^{X} \left| \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} \right|^2 \frac{dx}{x^3} \ll C \log x, \quad C = \sum_{|\gamma| \leq \theta} \frac{m^2(\gamma)}{|\gamma|^2},
\]
where \(T = T_x\) is a suitable function such that \(T \to \infty\) and \(\log T = o(\log x)\) as \(X \to \infty\). From (7) we get that, for such \(T\),
\[
\int_{x}^{X} \left| \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} \right|^2 \frac{dx}{x^3} = o(\log x).
\]
It follows that
\[
\int_{x}^{X} \left| \psi(x) - x \right|^2 \frac{dx}{x^3} \ll C \log x
\]
as \(X \to \infty\), for such \(T\). Combining this with
\[
\int_{x}^{X} \left| \psi(x) - x \right|^2 \frac{dx}{x^3} \ll \log T = o(\log x),
\]
which follows from (8), we get (4).

3. To prove (5), we use (6) to express \(\psi(x + h) - \psi(x) - h\) as
\[
- \sum_{|\gamma| \leq \theta} \left( \sum_{|\gamma| \leq \theta} \frac{y^\gamma}{\gamma} \right) \frac{dy}{y^3} + \sum_{|\gamma| \leq \theta} \frac{x^\gamma}{\gamma} - \sum_{|\gamma| \leq \theta} \frac{(x + h)^\gamma}{\gamma} + O(\log^2 x),
\]
for \(X < x \leq e^X\), \(h \leq X\), and \(T \leq X\). Putting
\[
S_1(y) = \sum_{|\gamma| \leq \theta} \frac{y^\gamma}{\gamma}, \quad S_2(y) = \sum_{|\gamma| \leq \theta} \frac{y^\gamma}{\gamma},
\]
this may be written as
\[
- \int \frac{S_1(y)}{y^2} \frac{dy}{y^3} + \alpha^{1/2} S_4(x) - (x + h)^{1/2} S_4(x + h) + O(\log^2 x).
\]
A simple argument shows that

$$\int_{C} \int_{a}^{a+h} \left| S_2(y) \right|^2 \frac{dy}{y^{1/2}} \, ds \ll h_{2,1} \int_{C} \left| S_1(y) \right|^2 \frac{dy}{y}.$$ 

The same method as in § 1 shows that the integral on the right is

$$\ll \int_{-(2+1)}^{T} \left( \sum_{t \leq x \leq t+1} 1 \right)^2 \, dt \ll T \log^2 T.$$ 

Also,

$$\int_{C} \left| \psi(x+h) - \psi(x) \right|^2 \, ds \ll X \int_{C} \left| S_1(x) \right|^2 \frac{dx}{x} \ll X^2 \log^2 T,$$

by (7); similarly,

$$\int_{C} \left| (x+h) \frac{S_2(x+h)}{h} \right|^2 \, dx \ll X \int_{C} \left| S_2(x+h) \right|^2 \, dx,$$

and since $h$ is increasing and $\ll x$, this is also

$$\ll X \int_{C} \left| S_2(y) \right|^2 \, dy \ll X^2 \frac{\log^2 T}{T}.$$ 

It follows that for $T \leq X$,

$$\int_{C} \left| \psi(x+h) - \psi(x) \right|^2 \, ds \ll h_{2,1}^2 T \log^2 T + X^2 \frac{\log^2 T}{T} + X \log X.$$

For $T = X^{1/2}$, this is $o(Xh_{2,1}^2)$, provided $h_{2,1}^2 \log^2 X \rightarrow \infty$. A simple splitting-up argument completes the proof of (5).

References