To finish, I state a few facts about Turán. He himself probably considered his "new method" to be his most important contribution. Once, several years before his death, I found him at his desk deeply absorbed in work. I asked him: "What are you working on?" He answered smiling "I am building my pyramid". Perhaps I should explain the meaning of this to the non-Hungarian reader. They refer to a play of a famous Hungarian writer Madách (the play, "The tragedy of Man", was translated into many languages, but is probably not very well known abroad). In one of the scenes, enacted in ancient Egypt, the Pharaoh to accomplish an immortal achievement is building his pyramid. Thus "building my pyramid" would mean: trying to accomplish an immortal achievement, which will live for ever. In fact, he was writing his book.

Several years later, in July 1976, at the meeting on combinatorics at Orsay in Paris, V. T. Sós (Mrs. Turán) gave me the terrible news (which she had known for 6 years) that Paul had leukemia. She told me that I should visit him as soon as possible and that I should be careful in talking to him because he did not know the true nature of his illness. My first reaction was to say that perhaps he should have been told so that he could "finish his pyramid". She said she felt that Paul loved life too much and with the death sentence hanging over him would not be able to live and work very well. (In fact, he could work very well under adverse conditions. For example, the theory of extremal graphs was started in a labour camp in 1940 in the nazi-fascist era.) Nevertheless, I am now fairly sure that her decision was right, since he clearly never tried to find out the true nature of his illness. In fact, a few days before his death, V. T. Sós and their son George (also mathematician) tried to persuade him to dictate some parts of his book to Halász or Pintz. He refused saying "I will write it when I feel better and stronger". Unfortunately he never had the chance. Fortunately his book was finished by his students G. Halász and J. Pintz and will soon appear.

It is always sad when a great man dies while still mentally in his prime.

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The number-theoretic work of Paul Turán

by

G. Halász* (Budapest)

Paul Turán made important contributions to many parts of mathematics but it was number theory that captivated his interest unabatedly throughout his life.

The power sum method. Turán made several attempts to solve the deepest problems of analytic number theory, first and foremost among them being the over 100 year old conjecture of Riemann on the zeros of the zeta function. The following has proved to be the most successful (not exactly for the purpose for which it was originally intended).

For s_0 on the vertical line $\mathrm{Re}\, s = \sigma_0 > 1$ let $r(s_0)$ denote the radius of the largest zero-free disc around s_0 of the zeta function $\zeta(s)$; Riemann's hypothesis is then equivalent to $r(s_0) \geqslant \sigma_0 - 1/2$. In other words, $r(s_0)$ is the radius of regularity for $\frac{\zeta'}{\zeta}(s)$ around s_0 (provided that $\mathrm{Im}\, s_0$ is sufficiently large, so that the pole at $s = 1$ does not come into play) and by Cauchy's elementary formula

$$\limsup_{\nu=\infty} \sqrt[\nu]{\frac{1}{\nu!}\left|\left(\frac{\zeta'}{\zeta}(s)\right)^{(\nu)}_{s=s_0}\right|} = \frac{1}{r(s_0)}.$$

By differentiating a classical approximation to $\frac{\zeta'}{\zeta}(s)$, the quantity under the νth root can be replaced by

$$\left|\sum_{\varrho}{}' \frac{1}{(s_0-\varrho)^{\nu+1}}\right|,$$

the summation being extended over a finite number of zeros ϱ of $\zeta(s)$ in the "vicinity" of s_0, and attempts to replace the limsup, impossible to calculate, by finite explicit estimations lead to general inequalities for sums of powers of complex numbers.

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Out of this "primitive" idea rose an important new theory with applications to many branches of mathematics. (See his book [66] in Hungarian, its translation into German [67] or an enlarged Chinese version [92]; a completely rewritten and essentially expanded English edition [241] will appear in the not too distant future.)

The above purpose is served by what he calls his second main theorem:

$$\max_{m+1\leqslant\nu\leqslant m+n}\Big|\sum_{i=1}^{n} b_i z_i^\nu\Big| \geqslant \Big(\frac{n}{8e(m+n)}\Big)^n \min_{l=1,\ldots,n}\Big|\sum_{i=1}^{l} b_i\Big|$$

for complex numbers b_i, z_i with $1 = |z_1| \geqslant \ldots \geqslant |z_n|$ and $m > 0$; (for this form see [116]).

Some of the most important applications of this and its one-sided version (see [123])

$$\max_{m+1\leqslant\nu\leqslant m+n(3+\pi/\varkappa)} \operatorname{Re}\sum_{i=1}^{n} b_i z_i^\nu$$
$$\geqslant \frac{1}{2n+1}\Big(\frac{n}{24e^3(m+n(3+\pi/\varkappa))}\Big)^{2n} \min_{l=1,\ldots,n}\Big|\operatorname{Re}\sum_{i=1}^{l} b_i\Big|$$

for $0 < \varkappa \leqslant |\arg z_i| \leqslant \pi$, $1 = |z_1| \geqslant \ldots \geqslant |z_n|$ will briefly be discussed below.

His first unsuccessful efforts to obtain such deep estimations, the proofs of which are based on interpolation theory, contour integration and a very skilful manipulation with polynomials led him to a better inequality, his first main theorem ([66], [67]),

$$\max_{m+1\leqslant\nu\leqslant m+n}\Big|\sum_{i=1}^{n} b_i z_i^\nu\Big| \geqslant \Big(\frac{n}{2e(m+n)}\Big)^n \Big|\sum_{i=1}^{n} b_i\Big|$$

under the stricter condition $|z_i| \geqslant 1$; he later found numerous applications of this theorem to gap power series, quasi analytic functions, differential equations, value distribution of certain entire functions, and via the latter, indirectly, to number theory: improving his results, Tijdeman learned that this was exactly what was needed to complete certain deep investigations of Gelfond on transcendental numbers. Incidentally, Turán considered his whole theory of power sum inequalities, to which besides his basic work many other mathematicians have now contributed, as part of number theory, namely as a new chapter on diophantine approximation where weakness of approximation is compensated by strong localization of the parameter ν.

Applications to zero free regions. The idea outlined above leads directly to the following theorem in [31].

Suppose that with positive numbers a, b, c we have for a t large enough the inequality

$$\Big|\sum_{N<n\leqslant N'} \Lambda(n) e^{it\log n}\Big| \leqslant \frac{N\log^c N}{|t|^b}$$

whenever $|t|^a \leqslant N < N' \leqslant 2N$. Then $\zeta(\sigma+it) \neq 0$ for $\sigma > 1-c'\dfrac{b^3}{a^2}$ with an absolute constant c'.

(Here $\Lambda(n)$ is the von Mangoldt symbol: $\log p$ if n is a power of the prime p and 0 otherwise.)

Unfortunately the methods for estimating trigonometric sums are not sufficiently strong to verify the assumption. Surprisingly enough, using Vinogradov's estimation, he proved it even with $b = 1/2$ for $\log n$ replaced in the exponent by $\log^\gamma n$ with any $\frac{1}{2} < \gamma < 2$ except $\gamma = 1$ ([34]), and connecting exponential sums like this with zeros as in the above theorem, he found that in order to have similar conclusions one would need exactly $b > 1/2$ ([66], [67]).

Although the power sum method has thus failed to produce zero-free regions unconditionally, the above result is of theoretic interest. In another version [187] the assumption is only needed for $N < |t|^d$, providing the first step towards the problem of Landau who, on the basis of an exact prime number formula in terms of zeros, supposed that there was a direct connection between certain zeros and certain primes. Still another version [48] answers a question of Littlewood, who asked how a single zero affects the remainder

$$\pi(x) - \operatorname{Li}x = \sum_{p\leqslant x} 1 - \int_2^x \frac{du}{\log u}$$

of the prime number theorem on a finite range, pointing out that no method existed to tackle such problems. (This research has been extended by Staś and more recently Pintz has given very precise answers to Littlewood's question.) The same can be said about Ingham's problem as to whether our present bound $O(x\exp(-\log^\alpha x))$ known to hold for any $\alpha < 1/(1+\gamma)$ for the remainder of the prime number theorem is the most that can be deduced from a zero-free region $\sigma > 1-c\log^{-\gamma}(|t|+2)$ (the best value of γ known today being $2/3+\varepsilon$). He was able to give the reassuring answer: denoting by α_0 the supremum of the α's and by γ_0 the infimum of the γ's, we have $\alpha_0 = 1/(1+\gamma_0)$ ([50]). This again has been extended by Staś and recently Pintz has proved precise results for general zero-free domains.)

Applications to density theorems. This is the name given to estimates of $N(\sigma, T)$, the number of zeros of $\zeta(s)$ in the rectangle

$\operatorname{Re} s \geqslant \sigma$, $|\operatorname{Im} s| \leqslant T$ for $\sigma > 1/2$, and here the method has produced not only results of theoretical interest.

Ingham deduced the "density hypothesis"

$$N(\sigma, T) = O(T^{2(1-\sigma)+\varepsilon}) \quad (\sigma > 1/2),$$

which sometimes can serve as a substitute for Riemann's hypothesis from Lindelöf's hypothesis

$$\zeta(1/2+it) = O(|t|^{\varepsilon}).$$

Turán replaced this by a much weaker hypothesis ([94]), showed also without any unproved assumption that the density hypothesis holds in a sense asymptotically as $\sigma \to 1-0$ ([54]), and stated as a conjecture in [77] that Lindelöf's hypothesis implies even

$$N(\sigma, T) = O(T^{\varepsilon}) \quad (\sigma > 1/2).$$

He and Halász have, in fact, proved this for $\sigma > 3/4$ and unconditionally demonstrated the truth of the density hypothesis and even much more for $\sigma_0 < \sigma < 1$ with a numerical $\sigma_0 < 1$ ([181]). By a different method but not independently of Turán's innovating work in this field, Montgomery and others have improved these results numerically.

Before the discovery of his power sum method, Turán deduced the upper bound k^c with a universal constant c for the least prime in any residue class l with $(l, k) = 1$ from an unproved hypothesis ([19]) and in an attempt to get rid of this assumption he was among the first to recognize the importance of density theorems for Dirichlet's L-functions $L(s, \chi, k)$ belonging to a varying modulus k ([19], [26]). Linnik proved the above result in a very complicated way without hypothesis, one of his two main lemmas being in fact a density estimation. By his power sum method Turán later gave a much simpler proof for the latter ([121]) and Knapowski for the other main lemma. Jutila used this new proof to improve Linnik's constant c. Fogels and Gallagher extended the validity of the density result in T and k, respectively.

Application to the comparative theory of primes. This is joint work with Knapowski ([131], [132], [133], [136], [137], [138], [140], [141], [145], [148], [149], [151], [153], [156], [157], [162], [164], [180], [209], [226], [229], [235], [241]). It was they who made a systematic study of discrepancies in the distribution of primes in different arithmetic progressions, although sporadic special results had been known in the literature.

A common premise in all the results is the knowledge of a $D = D(k)$ such that no L-function belonging to the modulus k vanishes for $0 < \sigma < 1$, $|t| \leqslant D$ (corresponding to the argument condition in the one-sided power-sum inequality). A special feature is the explicitness and numerical effectiveness of their estimations in terms of only k and this D. No other method has since been able to produce such results, even though the non-existence of positive zeros (which is verified so far only for a finite number of k) occurs naturally in all theorems of this kind.

A typical example is the following:

Each function $\psi(x, k, l_1) - \psi(x, k, l_2)$ $(l_1 \not\equiv l_2 \pmod k)$ where

$$\psi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l \pmod k}} \Lambda(n) \quad ((l, k) = 1)$$

changes sign in every interval $[\omega, \exp(2\sqrt{\omega})]$ provided

$$\omega \geqslant \max\left(\exp k^c, \exp(2/D)^3\right)$$

([140]). The universal constant c can be calculated numerically.

The case of

$$\pi(x, k, l_1) - \pi(x, k, l_2) = \sum_{\substack{p \leqslant x \\ p \equiv l_1 \pmod k}} 1 - \sum_{\substack{p \leqslant x \\ p \equiv l_2 \pmod k}} 1 \quad ((l_1, k) = (l_2, k) = 1)$$

is much deeper. Turán and Knapowski exhibited similar localized sign changes, even large positive and large negative values if l_1 and l_2 are both quadratic residues or non-residues, assuming also a weaker version of the generalized Riemann hypothesis for L-functions mod k ([140]). Especially difficult is to show the preponderance of a quadratic over a non-quadratic residue class. They have results of the above type only for $l_1 \equiv 1 \pmod k$ ([132], [133]). (For some lucky moduli k Grosswald was able to handle all the cases and Stark did it for the least modulus $k = 5$ left open by Turán and Knapowski. Kátai improved some aspects of certain cases of the results of Turán and Knapowski. All these other authors use different methods with concessions in the way of effectiveness.)

Tchebycheff asserted

$$\sum_{p > 2} (-1)^{(p-1)/2} e^{-px} \to -\infty \quad (x \to +0),$$

which, if interpreted as meaning that there are more primes $\equiv 3 \pmod 4$ than primes $\equiv 1 \pmod 4$, would explain the above difficulty. In an attempt to generalize the result of Hardy and Littlewood and/or Landau, according to which Tchebycheff's assertion is equivalent to the non-vanishing, for $\operatorname{Re} s > 1/2$, of the non-principal L-function mod4, Turán and Knapowski found it more appropriate to consider instead

$$\sum_{p \equiv l_1 \pmod k} \log p\, e^{-a \log^2 px} - \sum_{p \equiv l_2 \pmod k} \log p\, e^{-a \log^2 px}$$
$$((l_1, k) = (l_2, k) = 1,\ x \to +0).$$

When $l_1 = 1$ and l_2 is a quadratic non-residue, this tends to $-\infty$ if and only if $L(s, \chi, k) \neq 0$ whenever $\chi(l_2) \neq 1$ and $\operatorname{Re} s > 1/2$ ([148]), but the general problem remains unsolved except for some small values of k.

As to the comparison with the main term of the prime number theorem (with the restriction to the case $k = 1$), Littlewood proved the infiniteness of sign changes of $\pi(x) - \operatorname{Li} x$, and only much later was Skewes able to give a numerical bound for the first sign change. Knapowski observed that Turán's method can also yield such effective bounds, even effective lower bounds for the number of sign changes up to x. They improved his bound to $c \log\log\log\log x$ ([229]) and recently Pintz obtained $c\sqrt{\log x}/\log\log x$.

These are not all the questions Turán and Knapowski were able to tackle but many more still remain intact. A favourite problem of Turán's was the simultaneous comparison of more than two residue classes; he made unsuccessful attempts in this direction. A general question is whether there exist arbitrarily large values of x with

$$\pi(x, k, l_1) > \pi(x, k, l_2) > \ldots > \pi(x, k, l_h)$$

where $\{l_i\}_{i=1}^{h}$ $(h = \varphi(k))$ form in an arbitrary order a representative system of the reduced residue classes mod k.

Other attempts for the zeta function. There is an extensive literature on inequalities in terms of the coefficients for the smallest and largest modulus of zeros of power polynomials; the role of the modulus is motivated by the fact that level sets of powers z^k are circles around the origin. This gave Turán the idea of considering, with a view to applying it to the zeta function where real parts of zeros feature, expansion in terms of Hermite polynomials having "flat" level sets. This has led to interesting results on strips containing the zeros of polynomials ([64], [72], [104], [143], the latter jointly with Makai) but not, so far, to number theory.

He found a surprising connection between Riemann's hypothesis and zeros of the partial sums $\sum_{n=1}^{N} 1/n^s$ ([33], [112], [115], [139]). In the simplest case, Riemann's hypothesis is true if there is no zero in the half plane $\operatorname{Re} s > 1 + c/\sqrt{N}$. Bateman and Chowla, Wiener and Wintner, Haselgrove, Levinson, Voronin, Montgomery and others have extended Turán's results and/or examined the hypotheses. Even though the latter have all now been disproved, especially by Montgomery, who has exhibited zeros much further to the right, considering the investigations it has inspired, the idea has certainly proved fruitful.

Turán related power sums to the zeta function in the reverse direction to that discussed previously. A slightly better lower bound than that due to Tijdeman for

$$\max_{1 \leqslant \nu \leqslant n^B} \Big|\sum_{i=1}^{n} b_i z_i^\nu\Big| \quad \Big(b_i > 0,\ \sum_{i=1}^{n} b_i = 1,\ |z_i| = 1,\ B > 1\Big)$$

would disprove the Riemann hypothesis. Examples by Erdös and Rényi, Tijdeman, Montgomery leave, however, very little hope.

A little remark, a reformulation of a theorem of Erdös on additive functions, which nevertheless shows the subject in a different light, is the following characterization of the zeta function ([112], [160]).

Let, formally,

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with an Euler product

$$f(s) = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \ldots\right)$$

and with monotone coefficients a_n. Then $f(s) = \zeta(s+c)$ where c is a real constant.

The point is that in contrast to other characterizations the functional equation can here be dispensed with. Both the Euler product representation and the monotoneness of the coefficients are actually used in the standard proofs for zero-free regions.

The twin prime and the Goldbach problem. Turán used the following approach, which had apparently not been applied before to twin primes. It also applies to other additive problems. ([154], [158], [167], [170], [173], [175].)

Consider

$$\sum_{n \leqslant x} \Lambda(n)\Lambda(n+2),$$

the main contribution coming (as conjectured) from terms where n and $n+2$ are both primes. Using the sieve formula

$$\Lambda(n+2) = -\sum_{d \mid n+2} \mu(d) \log d,$$

one gets

$$\sum_{d \leqslant x+2} \mu(d) \log d \sum_{\substack{n \equiv -2 \pmod d \\ n \leqslant x}} \Lambda(n).$$

Expressing the inner sum by contour integration in terms of L-functions, one obtains the same heuristic main term as had been obtained by Hardy and Littlewood in their classical investigations and the connection with zeros of L-functions discovered by Hardy and Littlewood in a roundabout way is brought to light here by an explicit formula. Turán observed the surprising fact that the problem only depends on zeros up to a fixed height. The Goldbach problem, which has always been thought of as being almost equivalent, is different: it depends also on higher zeros corresponding to a narrow interval of large moduli.

Inserting appropriate kernels in the above formula, as proved so successful in his investigations discussed above, one is led to a new type of power sum situation, for which no theorem is available at present.

Probabilistic theory of additive functions. The first achievement which made Turán famous was a very simple proof of the theorem of Hardy and Ramanujan to the effect that $\nu(n)$, the number of prime divisors of n is usually close to $\log\log n$ ([3], [4]). As we say it today using the terms of probability, he calculated the variance

$$\frac{1}{x}\sum_{n\leqslant x}\big(\nu(n)-\log\log x\big)^2 \ll \log\log x$$

and applied Tchebycheff's inequality. He generalized this to a wide class of additive functions in place of $\nu(n)$, extended further by Kubilius to a general inequality, called the Turán-Kubilius inequality, which is a useful tool for additive and multiplicative functions, and the idea itself has been used by many authors, including Turán himself (see the next section), for different questions in number theory. But its true significance lies in the fact that this was the starting point of probabilistic number theory, a major part of which consists in applying results and principles of probability to arithmetic functions.

The Erdös–Kac theorem, which is the central limit theorem for $\nu(n)$, states e.g. that

$$\frac{1}{x}\sum_{\substack{n\leqslant x\\ \frac{\nu(n)-\log\log x}{\sqrt{\log\log x}}<y}} 1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}\,du+o(1)$$

uniformly in y. In a joint paper with Rényi [95] Turán later gave $O\big(1/\sqrt{\log\log x}\big)$ for $o(1)$ as the precise uniform error estimation improving the earlier results of Le Veque and Kubilius. (This also follows from a result by A. Selberg, published earlier.) The proof is an analytic version of a method from Turán's dissertation [3] on the theorem of Hardy and Ramanujan. The result has been generalized by a great many authors.

Statistical theory of groups and partitions. A systematic theory is mainly the creation of Turán and Erdös ([155], [168], [171], [174], [176],

[184], [189], [196], [198], [205], [208], [217], [218], [222], [232], [233], [234], [236] including some papers written jointly with Dénes and Szalay).

The group they were mostly concerned with is S_n, the symmetric group of order n in which every finite group of n elements can be embedded.

They showed that the order $O(P)$ of almost all, i.e. all but $o(n!)$, elements $P\in S_n$ satisfies

$$\big|\log O(P)-\tfrac{1}{2}\log^2 n\big| < \omega(n)\log^{3/2}n$$

whenever $\omega(n)\to\infty$ ([155]). (Compare with

$$\max_{P\in S_n}\log O(P) = \big(1+o(1)\big)\sqrt{n\log n}$$

according to Landau.) This is sharp: $\log O(P)$ even has a Gaussian limit distribution with expectation $(1/2)\log^2 n$ and variance $(1/\sqrt{3})\log^{3/2}n$. They also examined the order, depending only on the conjugate classes, for almost all conjugate classes ([208]). They found this to be much higher, which made them investigate the number of elements in almost all conjugate classes, showing by precise formulae that the majority of conjugate classes cover only a small part of S_n.

The number-theoretic properties of $O(P)$ are often more important than its magnitude. Starting from a question by Schinzel, Turán and Erdös discovered the strange fact that, for almost all $P\in S_n$, $O(P)$ is divisible by every prime power not exceeding

$$\frac{\log n}{\log\log n}\left(1+3\,\frac{\log\log\log n}{\log\log n}-\frac{\omega(n)}{\log n}\right)$$

whenever $\omega(n)\to\infty$ and $\omega(n)$ cannot be replaced by $-\omega(n)$ ([171]). They also found the magnitude of the maximal prime divisor of $O(P)$ ([171]) and corresponding but different results for conjugate classes ([196]).

A special estimation of theirs has been used by Dixon for proving an old conjecture: pairs, chosen at random, of elements in S_n generate S_n with probability $3/4$.

The theory of S_n is closely related to partitions of n as $n = n_1+\ldots+n_k$ with integers as summands. Generalizing a result of Erdös and Lehner on the number of terms k in almost all partitions, Turán and Erdös extended this to sequences of real numbers other than the integers (the most important case of applications to S_n being the sequence of prime powers) ([198]) and for the case of the integers, with a view to applying it to the theory of representations of S_n Turán and Szalay gave an almost complete description not only of the number of terms but also of the distribution of their magnitude for almost all partitions ([232], [233], [234]).

Another strange phenomenon Turán encountered in answering a special case of a question by Dénes is that asymptotically at least half of the terms coincide in almost all pairs of partitions, generally $(1/k)$th of the terms for k tuples ([217], [218]).

Uniform distribution. This is another probabilistic theory where Turán in collaboration with Erdös did some basic work. They had the following quantitative form of Weyl's criterion for uniform distribution, improving earlier results by van der Corput and Koksma, unsuitable for their purpose:

For an arbitrary set of N real numbers x_n

$$\Big|\sum_{a<x_n<\beta (\mathrm{mod}\, 2\pi)} 1-\frac{\beta-a}{2\pi}N\Big| \leqslant c\left(\frac{N}{m}+\sum_{k=1}^{m}\frac{|s_k|}{k}\right),$$

where

$$s_k = \sum_{n=1}^{N} e^{ikx_n}$$

and c is an absolute constant ([38]).

This is a discrete version of the Barry–Esseen inequality of probability (obtained independently). It has been generalized for higher dimension and distributions other than uniform by Koksma, Szüsz, Elliott, Fainleib, Niederreiter and Philips and others.

The inequality, which has many applications, was originally devised for investigating the distribution of zeros of polynomials ([20], [38], [45]), a problem Turán was again led to by the zeta function. Turán and Erdös proved, among other things the rather precise general inequality

$$\Big|\sum_{a<\arg z_i<\beta} 1-\frac{\beta-a}{2\pi}n\Big| \leqslant 16\sqrt{n\log\frac{M}{|a_0 a_n|}},$$

where

$$M = \max_{|z|=1} |f(z)|$$

for any polynomial

$$f(z) = a_0 z^n + \ldots + a_n = a_0\prod_{i=1}^{n}(z-z_i)$$

([38]). Many other investigators have joined in this research.

Combinatorics. We briefly include here Turán's classical graph theorem ([24], [70]), which, strictly speaking, does not belong to number theory.

The special graph consisting of the first n integers as vertices, two joined by an edge if and only if they are incongruent mod$(k-1)$, contains no complete subgraph of k vertices. The theorem states that this is the unique graph of n vertices with this property having the maximal number of edges.

The theorem had great influence and created a rich theory on extremal graphs. Turán later found beautiful applications to discrete geometry and potential theory partly in joint work with Erdös, Meir and Vera T. Sós ([188], [194], [197], [199], [204], [206], [207]).

It is, perhaps, not irrelevant to conclude this review, which is far from complete, with the following conjecture Turán and Erdös made a long time ago: Every sequence of integers with positive upper density contains arbitrarily long arithmetic progressions ([9]). After the first result by Roth on progressions of three terms, the full conjecture has been proved recently by Szemerédi. This is the most celebrated result of combinatorial number theory. Fürstenberg, led by it to deep ergodic investigations, has given a proof by means of ergodic theory.

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