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On a theorem of Erdös and Fuchs

by

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Dedicated to the memory of P. Turán

Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be infinite sequences of integers such that

$$0 \leqslant a_1 < a_2 < \ldots \quad \text{and} \quad 0 \leqslant b_1 < b_2 < \ldots \tag{1}$$

If n is a non-negative integer, let $r(n, A, B)$ denote the number of ordered pairs (i, j) of positive integers such that

$$a_i + b_j \leqslant n.$$

Erdös and Fuchs have proved in [2] that if $a > 0$ then

$$r(n, A, A) = an + o\left(n^{1/4}(\log n)^{-1/2}\right) \tag{2}$$

cannot hold.

Bateman, Kohlbecker and Tull in [1] and Vaughan in [3] have extended the original result in various directions. In particular, they have investigated the case where the main term in (2) is an arbitrary "nice" function (in place of an). However, in all these generalizations, the case $A = B$ is considered. The aim of this paper is to investigate the more general case $A \neq B$. To simplify the discussion, we investigate this case here only when the main term (i.e., the function approximating $r(n, A, A)$) is an.

Let A and B denote the sequences consisting of the integers of the form

$$\sum_k \varepsilon_k 2^{2k} \quad (\text{where } \varepsilon_k = 0 \text{ or } 1)$$

and

$$\sum_k \varepsilon_k 2^{2k+1} \quad (\text{where } \varepsilon_k = 0 \text{ or } 1),$$

respectively. Then for $n = 0, 1, 2, \ldots$, the equation $a_i + b_j = n$ has exactly one solution; hence

$$r(n, A, B) = n+1$$

for all n. Thus an Erdös–Fuchs type Ω-estimate for $r(n, A, B) - an$ does not exist in this case.

As this example shows, in order to obtain an Ω-estimate for $r(n, A, B) - an$, we must have some restrictions on the sequences A, B, saying that these sequences are "near" in a certain sense. In fact, we show that if $|a_k - b_k|$ is small then we have the same Ω-estimate as in the theorem of Erdös and Fuchs:

THEOREM. *If $a > 0$ and the sequences $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ of non-negative integers satisfy* (1) *and*

$$a_k - b_k = o\big(a_k^{1/2} (\log a_k)^{-1}\big), \tag{3}$$

then

$$r(n, A, B) = an + o\big(n^{1/4} (\log n)^{-1/2}\big) \tag{4}$$

cannot hold.

Proof. Let us assume that the sequences A and B satisfy (1) and (3) but (4) holds; we are going to deduce a contradiction from this indirect assumption.

We use the following notations: $\varepsilon = \varepsilon(a)$ denotes a small positive real number whose magnitude depends on a (but it is independent of the parameter n); n denotes a large positive integer whose magnitude depends on a and ε, and we put $m = [\varepsilon n^{1/2} (\log n)^{-1}]$. We put $e(\beta) = e^{2\pi i\beta}$ (for real β), $r = 1 - 1/n$, $z = \operatorname{Re}\beta$,

$$F(z) = \sum_{j=1}^{+\infty} z^{a_j}, \qquad G(z) = \sum_{j=1}^{+\infty} z^{b_j},$$

$$t_n = r(n, A, B) - an \quad \text{and} \quad T(z) = \sum_{j=0}^{+\infty} t_j z^j.$$

Then we have

$$\frac{F(z)G(z)}{1-z} = \sum_{j=0}^{+\infty} r(j, A, B) z^j = \sum_{j=0}^{+\infty} (aj + t_j) z^j = \frac{az}{(1-z)^2} + T(z);$$

hence

$$F(z)G(z) = \frac{az}{1-z} + (1-z)T(z). \tag{5}$$

We will estimate the integral

$$J = \int_0^1 |F(z)G(z)| \left|\frac{1-z^m}{1-z}\right|^2 d\beta.$$

We write

$$\left|\frac{1-z^m}{1-z}\right|^2 = \sum_{j=0}^{m-1} r^{2j} + \sum_{k=1}^{m-1} \Big(r^k \sum_{j=0}^{m-1-k} r^{2j}\Big)\big(e(k\beta) + e(-k\beta)\big) = \sum_{k=-\infty}^{\infty} d_k e(k\beta).$$

Then $d_k \geqslant 0$ for all k (in particular, $d_k = 0$ for $|k| \geqslant m$); thus

$$J \geqslant \left| \int_0^1 F(z)\overline{G(z)} \left|\frac{1-z^m}{1-z}\right|^2 d\beta \right| \tag{6}$$

$$= \left| \int_0^1 \sum_{i=1}^{+\infty} r^{a_i} e(a_i\beta) \sum_{j=1}^{+\infty} r^{b_j} e(-b_j\beta) \sum_{k=-\infty}^{+\infty} d_k e(k\beta)\, d\beta \right|$$

$$= \sum_{a_i - b_j + k = 0} r^{a_i + b_j} d_k \geqslant \sum_{\substack{a_i \leqslant n \\ a_i - b_i + k = 0}} r^{a_i + b_i} d_k = \sum_{a_i \leqslant n} r^{2a_i + (b_i - a_i)} d_{b_i - a_i}.$$

For large n, $a_i \leqslant n$ implies by (3) that

$$|b_i - a_i| < \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1} \quad (< m < n).$$

Thus, writing $A(n) = \sum_{a_i \leqslant n} 1$, we infer from (6) that for large n

$$J \geqslant \sum_{a_i \leqslant n} r^{3n} \min_{|k| < \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1}} d_k \tag{7}$$

$$= \Big(\sum_{a_i \leqslant n} 1\Big) \cdot r^{3n} \min_{0 \leqslant k \leqslant \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1}} r^k \sum_{j=0}^{m-1-k} r^{2j}$$

$$\geqslant A(n) r^{3n} \min_{0 \leqslant k \leqslant \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1}} r^k \sum_{j=0}^{m-1-k} r^{2m}$$

$$= A(n) r^{3n} \min_{0 \leqslant k \leqslant \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1}} r^{2m+k} \sum_{j=0}^{m-1-k} 1$$

$$\geqslant A(n) r^{3n} \min_{0 \leqslant k \leqslant \frac{\varepsilon}{3} n^{1/2} (\log n)^{-1}} r^{3n^{1/2}} (m-k)$$

$$\geq A(n) r^{4n}\left([\varepsilon n^{1/2}(\log n)^{-1}] - \frac{\varepsilon}{3} n^{1/2}(\log n)^{-1}\right)$$

$$> A(n)\left(1-\frac{1}{n}\right)^{4n} \frac{\varepsilon}{2} n^{1/2}(\log n)^{-1} > A(n)\cdot \frac{1}{100}\cdot\frac{\varepsilon}{2} n^{1/2}(\log n)^{-1}$$

since

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{4n} = \frac{1}{e^4} > \frac{1}{100}.$$

(1), (3) and (4) imply that, writing also $B(n) = \sum_{b_j\leqslant n} 1$, for large n we have

(8) $$\frac{a}{2} n < r(n-n^{1/2}, A, B) = \sum_{a_i+b_j\leqslant n-n^{1/2}} 1$$

$$\leqslant \Big(\sum_{a_i\leqslant n-n^{1/2}} 1\Big)\Big(\sum_{b_j\leqslant n-n^{1/2}} 1\Big) = \Big(\sum_{a_i\leqslant n-n^{1/2}} 1\Big)\Big(\sum_{a_j\leqslant n-n^{1/2}+(a_j-b_j)} 1\Big)$$

$$\leqslant \Big(\sum_{a_i\leqslant n} 1\Big)\Big(\sum_{a_j\leqslant n} 1\Big) = (A(n))^2;$$

hence

$$A(n) > (a/2)^{1/2} n^{1/2}.$$

(7) and (8) yield

$$J > (a/2)^{1/2} n^{1/2}\cdot\frac{1}{100}\cdot\frac{\varepsilon}{2} n^{1/2}(\log n)^{-1} > \varepsilon\frac{a^{1/2}}{300} n(\log n)^{-1}.$$

In order to give an upper estimate for J, we start from (5) and use Cauchy's inequality, Parseval's formula, and also the indirect assumption (4):

(9)

$$J = \int_0^1\left|\frac{az}{1-z}+(1-z)T(z)\right|\left|\frac{1-z^m}{1-z}\right|^2 d\beta$$

$$\leqslant \int_0^1\left|\frac{az}{1-z}\right|\left|\frac{1-z^m}{1-z}\right|^2 d\beta + \int_0^1\left|(1-z^m)\frac{1-z^m}{1-z}T(z)\right| d\beta$$

$$< a\int_0^1 \frac{1}{|1-z|}\left|\sum_{j=0}^{m-1} z^j\right|^2 d\beta + \int_0^1 2\left|\frac{1-z^m}{1-z}\right| |T(z)|\, d\beta$$

$$\leqslant a\int_0^1 \frac{1}{|1-z|}\Big(\sum_{j=0}^{m-1}|z|^j\Big)^2 d\beta + 2\left\{\left(\int_0^1\left|\frac{1-z^m}{1-z}\right|^2 d\beta\right)\left(\int_0^1 |T(z)|^2 d\beta\right)\right\}^{1/2}$$

$$< a\int_0^1 \frac{1}{|1-z|}\Big(\sum_{j=0}^{m-1} 1\Big)^2 d\beta +$$

$$+2\left\{\left(\int_0^1 \sum_{k=-(m-1)}^{m-1} d_k e(k\beta)\, d\beta\right)\left(\int_0^1 \Big|\sum t_j r^j e(j\beta)\Big|^2 d\beta\right)\right\}^{1/2}$$

$$= am^2\int_0^1 \frac{1}{|1-z|} d\beta + 2\Big(d_0\sum_{j=0}^{+\infty} t_j^2 r^{2j}\Big)^{1/2}$$

$$= 2am^2 \int_0^{1/2}\frac{1}{|(1-r\cos 2\pi\beta) - ir\sin 2\pi\beta|} d\beta +$$

$$+2\left\{\Big(\sum_{j=0}^{m-1} r^{2j}\Big)\Big(\sum_{j=2}^{+\infty} o\big(j^{1/2}(\log j)^{-1}\big) r^{2j}\Big)\right\}^{1/2}.$$

For $0\leqslant\beta\leqslant 1/2$, we have

(10) $$|(1-r\cos 2\pi\beta) - ir\sin 2\pi\beta|$$

$$= (1+r^2-2r\cos 2\pi\beta)^{1/2}$$

$$= \{(1-r)^2+2r(1-\cos 2\pi\beta)\}^{1/2} = (n^{-2}+2r\cdot 2\sin^2\pi\beta)^{1/2}$$

$$\geqslant \big(n^{-2}+2\cdot\tfrac{1}{2}\cdot 2\cdot(2\beta)^2\big)^{1/2} = (n^{-2}+8\beta^2)^{1/2}$$

$$\geqslant \begin{cases} n^{-1} & \text{for} \quad 0\leqslant\beta\leqslant n^{-1} \\ 2\beta & \text{for} \quad n^{-1}\leqslant\beta\leqslant 1/2.\end{cases}$$

Furthermore,

(11) $$\sum_{j=2}^{+\infty} o\big(j^{1/2}(\log j)^{-1}\big) r^{2j} = O\Big(\sum_{2\leqslant j\leqslant n^{1/2}} j^{1/2}\Big) + o\Big(\sum_{n^{1/2}<j} j^{1/2}(\log j)^{-1} r^{2j}\Big)$$

$$= O\Big(\sum_{2\leqslant j\leqslant n^{1/2}} n^{1/4}\Big) + o\Big(\sum_{n^{1/2}<j} j^{1/2}(\log n^{1/2})^{-1} r^{2j}\Big)$$

$$= O(n) + o\Big((\log n)^{-1}\sum_{j=1}^{+\infty} j^{1/2} r^{2j}\Big)$$

$$= O(n) + o\big((\log n)^{-1}(1-r^2)^{-3/2}\big)$$

$$= O(n) + o\big((\log n)^{-1} n^{3/2}\big) = o\big(n^{3/2}(\log n)^{-1}\big)$$

since, writing

$$(1-x)^{-3/2} = 1+\sum_{j=1}^{+\infty} q_j x^j \quad (\text{where } |x|<1),$$

we have $q_j > cj^{1/2}$ for $j = 1, 2, \ldots$ and for some absolute constant c.

By (10) and (11), (9) implies that for large n

$$(12)\quad J < 2am^2\left(\int_0^{1/n}\frac{1}{n^{-1}}\,d\beta+\int_{1/n}^{1/2}\frac{1}{2\beta}\,d\beta\right)+2\left\{\left(\sum_{j=0}^{m-1}1\right)\varepsilon^3 n^{3/2}(\log n)^{-1}\right\}^{1/2}$$

$$< 2am^2\left(1+\frac{1}{2}\int_{1/n}^{1}\frac{1}{\beta}\,d\beta\right)+2\{m\varepsilon^3 n^{3/2}(\log n)^{-1}\}^{1/2}$$

$$< 2a\big(\varepsilon n^{1/2}(\log n)^{-1}\big)^2\big(1+\tfrac{1}{2}\log n\big)+2\Big\{\big(\varepsilon n^{1/2}(\log n)^{-1}\big)\varepsilon^3 n^{3/2}(\log n)^{-1}\Big\}^{1/2}$$

$$< 2a\varepsilon^2 n(\log n)^{-2}\log n+2\varepsilon^2 n(\log n)^{-1}$$

$$= (2a+1)\varepsilon^2 n(\log n)^{-1}.$$

(11) and (12) yield

$$\varepsilon\frac{a^{1/2}}{300} < (2a+1)\varepsilon^2,$$

but this cannot hold for sufficiently small ε (depending on a), and this contradiction proves our theorem.

References

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[2] P. Erdös and W. H. J. Fuchs, *On a problem of additive number theory*, J. London Math. Soc. 31 (1956), pp. 67–73.

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Some consequences of the Riemann hypothesis

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In memory of Paul Turán

The main object of this note is to show that the remainder term in the prime number theorem, assuming the Riemann hypothesis, can be reduced from

$$(1)\qquad \psi(x) = x+O(x^{1/2}\log^2 x)$$

to

$$(2)\qquad \psi(x) = x+O\big(x^{1/2}(\log\log x)^2\big),$$

except on a set of finite logarithmic measure.

We also give short proofs of Cramér's conditional estimates ([1], [2]) of the mean value of the remainder term

$$(3)\qquad \int_1^X (\psi(x)-x)^2\frac{dx}{x} \ll X,$$

and

$$(4)\qquad \int_1^X (\psi(x)-x)^2\frac{dx}{x^2} \sim C\log X.$$

It follows from (3) that for each function $\varphi = \varphi_x$ for which $\varphi_x \to \infty$ as $x \to \infty$, we have

$$\psi(x) = x+O(x^{1/2}\varphi_x)$$

for almost all x, i.e. except on a set whose intersection with the interval $[1, X]$ has measure $o(X)$. The proof that for $\varphi_x = (\log\log x)^2$ the exceptional set has finite logarithmic measure is a combination of the arguments which prove (1) and (3). A similar method yields a short proof of Selberg's