By (29), (30), (31), (32), (34) and (12) we obtain

$$A^+ = M^+(D^\varepsilon, \mathscr{P}, u)\{M^+(D, \tilde{\mathscr{P}}, z) + O(\varepsilon^3 + \varepsilon^{-6}K^3e^L(\log D)^{-1})\} + \\ + O(E(\varepsilon, D, K, L)V(z))$$

$$\leqslant V(u)\left\{F\left(\frac{1}{\varepsilon}\right) + O\left(e^{\sqrt{K}-\varepsilon^{-1}}(\log D)^{-1/3}\right)\right\} \times \\ \times \frac{V(z)}{V(u)}\{F(s) + O(\varepsilon^3 + \varepsilon^{-6}K^3e^L(\log D)^{-1})\} + O(E(\varepsilon, D, K, L)V(z))$$

$$< V(z)\{F(s) + E(\varepsilon, D, K, L)\}.$$

The proof of (27) is complete. Much the same arguments give the proof of (28).

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On the maximal order in S_n and S_n^*

by

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To the memory of Professor Paul Turán

1. In what follows we are dealing with the maximal order of the elements of S_n, the symmetric group on n letters, resp. of S_n^*, the symmetric semigroup on n letters.

Let $O(P)$ denote the order of the element P of S_n. E. Landau proved (see [2]) for

(1.1) $$G(n) = \max_{P \in S_n} O(P)$$

the asymptotical relation

(1.2) $$\lim_{n\to\infty} \frac{\log G(n)}{\sqrt{n\log n}} = 1.$$

Dealing with the value distribution of $O(P)$, in his paper [6], Professor P. Turán posed the problem of the analogue of (1.1)–(1.2) for S_n^*.

In our paper [5] we proved that for $\varepsilon > 0$ and $n \geqslant n_0(\varepsilon)$ the following relation holds

(1.3) $$\log G(n) = \sqrt{n(\log n + \log\log n + \delta(n))}$$

where

(1.4) $$-1 + \frac{\log\log n - 2 - \varepsilon}{\log n} < \delta(n) < \frac{1}{4}.$$

Meanwhile we got to know about a paper by J.-L. Nicolas. In that paper (see [3]) J.-L. Nicolas proved — among other things — the asymptotical relation

(1.5) $$\nu(G(n)) \sim 2\sqrt{\frac{n}{\log n}}$$

($\nu(k)$ stands for the number of different prime factors of k) and mentioned that S. M. Shah had proved the relation

$$\delta(n) = -1+o(1) \tag{1.6}$$

(writing it according to (1.3)) in [4].

In his paper [4] S. M. Shah really proved the estimate

$$\delta(n) = O(1), \tag{1.7}$$

which is somewhat weaker than (1.4). (The o-sign and the O-sign refer to $n\to\infty$.)

Combining S. M. Shah's proof with ours from [5], we can improve both results. Namely, in this paper we prove that

$$\delta(n) = -1+\frac{\log\log n-2+o(1)}{\log n} \tag{1.8}$$

(cf. Theorem I). By the way, Corollary 2 gives an estimate of similar exactness for p_n, the nth prime number.

Some arguments of ours from [5] are unnecessary for Theorem I, but will be of use for Theorem II, which improves J.-L. Nicolas' result (1.5).

Using a theorem of J. Dénes (see [1]) and Theorem I, we shall prove Theorem III, which asserts an analogue of the estimate (1.3)–(1.8) for the symmetric semigroup S_n^*.

2. Throughout this paper p stands for (positive) prime numbers. For different p's and positive integers ν_p let

$$G(x) \overset{\text{def}}{=} \max_{\Sigma p^{\nu_p}\leqslant x} \prod p^{\nu_p}. \tag{2.1}$$

Then, as was shown by E. Landau in [2],

$$G(n) = \max_{P\in S_n} O(P). \tag{2.2}$$

As is well known, for suitable $\alpha>0$ and arbitrary β with $0<\beta<3/5$ we have

$$\pi(x) = \sum_{p\leqslant x} 1 = \int_2^x \frac{du}{\log u} + O\big(x\cdot\exp(-\alpha\log^\beta x)\big) \tag{2.3}$$

and

$$\vartheta(x) = \sum_{p\leqslant x}\log p = x+O\big(x\cdot\exp(-\alpha\log^\beta x)\big), \tag{2.4}$$

where $\exp(v)$ stands for e^v.

Firstly, we assert

LEMMA 1. *For $z>0$ and fixed $s\geqslant 0$ we have*

$$\sum_{p\leqslant z} p^s = \int_2^{z^{s+1}} \frac{du}{\log u} + O\big(z^{s+1}\exp(-\alpha\log^\beta z)\big). \tag{2.5}$$

Proof.

$$\sum_{p\leqslant z} p^s = 2^s + \int_2^z t^s d\pi(t)$$

$$= 2^s + \int_2^z t^s d\left(\pi(t) - \int_2^t \frac{du}{\log u}\right) + \int_2^z \frac{t^s}{\log t}\,dt$$

$$= 2^s + \left[t^s\left(\pi(t) - \int_2^t \frac{du}{\log u}\right)\right]_2^z - \int_2^z \left(\pi(t)-\int_2^t\frac{du}{\log u}\right)dt^s + \int_{2^{s+1}}^{z^{s+1}} \frac{du}{\log u}$$

$$= z^s\left(\pi(z) - \int_2^z \frac{du}{\log u}\right) + O\big(z\cdot\exp(-\alpha\log^\beta z)\cdot z^s\big) + O(1) + \int_2^{z^{s+1}}\frac{du}{\log u}$$

$$= \int_2^{z^{s+1}} \frac{du}{\log u} + O\big(z^{s+1}\cdot\exp(-\alpha\log^\beta z)\big),$$

since $t\cdot\exp(-\alpha\log^\beta t)$ is monotonically increasing for $t\geqslant t_0(\alpha,\beta)$.

From Lemma 1 we get by partial integration

COROLLARY 1. *For $\varepsilon>0$, $z\geqslant z_0(\varepsilon)$ and fixed $s\geqslant 0$ we have the inequalities*

$$\frac{z^{s+1}}{\log z^{s+1}} + \frac{z^{s+1}}{\log^2 z^{s+1}} + \frac{2z^{s+1}}{\log^3 z^{s+1}} < \sum_{p\leqslant z} p^s < \frac{z^{s+1}}{\log z^{s+1}} + \frac{z^{s+1}}{\log^2 z^{s+1}} + \frac{(2+\varepsilon)z^{s+1}}{\log^3 z^{s+1}}. \tag{2.6}$$

3. Now, we assert

LEMMA 2. *For $\varepsilon>0$, $z>0$,*

$$z^{s+1} = x\left(\log x+\log\log x-1+\frac{\log\log x-2-\varepsilon}{\log x}\right), \tag{3.1}$$

$x\geqslant x_1(\varepsilon)$ and fixed $s\geqslant 0$ we have the inequality

$$\sum_{p\leqslant z} p^s < x. \tag{3.2}$$

Proof. Since $\log(1+t) > t - t^2/2$ holds for $0 < t < 1$, we have for $x \geqslant x_{11}(\varepsilon)$

$$\log z^{s+1} > \log x + \log\log x + \log\left(1 + \frac{\log\log x - 1}{\log x}\right)$$

$$> \log x + \log\log x + \frac{\log\log x - 1 - \varepsilon/2}{\log x} \overset{\text{def}}{=} K(x) > \log x.$$

Now, we infer from Corollary 1 for $x \geqslant x_1(\varepsilon) \geqslant x_{11}(\varepsilon)$ that

$$\sum_{p\leqslant z} p^s < \frac{z^{s+1}}{\log z^{s+1}} + \frac{z^{s+1}}{\log^2 z^{s+1}} + \frac{(2+\varepsilon/2)z^{s+1}}{\log^3 z^{s+1}}$$

$$< \frac{z^{s+1}}{K(x)}\left(1 + \frac{1 + \frac{2+\varepsilon/2}{K(x)}}{K(x)}\right)$$

$$< x\left(1 - \frac{1 + \frac{1+\varepsilon/2}{K(x)}}{K(x)}\right)\left(1 + \frac{1 + \frac{2+\varepsilon/2}{K(x)}}{K(x)}\right)$$

$$< x\left(1 - \frac{\left(1 + \frac{1+\varepsilon/2}{K(x)}\right)\left(1 + \frac{2+\varepsilon/2}{K(x)}\right) - 1}{K^2(x)}\right) < x. \ \blacksquare$$

Next, we prove

LEMMA 3. *For* $\varepsilon > 0$, $y > 0$,

$$y^{s+1} = x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2 + \varepsilon}{\log x}\right), \tag{3.3}$$

$x \geqslant x_2(\varepsilon)$ *and fixed* $s \geqslant 0$ *we have the inequality*

$$\sum_{p\leqslant y} p^s > x. \tag{3.4}$$

Proof. Since $\log(1+t) < t$ holds for $t > 0$, we have for $x \geqslant x_{21}(\varepsilon)$

$$\log y^{s+1} < \log x + \log\log x + \frac{\log\log x - 1 + \varepsilon/2}{\log x} \overset{\text{def}}{=} L(x) < (1+\varepsilon/2)\log x.$$

Now, we infer from Corollary 1 for $0 < \varepsilon < 2$ and $x \geqslant x_2(\varepsilon) \geqslant x_{21}(\varepsilon)$ that

$$\sum_{p\leqslant y} p^s > \frac{y^{s+1}}{\log y^{s+1}} + \frac{y^{s+1}}{\log^2 y^{s+1}} + \frac{2y^{s+1}}{\log^3 y^{s+1}} > \frac{y^{s+1}}{L(x)}\left(1 + \frac{1 + \frac{2}{L(x)}}{L(x)}\right)$$

$$> x\left(1 - \frac{1 + \frac{1-(\varepsilon/2)^2}{L(x)}}{L(x)}\right)\left(1 + \frac{1 + \frac{2}{L(x)}}{L(x)}\right)$$

$$= x\left(1 + \frac{1 + \left(\frac{\varepsilon}{2}\right)^2 - \left(1 + \frac{1-(\varepsilon/2)^2}{L(x)}\right)\left(1 + \frac{2}{L(x)}\right)}{L^2(x)}\right) > x. \ \blacksquare$$

Applying Lemmas 2 and 3 for $s = 0$, we get

COROLLARY 2.

$$p_n = n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2 + o(1)}{\log n}\right). \tag{3.5}$$

4. Now we can prove

THEOREM I. *With notation* (2.1) *we have the relation*

$$G(x) = \exp\left\{\sqrt{x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2 + o(1)}{\log x}\right)}\right\}. \tag{4.1}$$

Proof. In order to get a lower bound for $\log^2 G(x)$ we apply Lemma 2 for $s = 1$. Let $\varepsilon > 0$, $z > 0$, $x \geqslant x_1(\varepsilon/2)$ and

$$z^2 = x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2 - \varepsilon/2}{\log x}\right);$$

then (3.2) yields the inequality

$$\sum_{p\leqslant z} p < x.$$

Hence,

$$G(x) \geqslant \prod_{p\leqslant z} p;$$

therefore, using (2.4), we get

$$\log^2 G(x) \geqslant \Big(\sum_{p\leqslant z} \log p\Big)^2 = \big(\vartheta(z)\big)^2 = \big(z + O(z\cdot\exp(-\alpha\log^\beta z))\big)^2$$

$$= z^2 + O\big(z^2\cdot\exp(-\alpha\log^\beta z)\big) = z^2 + O\left(x\log x\,\frac{1}{\log^3 x}\right)$$

$$> x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2 - \varepsilon}{\log x}\right)$$

for $x \geqslant x_0(\varepsilon)$.

Turning to the proof of the upper estimation, let

$$p_2 + p_3 + \ldots + p_{k-1} < x \leqslant p_2 + p_3 + \ldots + p_k$$

and let $q_1, \ldots, q_j$ be arbitrary (different, positive) primes with the property

$$q_1+q_2+\ldots+q_j \leqslant x.$$

Then, as was shown by S. M. Shah (see [4], Lemma 4), we have

$$q_1\cdot q_2\cdot \ldots \cdot q_j < p_1\cdot p_2\cdot \ldots \cdot p_k. \tag{4.2}$$

Using (4.2) and estimating the contribution of the higher prime powers by E. Landau's theorem (1.2), he proved the following

LEMMA 4 (S. M. Shah, [4]). *Defining the integer m by*

$$\sum_{p\leqslant m} p \leqslant x < \sum_{p\leqslant m+1} p, \tag{4.3}$$

we obtain the inequality

$$\log G(x) \leqslant \vartheta(m)+O(x^{1/4}\log^{1/4}x). \tag{4.4}$$

Lemmas 2 and 3 yield relatively precise estimations for m. For $\varepsilon>0$, $x\geqslant x_2(\varepsilon/2)$ and

$$y=\sqrt{x\left(\log x+\log\log x-1+\frac{\log\log x-2+\varepsilon/2}{\log x}\right)}. \tag{4.5}$$

Lemma 3 gives $m<y$ and, consequently,

$$\vartheta(m)\leqslant\vartheta(y). \tag{4.6}$$

Owing to $\log y>\frac{1}{2}\log x$, $y=O(\sqrt{x\log x})$ and (2.4), we have

$$\vartheta(y)=y+O(y\log^{-3}y)=y+O(\sqrt{x}\log^{-5/2}x). \tag{4.7}$$

Now, we get from (4.4), (4.6) and (4.7) the estimate

$$\log G(x)\leqslant y+O(\sqrt{x}\log^{-5/2}x).$$

Hence,

$$\log^2 G(x)\leqslant y^2+O(x\log^{-2}x).$$

Finally,

$$\log^2 G(x)\leqslant x\left(\log x+\log\log x-1+\frac{\log\log x-2+\varepsilon/2}{\log x}\right)+O\left(\frac{x}{\log^2 x}\right)$$
$$\leqslant x\left(\log x+\log\log x-1+\frac{\log\log x-2+\varepsilon}{\log x}\right)$$

for $x\geqslant x_3(\varepsilon)$. Thus Theorem I is completely proved.

5. Turning to the investigation of $\nu(G(n))$ we assert

THEOREM II. *For $n>n_0$ we have the relation*

$$\nu(G(n))=2\sqrt{\frac{n}{\log n}\left(1-\frac{\log\log n}{\log n}+\frac{\eta(n)}{\log n}\right)}, \tag{5.1}$$

where

$$1.75<\eta(n)<3.05. \tag{5.2}$$

Proof. We have

$$G(n)=\max_{\sum p^{\nu_p}\leqslant n}\prod p^{\nu_p}. \tag{5.3}$$

Let $k=\nu(G(n))$. Then we write the maximizing product as $\prod_k p^{\nu_p}$, resp. the condition as $\sum_k p^{\nu_p}\leqslant n$. We obviously have

$$\prod_k p^{\nu_p}\leqslant\left(\frac{\sum_k p^{\nu_p}}{k}\right)^k\leqslant\exp\{k(\log n-\log k)\}. \tag{5.4}$$

First, let us suppose that

$$k\leqslant 2\sqrt{\frac{n}{\log n}\left(1-\frac{\log\log n}{\log n}+\frac{1.75}{\log n}\right)}. \tag{5.5}$$

Since $k(\log n-\log k)$ is monotonically increasing in k for $1\leqslant k\leqslant n/e$, we get for sufficiently large n

$$k(\log n-\log k)$$
$$\leqslant 2\sqrt{\frac{n}{\log n}\left(1-\frac{\log\log n-1.75}{\log n}\right)}\cdot\Big\{\log n-\log 2-$$
$$-\tfrac{1}{2}\left(\log n-\log\log n+\log\left(1-\frac{\log\log n-1.75}{\log n}\right)\right)\Big\}$$
$$<\sqrt{\frac{n}{\log n}\left(1-\frac{\log\log n-1.75}{\log n}\right)}\cdot\{\log n+\log\log n-1.379\}$$
$$=\left\{n(\log n+\log\log n-1.379)\left(1-\frac{\log\log n-1.75}{\log n}\right)\left(1+\frac{\log\log n-1.379}{\log n}\right)\right\}^{1/2}$$
$$<\left\{n(\log n+\log\log n-1.379)\left(1+\frac{0.371}{\log n}\right)\right\}^{1/2}$$
$$<\left\{n\left(\log n+\log\log n-1.379+0.371+\frac{0.371\log\log n}{\log n}\right)\right\}^{1/2}$$
$$<\sqrt{n(\log n+\log\log n-1)}.$$

Therefore, using Theorem I we have

$$k(\log n - \log k) < \log G(n)$$

and (from (5.4))

$$\prod_k p^{\nu_p} < G(n),$$

in contradiction with the maximization. Thus the inequality $\eta(n) > 1.75$ holds.

Turning to the upper estimation, we infer from the condition $\sum_k p^{\nu_p} \leqslant n$ that

(5.6) $$n \geqslant \sum_k p^{\nu_p} \geqslant \sum_k p \geqslant \sum_{\mu=1}^{k} p_\mu = \sum_{p\leqslant p_k} p.$$

But Lemma 3 yields the inequality

(5.7) $$\sum_{p\leqslant y} p > n$$

for $n > n_2(\varepsilon)$ and

(5.8) $$y = \sqrt{n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2 + \varepsilon}{\log n}\right)}.$$

Now, it follows from (5.6) and (5.7) that

(5.9) $$p_k < y.$$

Further, Corollary 2 implies for $k \geqslant k_0(\varepsilon)$ that

(5.10) $$p_k \geqslant k\left(\log k + \log\log k - 1 + \frac{\log\log k - 2 - \varepsilon}{\log k}\right).$$

Now, let us suppose that

(5.11) $$k \geqslant 2\sqrt{\frac{n}{\log n}\left(1 - \frac{\log\log n}{\log n} + \frac{3.05}{\log n}\right)}.$$

Then, for sufficiently large n, we have

$$p_k > k(\log k + \log\log k - 1),$$

$$\log k \geqslant \log 2 + \tfrac{1}{2}\left\{\log n - \log\log n + \log\left(1 - \frac{\log\log n - 3.05}{\log n}\right)\right\}$$

$$> \tfrac{1}{2}(\log n - \log\log n),$$

$$\log\log k > -\log 2 + \log\log n + \log\left(1 - \frac{\log\log n}{\log n}\right)$$

and

$$\log k + \log\log k - 1 > \tfrac{1}{2}(\log n + \log\log n - 2.01).$$

Consequently,

$$p_k > \left\{n(\log n + \log\log n - 2.01)\cdot\left(1 + \frac{\log\log n - 2.01}{\log n}\right) \times \right.$$
$$\left. \times \left(1 - \frac{\log\log n - 3.05}{\log n}\right)\right\}^{1/2}$$
$$> \left\{n(\log n + \log\log n - 2.01)\left(1 + \frac{1.03}{\log n}\right)\right\}^{1/2}$$
$$> \{n(\log n + \log\log n - 1 + 0.02)\}^{1/2} > y$$

for $n > n_3$.

Thus we have (for sufficiently large n) the inequality

$$p_k > y,$$

in contradiction with (5.9). Hence, also the inequality $\eta(n) < 3.05$ holds and Theorem II is completely proved.

6. Now we are going to prove an analogue of Theorem I for the symmetric semigroup S_n^*.

S_n^* consists of all mappings of $X_n = \{x_1, x_2, \ldots, x_n\}$ into X_n. If $\alpha, \beta \in S_n^*$, then the product $\alpha\beta \in S_n^*$ is defined by $(\alpha\beta)(x) = \alpha(\beta(x))$ for all $x \in X_n$.

As is known, for each $\alpha \in S_n^*$ we can divide X_n into two classes: of cyclical and non-cyclical elements. An $x \in X_n$ is said to be *cyclical* under α if there is an $m > 0$ with $\alpha^m(x) = x$. Let C_α denote the set of the cyclical elements under α. Since $|X_n| = n$, for fixed $x \in X_n$ the elements

$$x = \alpha^0(x),\ \alpha(x),\ \alpha^2(x),\ \ldots,\ \alpha^n(x)$$

cannot be all different. Thus, there exist integers i, j such that

(6.1) $$0 \leqslant i < j \leqslant n$$

and

(6.2) $$\alpha^i(x) = \alpha^j(x) = \alpha^{j-i}(\alpha^i(x)).$$

Consequently, C_α is not empty; further, for arbitrary $\alpha \in S_n^*$ and $x \in X_n$ there exists an integer i such that

(6.3) $$0 \leqslant i \leqslant n-1, \quad \alpha^i(x) \in C_\alpha.$$

Also, there is a least integer $r = r_\alpha(x) \geqslant 0$ such that $\alpha^r(x) \in C_\alpha$. This r is called the α-height of x.

The height of α is defined by

(6.4) $$h(\alpha) = \max_{x\in X_n} r_\alpha(x).$$

For this $h(\alpha)$, (6.3) yields the useful inequality

(6.5) $$h(\alpha) \leqslant n-1.$$

For fixed $\alpha \in S_n^*$, let $C_\alpha = \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$. Then for each j with $1 \leqslant j \leqslant t$ there exists an $m_j > 0$ such that

(6.6) $$\alpha^{m_j}(x_{i_j}) = x_{i_j};$$

consequently,

$$\alpha^{m_j}\alpha(x_{i_j}) = \alpha(x_{i_j}).$$

Thus we have for $1 \leqslant j \leqslant t$

(6.7) $$\alpha(x_{i_j}) \in C_\alpha.$$

Let us define P_α by

(6.8) $$P_\alpha(x) = \begin{cases} \alpha(x), & \text{if } x \in C_\alpha, \\ x, & \text{if } x \notin C_\alpha \end{cases}$$

for all $x \in X_n$. Owing to (6.6) and (6.7) we get $P_\alpha \in S_n$.

In his paper [1], J. Dénes has called the restriction α^* of α to C_α the main permutation of α. It is obvious from (6.6), (6.7) and (6.8) that α^* is the restriction of P_α to C_α and

(6.9) $$O(\alpha^*) = O(P_\alpha).$$

Thus we can write P_α instead of α^* in the following theorem of J. Dénes. For $\alpha \in S_n^*$, he defined the order of α, $O(\alpha)$, as the number of distinct elements of S_n^* in the set $\{\alpha, \alpha^2, \alpha^3, \ldots\}$ and proved the following

LEMMA 5 (J. Dénes [1]). *For $\alpha \in S_n^*$, $O(\alpha)$ is the least integer m for which there exists an integer q such that $0 < q \leqslant m$ and $\alpha^q = \alpha^{m+1}$. Further,*

(6.10) $$O(\alpha) = O(P_\alpha) + \max\{0, h(\alpha)-1\}.$$

7. For our Theorem III it is enough to prove

LEMMA 6. *For $\alpha \in S_n^*$ we have the inequality*

(7.1) $$O(\alpha) < O(P_\alpha) + n.$$

Proof. (7.1) is an immediate consequence of Lemma 5 and (6.5).

Now, we can prove

THEOREM III.

(7.2) $$\max_{\alpha\in S_n^*} O(\alpha) = \exp\left\{\sqrt{n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2 + o(1)}{\log n}\right)}\right\}.$$

Proof. Owing to $S_n \subset S_n^*$, the lower estimate follows from Theorem I. In order to prove the upper estimate we use Lemma 6 and Theorem I as follows.

For sufficiently large n

$$O(\alpha) < O(P_\alpha) + n \leqslant \max_{P\in S_n} O(P) + n = G(n) + n$$
$$< G(n)\{1 + n\exp(-\sqrt{n})\} < G(n)\exp\{n\exp(-\sqrt{n})\};$$

hence,

$$\log O(\alpha) < \log G(n) + n\exp(-\sqrt{n}) < \log G(n) + n^{-1} = \left(\log^2 G(n) + o(1)\right)^{1/2}$$
$$= \sqrt{n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2 + o(1)}{\log n}\right)}$$

and Theorem III is completely proved.

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