wo

(13) $$\varrho_1 = \varrho + w + w_1$$

ist.

Zum Beweis benützen wir das Additionstheorem der Gammafunktion (vgl. Rademacher, loc. cit., S. 55 (28 61))

(14) $$\frac{\Gamma(j)}{(1+u)^j} = \int_{(\sigma)} \Gamma(j-t)\Gamma(t)u^{-t}dt$$

wobei $|\operatorname{arc} u| < \pi$ sei. Wir nehmen, wenn $j > 1$, $u = (\varrho + i(ha+k))/(w-\varrho)$, dann erhalten wir da § 1 (5) auch für $\sigma < 1$ gilt sofort (9), da alle auftretenden Reihen absolut und gleichmäßig konvergieren. Im Falle $0 < j < r/\mu$ nehmen wir außer dem oben definierten u noch

$$u_1 = (\varrho_1 + i(ha+k))/(w_1 - \varrho)$$

und wieder $\sigma < j$ und erhalten (12).

Es ist ja z.B. wenn man in (14) einsetzt

$$(w + i(ha+k))^{-j} = \int_{(\sigma)} C(j, t)\big((\varrho + i(ha+k))^{-t}\big)dt.$$

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A new form of the error term in the linear sieve

by

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To the memory of Professor P. Turán

1. Introduction. Let $\mathscr{A}$ be a finite sequence of integers and let $\mathscr{P}$ be a set of primes. One of the fundamental problems in sieve theory is to estimate from above and from below the so-called sifting function $S(\mathscr{A}, \mathscr{P}, z)$ which represents the number of elements in $\mathscr{A}$ that have no prime factors $p < z$ in $\mathscr{P}$. Letting

$$P(z) = \prod_{p<z, p\in\mathscr{P}} p$$

one can write

$$S(\mathscr{A}, \mathscr{P}, z) = \big|\{a \in \mathscr{A};\ (a, P(z)) = 1\}\big|.$$

In general theory the sequence $\mathscr{A}$ can be almost arbitrary. The relevant information that we need about $\mathscr{A}$ is a good approximation formula (in an average sense) for the quantity

$$|\mathscr{A}_d| = |\{a \in \mathscr{A};\ a \equiv 0 \,(\operatorname{mod} d)\}|$$

which represents the number of elements in $\mathscr{A}$ that are divisible by a squarefree number $d|P(z)$. It is supposed, what frequently turns out to take place in practice, that every $|\mathscr{A}_d|$ may be written in the form

$$|\mathscr{A}_d| = \frac{\omega(d)}{d} X + r(\mathscr{A}, d)$$

where $\omega(d)$ is multiplicative and $0 \leqslant \omega(p) < p$ for $p \in \mathscr{P}$, X is some positive number independent of d and $r(\mathscr{A}, d)$ is considered as an error term, small on average (so X approximates to $|\mathscr{A}|$).

If $\omega(p)$ is bounded on average, say by $\varkappa$, we then deal with $\varkappa$ dimensional sieve. In literature there are multitude of ways in which this fact can be expressed. In this paper we shall assume two inequalities

(1) $$\prod_{\substack{w\leqslant p<z\\ p\in\mathcal{P}}}\left(1-\frac{\omega(p)}{p}\right)^{-1}\leqslant\left(\frac{\log z}{\log w}\right)^{\varkappa}\left(1+\frac{K}{\log w}\right),$$

(2) $$\sum_{\substack{w\leqslant p<z\\ p\in\mathcal{P}}}\sum_{a\geqslant 2}\frac{\omega(p^a)}{p^a}\leqslant\frac{L}{\log 3w}$$

which are to hold for all $z>w\geqslant 2$ with some constants K, $L>1$. Let D be any parameter >1, $s=\log D/\log z$ and

$$V(z)=\prod_{p|P(z)}\left(1-\frac{\omega(p)}{p}\right).$$

The sieve estimates of $S(\mathcal{A},\mathcal{P},z)$ have usually the following form

(3) $$S(\mathcal{A},\mathcal{P},z)\leqslant V(z)X\{F(s)+A(\varkappa,s,K,D)\}+R(\mathcal{A},D),$$

(4) $$S(\mathcal{A},\mathcal{P},z)\geqslant V(z)X\{f(s)-A(\varkappa,s,K,D)\}-R(\mathcal{A},D)$$

where the functions $F(s)$ and $f(s)$ depend on the dimension $\varkappa$ and satisfy $0\leqslant f(s)<1<F(s)$. The first error term $A(\varkappa,s,K,D)$ tends to zero as s or D approaches infinity and $\varkappa$ and K remain constant. The second error term $R(\mathcal{A},D)$ is defined by

(5) $$R(\mathcal{A},D)=\sum_{d<D,\,d|P(z)}|r(\mathcal{A},d)|.$$

Sometimes the sum (5) is weighted by the factor $C^{\Omega(d)}$ with some constant $C>1$. Since $V(z)X$ should be the order of magnitude of $S(\mathcal{A},\mathcal{P},z)$ the second error term $R(\mathcal{A},D)$ is required to satisfy

(6) $$R(\mathcal{A},D)=o\big(V(z)X\big)\quad\text{as}\quad X\to\infty.$$

Note that every single error term $r(\mathcal{A},d)$ appears in $R(\mathcal{A},D)$ in absolute value, thus there cannot be any cancellation of the errors. For this reason every D satisfying (6) may be called the natural level of distribution of $\mathcal{A}$ in arithmetic progressions. The functions $F(s)$ and $f(s)$ monotonically converge to 1, therefore the larger level of distribution D we have the better the results we shall obtain. In general case the best functions $F(s)$ and $f(s)$ are not known. In the very important case of linear sieve ($\varkappa=1$) Jurkat and Richert [6] have proved (3) and (4) with optimal functions $F(s)$ and $f(s)$ in the sense that there exist sequences $\mathcal{A}$ of the length X such that neither (3) nor (4) would hold as $X\to\infty$ if $F(s)$ and $f(s)$ were replaced by smaller and larger functions of s respectively. Their functions $F(s)$ and $f(s)$ are the continuous solutions of the following system of differential-difference equations

$$\begin{aligned} sF(s)&=2e^C &&\text{if}\quad 0<s\leqslant 3,\\ sf(s)&=0 &&\text{if}\quad 0<s\leqslant 2,\\ \big(sF(s)\big)'&=f(s-1) &&\text{if}\quad s>3,\\ \big(sf(s)\big)'&=F(s-1) &&\text{if}\quad s>2,\end{aligned}$$

where $C=.577\ldots$ is the Euler constant.

The main purpose of this paper is to show a result which in many interesting problems enables one to apply (3) or (4) with a larger D than the natural level of distribution, thus in that way improving the Jurkat and Richert results. It becomes possible due to a new more flexible form of the error term $R(\mathcal{A},D)$.

THEOREM 1. *Let $0<\varepsilon<1/3$, $M>1$, $N>1$, $D=MN$. If* (1) *and* (2) *hold then for all $2\leqslant z\leqslant D^{1/2}$ we have*

(7) $$S(\mathcal{A},\mathcal{P},z)\leqslant V(z)X\{F(s)+E(\varepsilon,D,K,L)\}+R^{+}(\mathcal{A},M,N),$$

(8) $$S(\mathcal{A},\mathcal{P},z)\geqslant V(z)X\{f(s)-E(\varepsilon,D,K,L)\}-R^{-}(\mathcal{A},M,N)$$

where $s=\log D/\log z$, $E(\varepsilon,D,K,L)\ll\varepsilon+\varepsilon^{-8}e^{K+L}(\log D)^{-1/3}$ and for $\nu=\pm$

(9) $$R^{\nu}(\mathcal{A},M,N)=\sum_{l<\exp(8\varepsilon^{-3})}\sum_{\substack{m<M\\ m|P(z)}}\sum_{\substack{n<N\\ n|P(z)}}a^{\nu}_{m,l}(M,N,\varepsilon)\,b^{\nu}_{n,l}(M,N,\varepsilon)\,r(\mathcal{A},mn).$$

The coefficients $a^{\nu}_{m,l}$ and $b^{\nu}_{n,l}$ depend at most on M, N, ε and they are bounded by 1 *in absolute value. Moreover, the estimates* (7) *and* (8) *will remain true if the variables of summation m and n in the remainder term $R^{\nu}(\mathcal{A},M,N)$ are assumed to satisfy $mn|P(z)$.*

Throughout this paper all constants implied by the symbols $\ll$ and O will be absolute.

There are several methods for working with $R^{\nu}(\mathcal{A},M,N)$ leading to (6) with values of MN larger than those possible for the conventional $R(\mathcal{A},MN)$. One way of treating $R^{\nu}(\mathcal{A},M,N)$ is related to an idea of expanding every single error term $r(\mathcal{A},mn)$ into Fourier series. On application of Cauchy–Schwarz's inequality one can change the coefficients a^{ν}_m and b^{ν}_n and one arrives then at exponential sums which can be estimated by various methods familiar from the analytic number theory. Other method is based on expressing $r(\mathcal{A},d)$ by the Perron integral of Dirichlet's generating function for the sequence $\mathcal{A}_d$. The double sum $\sum_{m,n}$ is then the integral of the product of three generating functions for the sequences $\mathcal{A}$, (a_m) and (b_n) respectively. Since the lenghts M and N of the generating functions for (a_m) and (b_n) are at our disposal one can very effectively apply the mean value theorem or the Halász–Montgomery–Huxley inequality for Dirichlet's polynomials. Still another approach to $R^{\nu}(\mathscr{A}, M, N)$ is offered by Linnik's dispersion method.

As an application we present the following two results (the proofs will appear elsewhere).

THEOREM 2. *For any irreducible polynomial* $g(n) = an^2+bn+c$ *with* $a > 0$ *and* $c \equiv 1 \pmod 2$ *there exist infinitely many integers* n *such that* $g(n)$ *has at most two prime factors.*

THEOREM 3. *For any* $\varepsilon > 0$ *and all* $x > x_0(\varepsilon)$ *we have*

$$\pi(x; q, a) < (2+\varepsilon)\frac{x}{\varphi(q)\log D(x, q)}$$

where

$$D(x, q) = \begin{cases} xq^{-1/2} & \text{if} \quad 1 < q < x^{1/2-\varepsilon}, \\ xq^{-3/8} & \text{if} \quad 1 < q < x^{8/19-\varepsilon}, \\ x^{4/3}q^{-3/2} & \text{if} \quad x^{1/3} < q < x^{2/3-\varepsilon}. \end{cases}$$

Moreover, on the Lindelöf conjecture $L(\frac{1}{2}+it, \chi) \ll (|t|+1)q^{\varepsilon}$ *one can take*

$$D(x, q) = x \quad \text{if} \quad 1 < q < x^{1/2-\varepsilon}.$$

Also, if the Hooley conjecture (see [3]) *concerning incomplete Kloosterman's sums is true, then one can take*

$$D(x, q) = (x/q)^{6/5} \quad \text{if} \quad x^{4/9} < q < x^{1-\varepsilon}.$$

In the paper we shall prove more precise inequalities than (7) and (8). Theorem 1 will follow as a corollary. That will have significance in some applications, for example in the problem of the difference between consecutive primes.

For a given integer $R \geqslant 1$ let us denote

$$W_R^+(\mathscr{A}, \mathscr{P}, z) = S(\mathscr{A}, \mathscr{P}, z) + \sum_{r=0}^{R} \sum_{w^+(p_1, \ldots, p_{2r+1})} S(\mathscr{A}_{p_1\ldots p_{2r+1}}, \mathscr{P}, p_{2r+1})$$

and

$$W_R^-(\mathscr{A}, \mathscr{P}, z) = S(\mathscr{A}, \mathscr{P}, z) - \sum_{r=1}^{R} \sum_{w^-(p_1, \ldots, p_{2r})} S(\mathscr{A}_{p_1\ldots p_{2r}}, \mathscr{P}, p_{2r})$$

where the symbol $w^+(p_1, \ldots, p_{2r+1})$ indicates that the summation is over the primes $p_1, \ldots, p_{2r+1}$ from $\mathscr{P}$ satisfying the simultaneous conditions

$$(w^+) \quad \begin{aligned} & p_{2r+1} < \ldots < p_1 < z, \\ & p_{2l+1}^3 p_{2l} \ldots p_1 < D \text{ for all } l < r, \\ & p_{2r+1}^3 p_{2r} \ldots p_1 \geqslant D \end{aligned}$$

and the symbol $w^-(p_1, \ldots, p_{2r})$ indicates that the summation is over the primes $p_1, \ldots, p_{2r}$ from $\mathscr{P}$ satisfying the simultaneous conditions

$$(w^-) \quad \begin{aligned} & p_{2r} < \ldots < p_1 < z, \\ & p_{2l}^3 p_{2l-1} \ldots p_1 < D \text{ for all } l < r, \\ & p_{2r}^3 p_{2r-1} \ldots p_1 \geqslant D. \end{aligned}$$

The notation $W^{\pm}(\mathscr{A}, \mathscr{P}, z)$ will stand for $W_{\infty}^{\pm}(\mathscr{A}, \mathscr{P}, z)$.

Let $0 < \varepsilon < 1/3$ and $D \geqslant 2$. Define $\eta = \varepsilon^9$,

$$\begin{aligned} \mathscr{G} &= \{D^{\varepsilon^2(1+\eta)^n};\ n \geqslant 0\}, \\ \mathscr{H} &= \{(D_1, \ldots, D_r);\ r \geqslant 1,\ D_l \in \mathscr{G} \text{ for } 1 \leqslant l \leqslant r,\ D_r \leqslant \ldots \leqslant D_1 < D^{1/2}\}, \\ \mathscr{D}^+ &= \{(D_1, \ldots, D_r) \in \mathscr{H};\ D_1 \ldots D_{2l}D_{2l+1}^3 < D \text{ for all } 0 \leqslant l \leqslant (r-1)/2\}, \\ \mathscr{D}^- &= \{(D_1, \ldots, D_r) \in \mathscr{H};\ D_1 \ldots D_{2l-1}D_{2l}^3 < D \text{ for all } 1 \leqslant l \leqslant r/2\}. \end{aligned}$$

Note that $|\mathscr{D}^{\nu}| < \exp(8\varepsilon^{-3})$, $\nu = \pm$.

THEOREM 4. *Let* $R \geqslant 1$, $0 < \varepsilon < 3^{-R}$ *and* $D \geqslant 2$. *If* (1) *and* (2) *hold then*

$$W_R^+(\mathscr{A}, \mathscr{P}, z) \leqslant V(z)X\{F(s)+E\} + R_1^+ + R^+ \quad \text{if} \quad z \leqslant D, \tag{10}$$

$$W_R^-(\mathscr{A}, \mathscr{P}, z) \geqslant V(z)X\{f(s)-E\} - R_1^- - R^- \quad \text{if} \quad z \leqslant D^{1/2}, \tag{11}$$

where $s = \log D/\log z$, $E = E(\varepsilon, D, K, L) \ll \varepsilon + \varepsilon^{-8}e^{K+L}(\log D)^{-1/3}$,

$$R_1^{\nu} = \sum_{\substack{d < D^{\varepsilon} \\ d \mid P(D^{\varepsilon^2})}} \varphi_d^{\nu}(D^{\varepsilon})r(\mathscr{A}, d),$$

$$R^{\nu} = \sum_{(D_1, \ldots, D_r) \in \mathscr{D}^{\nu}} \sum_{\substack{d < D^{\varepsilon} \\ d \mid P(D^{\varepsilon^2})}} \lambda_d^{\nu}(\varepsilon, D_1, \ldots, D_r)H_d(\mathscr{A}, \mathscr{P}, z; \varepsilon, D_1, \ldots, D_r)$$

with some coefficients $\varphi_d^{\nu}(D^{\varepsilon})$ *and* $\lambda_d^{\nu}(\varepsilon, D_1, \ldots, D_r)$ *bounded by* 1 *in absolute value and*

$$H_d(\mathscr{A}, \mathscr{P}, z; \varepsilon, D_1, \ldots, D_r) = \sum_{\substack{D_1 \leqslant p_1 < D_1^{1+\eta} \\ p_1 \mid P(z)}} \ldots \sum_{\substack{D_r \leqslant p_r < D_r^{1+\eta} \\ p_r \mid P(z) \\ p_i \neq p_j \text{ for } (i,j) \in I(D_1, \ldots, D_r)}} r(\mathscr{A}, dp_1 \ldots p_r).$$

Here $I(D_1, \ldots, D_r)$ *is an arbitrarily chosen (not necessarily complete) set of pairs of indices* (i, j) *with* $i \neq j$, $1 \leqslant i, j \leqslant r$.

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2. Proof of Theorem 1. In this section we shall derive Theorem 1 from Theorem 4.

Every sequence $(D_1, \ldots, D_r)$ in $\mathscr{D}^{\nu}$, $\nu = \pm$ will be called *ν-admissible.* The empty sequence will also be considered as a ν-admissible.

LEMMA 1. *Every ν-admissible sequence $(D_1, \dots, D_r)$ is a union of two disjoint ν-admissible sequences $(M_1, \dots, M_s)$ and $(N_1, \dots, N_t)$ such that $M_1 \dots M_s \leqslant M$ and $N_1 \dots N_t \leqslant N$ subject to $M > 1$, $N > 1$, $MN = D$ and $s+t = r$.*

Proof. If $r = 1$ then the result is trivial because $D_1 \leqslant D^{1/2}$ and either M or N is $\geqslant D^{1/2}$. Assume that $r \geqslant 2$. Since the sequence $(D_1, \dots, D_{r-1})$ is ν-admissible we can write it, by induction hypothesis, as a union of two sub-sequences $(M_1, \dots, M_{s'})$ and $(N_1, \dots, N_{t'})$ such that $M_1 \dots M_{s'} \leqslant M$ and $N_1 \dots N_{t'} \leqslant N$. But $D_1 \dots D_{r-1} D_r^2 \leqslant D$, thus either $M_1 \dots M_{s'} D_r \leqslant M$ or $N_1 \dots N_{t'} D_r \leqslant N$. Therefore one of the above sequences can be extended by D_r giving the desired decomposition of $(D_1, \dots, D_r)$.

By Theorem 4 and Lemma 1 we deduce that (7) and (8) hold for $z \leqslant D$ and $z \leqslant D^{1/2}$ respectively with the remainder terms $R^{\pm}(\mathscr{A}, D^{\varepsilon} M^{1+\eta}, N^{1+\eta})$. Assume that $M \geqslant N$. Then reinterpreting the parameters D, M, N by $D^{1+\varepsilon+\eta}$, $M^{1+\eta} D^{\varepsilon}$, $N^{1+\eta}$, respectively we find that s goes into $(1+\varepsilon+\eta)s$ and hence that (7) holds for $s \geqslant 1+\varepsilon+\eta$, *a fortiori*, for $s \geqslant 2$ while (8) holds for $s \geqslant 2(1+\varepsilon+\eta)$. Notice that the variation of the argument s in the main terms $f(s)$ and $F(s)$ does not matter because of the error $E(\varepsilon, K, L, D)$. To prove (8) for all $z \leqslant D^{1/2}$ we appeal to Buchstab's identity. Letting $\Delta = D^{1/2(1+\varepsilon+\eta)}$, $z_0 = \min(\Delta, z)$ we derive

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant S(\mathscr{A}, \mathscr{P}, z_0) - \sum_{\Delta \leqslant p < D^{1/2}, p|P(z)} S(\mathscr{A}_p, \mathscr{P}, D^{1/4})$$

$$\geqslant V(z) X \{f(s) - E\} - \sum_{1 \leqslant l < \exp(8\varepsilon^{-3})} \sum_{\substack{m<M \\ m,n|P(z)}} \sum_{n<N} \alpha_{m,l}(M, N, \varepsilon) \times$$

$$\times \beta_{m,l}(M, N, \varepsilon) r(\mathscr{A}, mn) +$$

$$+ O\big(V(z) XE\big) - \sum_{\Delta \leqslant p < D^{1/2}, p|P(z)} \sum_{d < D^{1/4}, d|P(z)} \varphi_d^{+} r(\mathscr{A}, pd),$$

the second line resulting from application of (8) to $S(\mathscr{A}, \mathscr{P}, z_0)$ and the last one from estimating $S(\mathscr{A}_p, \mathscr{P}, D^{1/4})$ individually by Rosser's upper bound sieve with weights φ_d^{+} such that $|\varphi_d^{+}| \leqslant 1$, $\varphi_d^{+} = 0$ for $d \geqslant D^{4/1}$ (see Lemma 4). On multiplying the coefficients α_m, β_n by the characteristic function of numbers free of prime factors $\geqslant \Delta$ the condition $m, n | P(z_0)$ can be changed into equivalent one $m, n | P(z)$ thus giving the required shape for the first remainder term. The second remainder term has a bilinear form as desired. To see that consider p and d as two different variables when $N \geqslant D^{1/4}$, so $p < M$ and $d < N$. In case $N < D^{1/4}$ take pd as one variable, so $pd < D^{3/4} < M$ and the constant 1 as the second variable, so $1 < M$. This completes the proof of Theorem 1.

3. Main lemmas. The main results that we shall utilize in the proof of Theorem 4 can be found in [5].

Put $\mu_1^{+} = 1$, $\mu_1^{-} = 1$ and for all squarefree numbers $d = p_1 \dots p_r$, $p_1 > \dots > p_r$, $r \geqslant 1$, define

$$\mu_d^{+} = \begin{cases} (-1)^r & \text{if } p_1 \dots p_{2l} p_{2l+1}^3 < D \text{ for all } 0 \leqslant l \leqslant (r-1)/2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_d^{-} = \begin{cases} (-1)^r & \text{if } p_1 \dots p_{2l-1} p_{2l}^3 < D \text{ for all } 1 \leqslant l \leqslant r/2, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1 of [5] we have

LEMMA 2. *For any sieve $(\mathscr{A}, \mathscr{P}, z)$ the following identities hold*

$$W^{+}(\mathscr{A}, \mathscr{P}, z) = \sum_{d|P(z)} \mu_d^{+} |\mathscr{A}_d|, \quad W^{-}(\mathscr{A}, \mathscr{P}, z) = \sum_{d|P(z)} \mu_d^{-} |\mathscr{A}_d|.$$

If we introduce for each $|\mathscr{A}_d|$ its approximate value $\frac{\omega(d)}{d} X$ then the following sums will arise

$$M^{\pm}(D, \mathscr{P}, z) = \sum_{d|P(z)} \mu_d^{\pm} \frac{\omega(d)}{d}.$$

For these sums, by Lemmas 18 and 20 of [5] we have

LEMMA 3. *If condition* (1) *holds then*

(12) $\quad M^{+}(D, \mathscr{P}, z) \leqslant V(z) \{F(s) + O(e^{\sqrt{K}-s} (\log D)^{-1/3})\} \quad$ *if* $\quad z \leqslant D$,

(13) $\quad M^{-}(D, \mathscr{P}, z) \geqslant V(z) \{f(s) + O(e^{\sqrt{K}-s} (\log D)^{-1/3})\} \quad$ *if* $\quad z \leqslant D^{1/2}$.

Remark 1. It was shown in [4] (in the special case $\omega(d) = 1$) that the exponent $-1/3$ in the error terms of (12) and (13) can be replaced by -1. Moreover, it was shown there that $\sum_{d<D} |\mu_d^{\pm}| \ll D(\log D)^{-2}$, the bound $O(D)$ is of course trivial. This seems to be best possible because any sharper estimate would lead one to an extraordinary result about Siegel's zero.

Remark 2. Much the same proof as that of Lemma 1 leads to the following: if $z < D^{1/2}$, $d | P(z)$ and either $\mu_d^{+} \neq 0$ or $\mu_d^{-} \neq 0$ then for any $M > 1$ and $N > 1$ such that $MN = D$ one can write

(14) $\quad d = d_1 d_2 \quad$ with $\quad d_1 < M$ and $d_2 < N$.

A similar factorization exists in the case of larger dimensions and results analogous to Theorem 1 can be proved. It is worth remarking that if $M = N = D^{1/2}$ then the factorization (14) is also possible for

d's from the support of Selberg's λ's because they are of the form

$$\lambda_d = \sum_{\substack{[d_1, d_2] = d \\ d_1, d_2 < D^{1/2}}} \varrho_{d_1} \varrho_{d_2}.$$

This structure of Selberg's λ's plays a crucial rôle in Motohashi's improvement of the Brun–Titchmarsh theorem. The present work was just inspired by his pioneering work [7] (see also [1] and [3]).

4. Handling of the remainder terms $R^{\pm}$. If $z < D^{-1/\log\varepsilon}$ then the estimates (10) and (11) follow from these for $z_1 = D^{-1/\log\varepsilon}$. To this end, apply Theorem 4 for the sieve $(\mathscr{A}, \mathscr{P}_z, z_1)$ with $\mathscr{P}_z = \{p;\ p | P(z)\}$. We obtain

$$W_R^{+}(\mathscr{A}, \mathscr{P}, z) = W_R^{+}(\mathscr{A}, \mathscr{P}_z, z_1) \leqslant V(z) X \{F(s_1) + E(\varepsilon, D, K, L)\} + R_1^{+} + R^{+}$$

and

$$W_R^{-}(\mathscr{A}, \mathscr{P}, z) = W_R^{-}(\mathscr{A}, \mathscr{P}_z, z_1) \geqslant V(z) X \{f(s_1) - E(\varepsilon, D, K, L)\} - R_1^{-} - R^{-}.$$

Here $s_1 = \log D / \log z_1 = -\log\varepsilon$, thus

$$F(s_1) = 1 + O(e^{-s_1}) = 1 + O(\varepsilon) \leqslant F(s) + O(\varepsilon),$$
$$f(s_1) = 1 + O(e^{-s_1}) = 1 + O(\varepsilon) \geqslant f(s) + O(\varepsilon).$$

This completes the proof of implication in question.

In what follows we shall assume that $D^{-1/\log\varepsilon} \leqslant z < D^{1/2}$. Define

$$u = D^{\varepsilon^2}, \quad P(z, u) = P(z)/P(u), \quad V(z, u) = V(z)/V(u),$$
$$\tilde{\mathscr{A}} = \{a \in \mathscr{A};\ (a, P(u)) = 1\}, \quad \tilde{\mathscr{P}} = \{p \in \mathscr{P};\ p \nmid P(u)\}.$$

We shall make use of Lemma 2 for the sieve $(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z)$. But first we show that

$$W_R^{+}(\mathscr{A}, \mathscr{P}, z) \leqslant W^{+}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z), \tag{15}$$

$$W_R^{-}(\mathscr{A}, \mathscr{P}, z) \geqslant W^{-}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z). \tag{16}$$

For, we observe that if $w^{+}(p_1, \ldots, p_{2r+1})$ holds and $r \leqslant R$ then $p_1 \ldots p_{2r} < D^{1-3^{-r}}$ and $p_{2r+1} > D^{3^{-r-1}} > u$. Analogously, if $w^{-}(p_1, \ldots, p_{2r})$ holds and $r \leqslant R$ then $p_1 \ldots p_{2r-1} < D^{1-\frac{3}{2}3^{-r}}$ and $p_{2r} > D^{\frac{1}{2}3^{-r}} > u$. Therefore, if $S(\mathscr{A}_{p_1 \ldots p_t}, \mathscr{P}, p_t)$ contributes to $W_R^{\pm}(\mathscr{A}, \mathscr{P}, z)$ then it is equal to $S(\tilde{\mathscr{A}}_{p_1 \ldots p_t}, \tilde{\mathscr{P}}, p_t)$, so it contributes to $W^{\pm}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z)$ as well. This yields (15) and (16).

Now, by Lemma 2 when applied to $(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z)$ we obtain

$$W^{\pm}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z) = \sum_{d | P(z,u)} \mu_d^{\pm} S(\mathscr{A}_d, \mathscr{P}, u),$$

because $|\tilde{\mathscr{A}}_d| = S(\mathscr{A}_d, \mathscr{P}, u)$ for all $d | P(z, u)$. We divide the interval $[u, D^{1/2})$ into $\ll \varepsilon^{-7} \log \varepsilon^{-1}$ subintervals by points from $\mathscr{G}$. If $d = p_1 \ldots p_r$, $D^{1/2} > p_1 > \ldots > p_r \geqslant u$ then we say that d *belongs to the sequences* $(D_1, \ldots, D_r)$ iff

$$D_1 \leqslant p_1 < D_1^{1+\eta}, \quad \ldots, \quad D_r \leqslant p_r < D_r^{1+\eta}.$$

Here and in the sequel the sequence $(D_1, \ldots, D_r)$ will always be taken from $\mathscr{H}$. It is easy to see that if d belongs to $(D_1, \ldots, D_r)$ and $\mu_d^{+} \neq 0$ then $D_1 \ldots D_{2l} D_{2l+1}^3 < D$ for all $0 \leqslant l \leqslant (r-1)/2$ and analogously, if d belongs to $(D_1, \ldots, D_r)$ and $\mu_d^{-} \neq 0$ then $D_1 \ldots D_{2l-1} D_{2l}^3 < D$ for all $1 \leqslant l \leqslant r/2$. Therefore we have

$$W^{+}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z) \leqslant S(\mathscr{A}, \mathscr{P}, u) - \sum_{r \geqslant 0} \sum_{\substack{D_1 > \ldots > D_{2r+1} \\ D_1 \ldots D_{2l} D_{2l+1}^3 < D^{1/(1+\eta)} \\ \text{for all } 0 \leqslant l \leqslant r}} H_{(D_1, \ldots, D_{2r+1})}(\mathscr{A}, \mathscr{P}, u) + \sum_{r \geqslant 1} \sum_{\substack{D_1 \geqslant \ldots \geqslant D_{2r} \\ D_1 \ldots D_{2l} D_{2l+1}^3 < D \\ \text{for all } 0 \leqslant l < r}} H_{(D_1, \ldots, D_{2r})}(\mathscr{A}, \mathscr{P}, u) \tag{17}$$

and

$$W^{-}(\tilde{\mathscr{A}}, \tilde{\mathscr{P}}, z) \geqslant S(\mathscr{A}, \mathscr{P}, u) - \sum_{r \geqslant 0} \sum_{\substack{D_1 \geqslant \ldots \geqslant D_{2r+1} \\ D_1 \ldots D_{2l-1} D_{2l}^3 < D \\ \text{for all } 1 \leqslant l \leqslant r}} H_{(D_1, \ldots, D_{2r+1})}(\mathscr{A}, \mathscr{P}, u) + \sum_{r \geqslant 1} \sum_{\substack{D_1 > \ldots > D_{2r} \\ D_1 \ldots D_{2l-1} D_{2l}^3 < D^{1/(1+\eta)} \\ \text{for all } 1 \leqslant l \leqslant r}} H_{(D_1, \ldots, D_{2r})}(\mathscr{A}, \mathscr{P}, u) \tag{18}$$

where

$$H_{(D_1, \ldots, D_r)}(\mathscr{A}, \mathscr{P}, u) = \sideset{}{^*}\sum_{\substack{D_1 \leqslant p_1 < D_1^{1+\eta} \\ \cdots\cdots \\ D_r \leqslant p_r < D_r^{1+\eta}}} S(\mathscr{A}_{p_1 \ldots p_r}, \mathscr{P}, u). \tag{19}$$

The symbol $\sum^*$ means that the variables of summation $p_1, \ldots, p_r$ run over prime divisors of $P(z, u)$ such that $p_i \neq p_j$ for $(i, j) \in I(D_1, \ldots, D_r)$. To evaluate $S(\mathscr{A}_q, \mathscr{P}, u)$ we need the following

Lemma 4. *There exist two sequences $\{\varphi_d^+\}$ and $\{\varphi_d^-\}$ such that*

$$(20)\qquad \varphi_1^{\pm}=1,\quad |\varphi_d^{\pm}|\leqslant 1 \quad \text{and} \quad \varphi_d^{\pm}=0 \quad \text{if} \quad d\geqslant D^{\varepsilon},$$

$$(21)\qquad \varphi^-*1\leqslant \mu*1\leqslant \varphi^+*1,$$

$$(22)\qquad \sum_{d|P(u)}\varphi_d^{\pm}\frac{\omega(d)}{d}=V(u)\left\{1+O\left(e^{-1/\varepsilon}+e^{\sqrt{K}-1/\varepsilon}(\varepsilon\log D)^{-1/3}\right)\right\}.$$

Proof. It is well known in the literature as the Fundamental Lemma (see [2]). To prove the above version take for φ^+ and φ^- the functions μ^+ and μ^- corresponding to the parameter D^{ε} in place of D. Then (20) and (21) are obvious while (22) follows from Lemma 3.

By Lemma 4 we get

$$\sum_{d|P(u)}\varphi_d^-|\mathscr{A}_{dq}|\leqslant S(\mathscr{A}_q,\mathscr{P},u)\leqslant \sum_{d|P(u)}\varphi_d^+|\mathscr{A}_{dq}|,$$

so

$$\pm H_{(D_1,\ldots,D_r)}(\mathscr{A},\mathscr{P},u)\leqslant \sum_{d|P(u)}\varphi_d^{\pm}\sum^*_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_r\leqslant p_r<D_r^{1+\eta}}}|\mathscr{A}_{dp_1\ldots p_r}|.$$

If we introduce these inequalities to (17) and (18) and then replace the quantities $|\mathscr{A}_q|$ by $\dfrac{\omega(q)}{q}X+r(\mathscr{A},q)$ we shall arrive at the estimates

$$(23)\qquad W^+(\tilde{\mathscr{A}},\tilde{\mathscr{P}},z)\leqslant X\Lambda^++R_1^++R^+,$$

$$(24)\qquad W^-(\tilde{\mathscr{A}},\tilde{\mathscr{P}},z)\geqslant X\Lambda^--R_1^--R^-$$

with the remainder terms $R^+=R^+(\mathscr{A},\mathscr{P},z,\varepsilon,D)$ and $R^-=R^-(\mathscr{A},\mathscr{P},z,\varepsilon,D)$ of the required type. The main terms Λ^+ and Λ^- are equal to

$$(25)\qquad \Lambda^+=\sum_{d|P(u)}\varphi_d^+\frac{\omega(d)}{d}-\sum_{r\geqslant 0}\sum_{\substack{D_1>\ldots>D_{2r+1}\\ D_1\ldots D_{2l}D_{2l+1}^3<D^{1/(1+\eta)}\\ \text{for all } 0\leqslant l\leqslant r}}\sum_{d|P(u)}\varphi_d^-\sum_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r+1}\leqslant p_{2r+1}<D_{2r+1}^{1+\eta}}}\frac{\omega(dp_1\ldots p_{2r+1})}{dp_1\ldots p_{2r+1}}+$$
$$+\sum_{r\geqslant 1}\sum_{\substack{D_1\geqslant\ldots\geqslant D_{2r}\\ D_1\ldots D_{2l}D_{2l+1}^3<D\\ \text{for all } 0\leqslant l<r}}\sum_{d|P(u)}\varphi_d^+\sum^*_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r}\leqslant p_{2r}<D_{2r}^{1+\eta}}}\frac{\omega(dp_1\ldots p_{2r})}{dp_1\ldots p_{2r}}$$

and

$$(26)\qquad \Lambda^-=\sum_{d|P(u)}\varphi_d^-\frac{\omega(d)}{d}-\sum_{r\geqslant 0}\sum_{\substack{D_1\geqslant\ldots\geqslant D_{2r+1}\\ D_1\ldots D_{2l-1}D_{2l}^3<D\\ \text{for all } 1\leqslant l\leqslant r}}\sum_{d|P(u)}\varphi_d^+\sum^*_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r+1}\leqslant p_{2r+1}<D_{2r+1}^{1+\eta}}}\frac{\omega(dp_1\ldots p_{2r+1})}{dp_1\ldots p_{2r+1}}+$$
$$+\sum_{r\geqslant 1}\sum_{\substack{D_1>\ldots>D_{2r}\\ D_1\ldots D_{2l-1}D_{2l}^3<D\\ \text{for all } 1\leqslant l\leqslant r}}\sum_{d|P(u)}\varphi_d^-\sum_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r}\leqslant p_{2r}<D_{2r}^{1+\eta}}}\frac{\omega(dp_1\ldots p_{2r})}{dp_1\ldots p_{2r}}.$$

5. Estimates of the main terms. To complete the proof of Theorem 4 it remains to show that

$$(27)\qquad \Lambda^+\leqslant V(z)\{F(s)+E(\varepsilon,D,K,L)\}$$

and

$$(28)\qquad \Lambda^-\geqslant V(z)\{f(s)-E(\varepsilon,D,K,L)\}.$$

We shall treat in great detail Λ^+ only, the case of Λ^- being similar. If one replaces in (25) φ^- by φ^+ we make an error which is less than (in absolute value)

$$\left(\sum_{d|P(u)}\varphi_d^+\frac{\omega(d)}{d}-\sum_{d|P(u)}\varphi_d^-\frac{\omega(d)}{d}\right)\left(\sum_{q|P(z,u)}\frac{\omega(q)}{q}\right)$$
$$\ll V(u)e^{-1/\varepsilon}\left(1+e^{\sqrt{K}}(\varepsilon\log D)^{-1/3}\right)\frac{V(u)}{V(z)}$$
$$\ll V(z)e^{-1/\varepsilon}\left(1+e^{\sqrt{K}}(\varepsilon\log D)^{-1/3}\right)\varepsilon^{-4}\left(1+\frac{K}{\varepsilon^2\log D}\right)^2$$
$$\ll E(\varepsilon,D,K,L)V(z).$$

Therefore

$$(29)\qquad \Lambda^+=M^+(D^{\varepsilon},\mathscr{P},u)\times$$
$$\times\Bigg\{1-\sum_{r\geqslant 0}\sum_{\substack{D_1>\ldots>D_{2r+1}\\ D_1\ldots D_{2l}D_{2l+1}^3<D^{1/(1+\eta)}\\ \text{for all } 0\leqslant l\leqslant r}}\sum_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r+1}\leqslant p_{2r+1}<D_{2r+1}^{1+\eta}}}\frac{\omega(p_1\ldots p_{2r+1})}{p_1\ldots p_{2r+1}}+$$
$$+\sum_{r\geqslant 1}\sum_{\substack{D_1\geqslant\ldots\geqslant D_{2r}\\ D_1\ldots D_{2l}D_{2l+1}^3<D\\ \text{for all } 0\leqslant l<r}}\sum^*_{\substack{D_1\leqslant p_1<D_1^{1+\eta}\\ \cdots\cdots\\ D_{2r}\leqslant p_{2r}<D_{2r}^{1+\eta}}}\frac{\omega(p_1\ldots p_{2r})}{p_1\ldots p_{2r}}\Bigg\}+O\left(E(\varepsilon,D,K,L)V(z)\right)$$
$$=M^+(D^{\varepsilon},\mathscr{P},u)L^+\left(\varepsilon,D,P(z)\right)+O\left(E(\varepsilon,D,K,L)V(z)\right),\ \text{say}.$$

To estimate $L^+(\varepsilon, D, P(z))$ it is convenient to compare it with $M^+(D, \tilde{\mathscr{P}}, z)$. In both sums L^+ and M^+ there are terms of the same type $\omega(q)/q$ with $q = p_1 \ldots p_l$, $p_l | P(z, u)$. We shall classify all q that occur in one sum only into four classes A, B, C and D. The first class consists of these q that are not squarefree:

$$A = \{q = p_1 \ldots p_t;\ p_l | P(z, u) \text{ for } 1 \leqslant l \leqslant t,\ \mu(q) = 0\}.$$

Now, let us take any $q | P(z, u)$. If $\omega(q)/q$ contributes to $M^+(D, \tilde{\mathscr{P}}, z)$ and it does not contribute to $L^+(\varepsilon, D, P(z))$ then either q belongs to

$$B = \{q;\ q | P(z, u),\ q \text{ has at least two prime factors in one interval of the type } [B_n, B_{n+1})\}$$

where $B_n = D^{\varepsilon^2(1+\eta)^n}$, $n \geqslant 0$, or it belongs to some $+$ admissible sequence $(D_1, \ldots, D_{2r+1})$ such that $D_1 \ldots D_{2l} D_{2l+1}^3 \geqslant D^{1/(1+\eta)}$ for some $0 \leqslant l \leqslant r$. In the latter case such q must belong to

$$C = \{q;\ q | P(z, u),\ q = p_1 \ldots p_{2r+1},\ p_1 > \ldots > p_{2r+1},\ D^{1/(1+\eta)} \leqslant p_1 \ldots p_{2l} p_{2l+1}^3 < D \text{ for some } 0 \leqslant l \leqslant r\}.$$

To complete the classification it remains to consider q's such that $\omega(q)/q$ contributes to $L^+(\varepsilon, D, P(z))$ and it does not contribute to $M^-(D, \tilde{\mathscr{P}}, z)$. Obviously, every q in question belongs to some $+$ admissible sequence $(D_1, \ldots, D_{2r})$ and, writing $q = p_1 \ldots p_{2r}$ with $p_1 > \ldots > p_{2r}$, it must hold $p_1 \ldots p_{2l} p_{2l+1}^3 \geqslant D$ for some $0 \leqslant l \leqslant (r-1)/2$. Hence we see that q is in the class

$$D = \{q;\ q | P(z, u),\ q = p_1 \ldots p_{2r},\ p_1 > \ldots > p_{2r},\ D \leqslant p_1 \ldots p_{2l} p_{2l+1}^3 < D^{1+\eta} \text{ for some } 0 \leqslant l < r\}.$$

By the above discussion we get

$$(30) \qquad \big|L^+(\varepsilon, D, P(z)) - M^+(D, \tilde{\mathscr{P}}, z)\big| \leqslant \sum_{q \in A \cup B \cup C \cup D} \frac{\omega(q)}{q}.$$

A. *Estimate of* $\sum_{q \in A}$. We have

$$(31) \qquad \sum_{q \in A} \leqslant \Big(\sum_{p | P(z,u)} \sum_{\alpha \geqslant 2} \frac{\omega(p^\alpha)}{p^\alpha}\Big) \prod_{p | P(z,u)} \Big(\sum_{\alpha \geqslant 0} \frac{\omega(p^\alpha)}{p^\alpha}\Big) \leqslant U e^U \frac{V(u)}{V(z)} < \frac{L e^{L \log 3}}{\log u} \frac{\log z}{\log u} \Big(1 + \frac{K}{\log u}\Big) \leqslant \varepsilon^{-4} K e^L (\log D)^{-1},$$

where for simplicity we denoted $\sum'_{q \in A} \frac{\omega(q)}{q}$ by $\sum_{q \in A}$, $\sum_{p | P(z,u)} \sum_{\alpha \geqslant 2} \frac{\omega(p^\alpha)}{p^\alpha}$ by U.

B. *Estimate of* $\sum_{q \in B}$. We have

$$\sum_{q \in B} \leqslant \sum_{\substack{G \in \mathscr{G} \\ u \leqslant G < z}} \Big(\sum_{\substack{G \leqslant p < G^{1+\eta} \\ p \in \mathscr{P}}} \frac{\omega(p)}{p}\Big)^2 \sum_{q | P(z,u)} \frac{\omega(q)}{q}.$$

But we have

$$\sum_{\substack{G \leqslant p < G^{1+\eta} \\ p \in \mathscr{P}}} \frac{\omega(p)}{p} \leqslant \log \frac{V(G)}{V(G^{1+\eta})} \leqslant \log(1+\eta)\Big(1 + \frac{K}{\log u}\Big) < \varepsilon^9 + \frac{K}{\log u},$$

$$\sum_{\substack{u \leqslant p < z \\ p \in \mathscr{P}}} \frac{\omega(p)}{p} \leqslant \log \frac{V(u)}{V(z)} < -2 \log \varepsilon + \frac{K}{\log u}$$

and

$$\sum_{q | P(z,u)} \frac{\omega(q)}{q} \leqslant \frac{V(u)}{V(z)} < \varepsilon^{-2} \Big(1 + \frac{K}{\log u}\Big).$$

Gathering these estimates together we obtain

$$(32) \qquad \sum_{q \in B} \leqslant \Big(\varepsilon^9 + \frac{K}{\log u}\Big)\Big(\log \varepsilon^{-2} + \frac{K}{\log u}\Big) \varepsilon^{-2} \Big(1 + \frac{K}{\log u}\Big) \ll \varepsilon^6 + \varepsilon^{-5} K^3 (\log D)^{-1}.$$

CD. *Estimates of* $\sum_{q \in C}$ *and* $\sum_{q \in D}$. We have

$$\sum_{q \in C \cup D} \leqslant \sum_{\substack{mpn | P(z,u) \\ D^{1/(1+\eta)} \leqslant mp^3 < D^{1+\eta}}} \frac{\omega(mpn)}{mpn} \leqslant \Big(\sum_{m | P(z,u)} \frac{\omega(m)}{m} \sum_{\substack{m_1 \leqslant p^3 < m_2 \\ p | P(z,u)}} \frac{\omega(p)}{p}\Big)\Big(\sum_{n | P(z,u)} \frac{\omega(n)}{n}\Big)$$

where $m_1 = \max\{u^3, D^{1/(1+\eta)}/m\}$ and $m_2 = \min\{z^3, D^{1+\eta}/m\}$. By (1) we obtain

$$\sum_{\substack{m_1 \leqslant p^3 < m_2 \\ p \in \mathscr{P}}} \frac{\omega(p)}{p} \leqslant \log \frac{V(m_1^{1/3})}{V(m_2^{1/3})} \leqslant \log\Big(\frac{\log m_2}{\log m_1}\Big) + \frac{K}{\log u} < \varepsilon^7 + \frac{K}{\log u}.$$

Hence

$$(33) \qquad \sum_{q \in C \cup D} \leqslant \Big(\frac{V(u)}{V(z)}\Big)^2 \Big(\varepsilon^7 + \frac{K}{\log u}\Big) < \varepsilon^{-4} \Big(1 + \frac{K}{\log u}\Big)^2 \Big(\varepsilon^7 + \frac{K}{\log u}\Big) \ll \varepsilon^3 + \varepsilon^{-6} K^3 (\log D)^{-1}.$$

By (29), (30), (31), (32), (34) and (12) we obtain

$$\Lambda^+ = M^+(D^\varepsilon, \mathscr{P}, u)\{M^+(D, \tilde{\mathscr{P}}, z) + O(\varepsilon^3 + \varepsilon^{-6}K^3 e^L (\log D)^{-1})\} + \\ + O(E(\varepsilon, D, K, L)V(z))$$

$$\leqslant V(u)\left\{F\left(\frac{1}{\varepsilon}\right) + O\left(e^{\sqrt{K} - \varepsilon^{-1}}(\log D)^{-1/3}\right)\right\} \times$$

$$\times \frac{V(z)}{V(u)}\{F(s) + O(\varepsilon^3 + \varepsilon^{-6}K^3 e^L (\log D)^{-1})\} + O(E(\varepsilon, D, K, L)V(z))$$

$$< V(z)\{F(s) + E(\varepsilon, D, K, L)\}.$$

The proof of (27) is complete. Much the same arguments give the proof of (28).

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On the maximal order in S_n and S_n^*

by

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To the memory of Professor Paul Turán

1. In what follows we are dealing with the maximal order of the elements of S_n, the symmetric group on n letters, resp. of S_n^*, the symmetric semigroup on n letters.

Let $O(P)$ denote the order of the element P of S_n. E. Landau proved (see [2]) for

$$G(n) = \max_{P \in S_n} O(P) \tag{1.1}$$

the asymptotical relation

$$\lim_{n\to\infty} \frac{\log G(n)}{\sqrt{n \log n}} = 1. \tag{1.2}$$

Dealing with the value distribution of $O(P)$, in his paper [6], Professor P. Turán posed the problem of the analogue of (1.1)–(1.2) for S_n^*.

In our paper [5] we proved that for $\varepsilon > 0$ and $n \geqslant n_0(\varepsilon)$ the following relation holds

$$\log G(n) = \sqrt{n(\log n + \log\log n + \delta(n))} \tag{1.3}$$

where

$$-1 + \frac{\log\log n - 2 - \varepsilon}{\log n} < \delta(n) < \frac{1}{4}. \tag{1.4}$$

Meanwhile we got to know about a paper by J.-L. Nicolas. In that paper (see [3]) J.-L. Nicolas proved — among other things — the asymptotical relation

$$\nu(G(n)) \sim 2\sqrt{\frac{n}{\log n}} \tag{1.5}$$

($\nu(k)$ stands for the number of different prime factors of k) and mentioned