so that

$$\sum_{n\leqslant x} n^{-1}|f(n+2d)-f(n)|^2 = O(\log x), \qquad x\geqslant 2.$$

The outstanding case of Theorem 2 may thus be deduced from Theorem 1.

This completes our proof of Theorem 2.

In the latter stages of this proof the influence of Professor Turán's ideas is clearly visible.

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A lower bound for linear forms in logarithms

by

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Dedicated to the memory of Paul Turán

We give an explicit lower bound for a non-homogeneous linear form in logarithms of algebraic numbers with algebraic coefficients. We pay a special attention to the dependence on the degree of the algebraic numbers, and on the number of terms in the linear form.

1. The main result. We consider a linear form in logarithms of algebraic numbers

$$\Lambda = \beta_0+\beta_1\log\alpha_1+\ldots+\beta_n\log\alpha_n,$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, and $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers. Our aim is to prove a new lower bound for $|\Lambda|$ assuming that it does not vanish. For a complete history of this subject, we refer to [2].

When K is a number field, we denote by $\mathscr{L}_K$ the set of the logarithms of the elements of K^*:

$$\mathscr{L}_K = \{l\in \boldsymbol{C};\ e^l\in K\}.$$

When $l\in\mathscr{L}_K$ and $\alpha = e^l$, we write $l = \log\alpha$. We use the "absolute logarithmic height" $h(\alpha)$ of Néron and Lang [6] (the definition, and connections with Mahler's measure and with the usual height, are detailed in § 2 below).

Our main result is the following.

THEOREM. *Let K be a number field of degree D over $\boldsymbol{Q}$, $l_1, \ldots, l_n$ be non-zero elements of $\mathscr{L}_K$, and $\beta_0, \ldots, \beta_n$ be elements of K. Define $\alpha_j = e^{l_j}$, $(1\leqslant j\leqslant n)$, and*

$$\Lambda = \beta_0+\beta_1\log\alpha_1+\ldots+\beta_n\log\alpha_n.$$

Let $V_1, \ldots, V_n$, W, E be positive real numbers, satisfying

$$1/D \leqslant V_1 \leqslant \ldots \leqslant V_n,$$
$$V_j \geqslant \max\{h(\alpha_j); |\log \alpha_j|/D\} \quad (1 \leqslant j \leqslant n),$$
$$W \geqslant \max_{1\leqslant j\leqslant n} \{h(\beta_j)\},$$

and

$$1 < E \leqslant \min\{e^{DV_1}; \min_{1\leqslant j\leqslant n} 4DV_j/|\log \alpha_j|\}.$$

Finally, define $V_j^+ = \max\{V_j, 1\}$ for $j = n$ and $j = n-1$, with $V_0^+ = 1$ in the case $n = 1$.

If $\Lambda \neq 0$, then

$$|\Lambda| > \exp\{-C(n)D^{n+2}V_1 \ldots V_n(W + \operatorname{Log}(EDV_n^+))(\operatorname{Log}(EDV_{n-1}^+)) \times \times (\operatorname{Log} E)^{-n-1}\},$$

with

$$C(n) \leqslant 2^{8n+51} n^{2n}.$$

The method of proof is that of Baker's sharpening III [1]; see also [2], [4], [9], [10], [6]. The first part is a transcendence argument which deals with the case of strong independence of the α's: we assume that there exists a prime q such that

$$[K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^n,$$

where $\alpha_j^{1/q} = \exp(l_j/q)$. In this case we prove the desired result with $C(n)$ replaced by $C_1(n, q)$, where

$$C_1(n, q) \leqslant 2^{6n+24} n^{n+4} q^{3n+2}.$$

(See Proposition 3.8 below.)

In the second part, we remove this assumption, by using the case $q = 2$. Up to now, this reduction (the "final descent" of [6]) was not very accurate, especially in terms of the degree. For instance the only lower bound which was very precise in this respect was that of [8] (concerning the case $n = 2$, $\beta_0 = 0$), but there the final descent is rather straightforward. Here we perform this reduction without losing anything more than $20n^5 n!$. To achieve such an estimate, our systematic use of the absolute logarithmic height is essential (as far as the first part is concerned, we use the refined arguments of [8]).

Incidentally, our dependence in n improves earlier known results. In [12], T. N. Shorey was the first one to give a dependence in n of the shape n^{Cn} rather than e^{Cn^2}. He was able to get such a sharp result by using smaller steps than usual in the induction procedure. In [9], Loxton and van der Poorten obtained $n^{n+o(n)}$ in the case of strong linear independence. Here, we modify slightly the argument to get our estimate. However it is likely that the same bound holds with $C(n) \leqslant C^n$ for some absolute constant C, and this would be useful for applications. We remark that the method of [4], which avoids the extrapolation procedure, would give essentially the same bound as our theorem, apart from the term n^n which would be replaced by q^{n^2}, but on the other hand the absolute constants would be slightly smaller; therefore this alternative procedure is sharper for small values of n.

Finally, we develop an idea of Shorey [11], who derived a surprisingly strong result in the special case where all the numbers $\alpha_1, \ldots, \alpha_n$ are close to 1, by taking a large radius in the interpolation formula. Here, we use the parameter E (already introduced in [8]) which enables us to improve earlier results when the number $\max_{1\leqslant j\leqslant n} |\log \alpha_j|$ is small, or even only bounded.

All these refinements are useful in applications. In a subsequent paper,(1) we derive several results of diophantine approximation (transcendence measures and simultaneous approximations) where the dependence on D and E is essential.

2. Preliminary lemmas. Let α be an algebraic number; denote by

$$P(X) = a_0 X^D + \ldots + a_D = a_0 \prod_{j=1}^{D} (X - \alpha_j)$$

its minimal polynomial over $\mathbf{Z}$. The "usual height" of α is

$$H(\alpha) = \max_{0\leqslant j\leqslant D} |a_j|,$$

and the "measure" of α (see [7]) is

$$M(\alpha) = a_0 \prod_{j=1}^{D} \max(1, |\alpha_j|) = \exp \int_0^1 \log |P(e^{2i\pi t})|\, dt.$$

Let K be a number field containing α, and let M_K be the set of absolute values of K, suitably normalized to satisfy the product formula. Following [6], we define

$$H_K(\alpha) = \prod_{v\in M_K} \max(1, |\alpha_v|^{n_v}),$$

where n_v is the local degree of v. The "absolute logarithmic height" of α is the number

$$h(\alpha) = \frac{1}{[K:\mathbf{Q}]} \operatorname{Log} H_K(\alpha),$$

(1) J. Austral. Math. Soc. 25(1978), pp. 445–478.

which is independent of K. The relation

$$M(a) = H_{Q(a)}(a)$$

(see for instance [3], Lemma 11) shows that

$$h(a) = \frac{1}{D}\operatorname{Log} M(a),$$

where $D = [Q(a):Q]$. Further, for any algebraic numbers a, β and any non-zero rational integer m,

$$h(a\cdot\beta) \leqslant h(a)+h(\beta) \tag{2.1}$$

and

$$h(a^m) = mh(a). \tag{2.2}$$

Furthermore, if $a_1, \ldots, a_n$ are any algebraic numbers, then

$$h(a_1+\ldots+a_n) \leqslant h(a_1)+\ldots+h(a_n)+\operatorname{Log} n. \tag{2.3}$$

From the inequality

$$M(a) \leqslant (D+1)^{1/2}H(a)$$

which is proved in [7], we see that

$$h(a) \leqslant \frac{1}{D}\left(\operatorname{Log} H(a)+\tfrac{1}{2}\operatorname{Log}(D+1)\right).$$

But since $x+1 \leqslant x^2$ for $x \geqslant 2$ and $h(p/q) = \operatorname{Log} H(p/q)$ for $p/q \in Q$, it follows that

$$h(a) \leqslant \frac{1}{D}\left(\operatorname{Log} H(a)+\operatorname{Log} D\right).$$

We now give a rather precise version of Siegel's lemma, then a refined Liouville inequality (essentially due to N. I. Fel'dman), both results in terms of the absolute logarithmic height. We then give a consequence of a classical interpolation formula, a lemma on Fel'dman's polynomials $\Delta(z;k)$, a combinatorial lemma, a recent result of Dobrowolski on the Schinzel–Zassenhaus conjecture, and a rather trivial consequence of (2.3). For the next lemma, we denote by $L(P)$ the length of a polynomial P (i.e. the sum of the absolute values of its coefficients).

LEMMA 2.1. *Let $\theta_1, \ldots, \theta_r$ be algebraic numbers in a number field of degree d. Let*

$$P_{i,j} \in Z[X_1, \ldots, X_r] \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$$

be polynomials of degree at most $N_{j,k}$ in X_k (for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant r$). Define

$$X = \max_{1\leqslant j\leqslant m}\left\{\Big(\sum_{i=1}^{n} L(P_{i,j})\Big)\exp\Big(\sum_{k=1}^{r} N_{j,k}h(\theta_k)\Big)\right\},$$

and

$$\gamma_{i,j} = P_{i,j}(\theta_1, \ldots, \theta_r) \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m).$$

If $n > md$, then there exist rational integers $x_1, \ldots, x_n$, not all zero, such that

$$\sum_{i=1}^{n} \gamma_{i,j}x_i = 0 \quad (1 \leqslant j \leqslant m),$$

and

$$\max_{1\leqslant i\leqslant n} |x_i| \leqslant 2+(2X)^{md/(n-md)}.$$

Proof of Lemma 2.1. See [8], Lemma 4.

LEMMA 2.2. *Let $\theta_1, \ldots, \theta_r$ be algebraic numbers in a number field of degree d. Let $P \in Z[X_1, \ldots, X_r]$ have degree at most N_k in X_k $(1 \leqslant k \leqslant r)$. If $P(\theta_1, \ldots, \theta_r) \neq 0$, then*

$$|P(\theta_1, \ldots, \theta_r)| \geqslant L(P)^{1-d}\exp\Big\{-\sum_{k=1}^{r} dN_k h(\theta_k)\Big\}.$$

Proof of Lemma 2.2. See [8], Lemma 3.

LEMMA 2.3. *Let f be an analytic function in a disc $|z| \leqslant R$ of the complex plane. Let $\mathscr{E}$ be a finite subset of the disc $|z| \leqslant r$ with $r \leqslant R/2$, consisting of l points which are lying on a straight line and have a mutual distance at least δ with $\delta \leqslant \min(r/2, 1)$. Let t be a positive integer. Then*

$$|f|_{2r} \leqslant 2\,|f|_R\left(4\frac{r}{R}\right)^{lt}+5\left(\frac{18r}{\delta l}\right)^{lt}\max_{\substack{s\in\mathscr{E}\\ 0\leqslant m\leqslant t-1}}\left|\frac{f^{(m)}(s)}{m!}\right|.$$

Proof. This is Lemma 2 of [4].

LEMMA 2.4. *For any positive integer k, let $\nu(k)$ be the least common multiple of $1, 2, \ldots, k$. Define, for $z \in C$,*

$$\Delta(z;k) = (z+1)\ldots(z+k)/k! \quad (k \in Z,\ k \geqslant 1), \quad \text{and} \quad \Delta(z;0) = 1,$$

and, for l and m non-negative integers,

$$\Delta(z;k;l;m) = \frac{d^m}{dz^m}\left(\Delta(z;k)\right)^l.$$

For any integers $l \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$, and any $z \in C$,

$$|\Delta(z;k;l;m)| \leqslant m!\left(\frac{|z|+k+1}{k}\right)^{kl}(2e)^{kl}.$$

Moreover, let q be a positive integer, and let x be a rational number such that qx is a positive integer. Then

$$q^{2kl}(\nu(k))^m \frac{1}{m!} \Delta(x; k; l; m)$$

is a positive integer, and we have $\nu(k) \leqslant 3^k$.

Finally, for any positive integers k, R and L with $k \geqslant R$, the polynomials $(\Delta(z+r; k))^l$ $(r = 0, 1, \ldots, R-1;\ l = 1, \ldots, L)$, are linearly independent.

Proof. This is Lemmas 3 and 4 of [4] apart from the fact that we bound $\binom{[|z|]+k+1}{k}$ by $\left(\frac{|z|+k+1}{k}\right)^k e^k$, and $\nu(k)$ by 3^k.

LEMMA 2.5. *Let M and T be non-negative integers, let a, b and E_{mt} $(m = 0, 1, \ldots, M;\ t = 0, 1, \ldots, T)$ be complex numbers.*

If

$$\sum_{m=0}^{M} \sum_{\tau=0}^{t} \binom{t}{\tau} \big((a+m)b\big)^{t-\tau} E_{m\tau} = 0$$

for $t = 0, 1, \ldots, T$, then

$$\sum_{m=0}^{M} \sum_{\tau=0}^{t} \binom{t}{\tau} (mb)^{t-\tau} E_{m\tau} = 0$$

for $t = 0, 1, \ldots, T$.

Proof. This is Lemma 5 of [4].

LEMMA 2.6. *Let $D \geqslant 2$ be an integer. There exists a constant $C_0(D)$ satisfying $C_0(D) \leqslant 6D^2(\operatorname{Log} D)^{-1}$ such that if α is a non-zero algebraic integer of degree $\leqslant D$ which is not a root of unity, then*

$$\operatorname{Log} \overline{|\alpha|} \geqslant 1/C_0(D).$$

Proof of Lemma 2.6. This is the main result of [5]. Notice that $C_0(D) \geqslant D/\operatorname{Log} 2$ (cf. [10], Lemma 3). For convenience, we define $C_0(1) = (\operatorname{Log} 2)^{-1}$. Therefore $C_0(D) \leqslant 9D^2$ for all $D \geqslant 1$.

LEMMA 2.7. *Let $\theta_1, \ldots, \theta_n$ be algebraic numbers, $b_1, \ldots, b_n$ non-zero rational integers, and $d_1, \ldots, d_n$ positive integers. Define*

$$\theta = \sum_{j=1}^{n} b_j \theta_j / d_j$$

and

$$B = \max_{1 \leqslant j \leqslant n} \{|b_j|, d_j\}.$$

Then

$$h(\theta) \leqslant n(n+1)\operatorname{Log} B + \operatorname{Log} n + \sum_{j=1}^{n} h(\theta_j).$$

Proof of Lemma 2.7. We define

$$d = \prod_{k=1}^{n} d_k \quad \text{and} \quad b_j' = b_j d d_j^{-1} \quad (1 \leqslant j \leqslant n).$$

Thus

$$d\theta = \sum_{j=1}^{n} b_j' \theta_j,$$

and from (2.3) we deduce

$$h(d\theta) \leqslant \operatorname{Log} n + \sum_{j=1}^{n} h(b_j'\theta_j).$$

Now clearly we have

$$h(d\theta) \geqslant h(\theta) - \operatorname{Log} d,$$

$$h(b_j'\theta_j) \leqslant h(\theta_j) + \operatorname{Log}|b_j'| \quad (1 \leqslant j \leqslant n),$$

and

$$|b_j'| \leqslant B^n \quad (1 \leqslant j \leqslant n), \qquad d \leqslant B^n.$$

The desired result clearly follows.

3. The conditional inequality. In this section, we assume that the hypotheses of the theorem are fulfilled, and we assume that there is a prime number q such that the numbers

$$\alpha_j^{1/q} = \exp\left(\frac{1}{q}\log\alpha_j\right) \quad (1 \leqslant j \leqslant n)$$

generate a field

$$K_1 = K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$$

over K of degree q^n. We give a lower bound for $|\Lambda|$ (see Proposition 3.8 below) which depends explicitly of this prime number q. Later (§ 5) we will remove this assumption by using the case $q = 2$.

3.1. Statement of the technical result. Let $c_0, c_0', c_1, c_2, c_3, c_4$ be positive real numbers satisfying the following inequalities

$$\begin{aligned}(3.1)\qquad 1 \geqslant\ & 20c_2^{-1} + \left(19 + 37(2n)^{-1} + \frac{22}{5}(\operatorname{Log} 4)^{-n-1}\right)c_3^{-1} + \\ & + (14 + 20n^{-1})c_4^{-1} + \left(272 + 813(2n)^{-1} + c_3 n^{-2}\right)c_1 2^{-22} + \\ & + \frac{1}{c_0 - 1}\left\{3c_2^{-1} + \left(18 + 75(4n)^{-1} + \frac{33}{5}(\operatorname{Log} 4)^{-n-1}\right)c_3^{-1} + \right. \\ & \left. + (18 + 24n^{-1})c_4^{-1} + (354 + 483n^{-1})c_1 2^{-22}\right\}.\end{aligned}$$

$$(3.2)\qquad c_0' = c_0 + 2^{-9}; \quad c_0 > 1; \quad c_2 \leqslant 2^8; \quad c_0 c_4 \leqslant 2^{12}; \quad c_3 \leqslant 2^{24}.$$

$$(3.3)\qquad c_1 = 98/25 \text{ if } q = 2, \quad c_1 = 43/10 \text{ if } q \geqslant 3.$$

For instance $c_0 = c_2 = c_3 = c_4 = 2^8$ is a reasonable choice. We shall use other values of these parameters in Section 3.6.

We define

$$E_1 = \min\left\{\exp(qDV_1),\ \min_{1\leqslant j\leqslant q}\{2qDV_j|\log\alpha_j|^{-1}\}\right\},$$

$$V_{n-1}^* = (2^{13}nq^2DV_{n-1}^+E_1)^n,$$

$$W^* = \max\left\{W,\ n\operatorname{Log}(2^{11}nq^2DV_n^+),\ \frac{q}{nD}\operatorname{Log}E_1\right\},$$

$$U_1 = 2^{22}n^2q^{n+1}D^2\max\{W^*,\ V_n^+,\ W^*V_n^+(\operatorname{Log}E_1)^{-1},\ \operatorname{Log}E_1\},$$

$$U_2 = c_0'c_1c_2^nc_3c_4q^{3n}(q-1)n^{2n+1}(n!)^{-1}D^{n+2}V_1\ldots V_nW^*(\operatorname{Log}V_{n-1}^*)(\operatorname{Log}E_1)^{-n-1}$$

and

$$U = \max\{U_1,\ U_2\}.$$

PROPOSITION 3.1. *If* $\beta_n = -1$ *and* $K = \mathbf{Q}(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_{n-1})$, *then*

$$|\Lambda| > e^{-U}.$$

3.2. Notations. We assume $\Lambda \leqslant e^{-U}$, and we shall eventually obtain a contradiction. We define non-negative integers $S, T, L_{-1}, L_0, \ldots, L_n$ in the following way:

$$S = q[c_3nDW^*(\operatorname{Log}E_1)^{-1}];$$

$$T = [U/c_1c_3q^nDW^*];$$

$$L_{-1} = [W^*(\operatorname{Log}E_1)^{-n-1}];$$

$$L_0 = [U/c_1c_4q^nD(L_{-1}+1)\operatorname{Log}V_{n-1}^*];$$

$$L_j = [U/c_1c_2nq^{n+1}DSV_j] \quad (1 \leqslant j \leqslant n).$$

The following inequalities will be used repeatedly in the sequel.

$$S\sum_{j=1}^{n}L_jV_j \leqslant U/c_1c_2q^{n+1}D; \tag{3.4}$$

$$E_1S\sum_{j=1}^{n}L_j|\log\alpha_j| \leqslant 2U/c_1c_2q^n; \tag{3.5}$$

$$(L_{-1}+1)(L_0+1)\ldots(L_n+1) \geqslant c_0\left(1-\frac{1}{q}\right)S\binom{T+n}{n} \tag{3.6}$$

(notice that $T \geqslant c_0c_2^nc_4q^{n+1}n^{n+1}(n+1)$, hence $\binom{T+n}{n} \leqslant c_0'T^n(c_0n!)^{-1}$).

Using the inequalities

$$a + \operatorname{Log}x \leqslant a(1+\operatorname{Log}x) \leqslant ax \quad \text{for } a \geqslant 1 \text{ and } x \geqslant 1,$$

we easily deduce

$$\operatorname{Log}V_{n-1}^* \leqslant \frac{19}{2}qn^2DV_{n-1}^+ + n\operatorname{Log}E_1 \leqslant \frac{19}{2}qn^2DV_{n-1}^+\operatorname{Log}E_1, \tag{3.7}$$

and

$$W^* \leqslant \max\left\{W,\ 8qn^2DV_n^+,\ \frac{q}{nD}\operatorname{Log}E_1\right\}. \tag{3.8}$$

The following inequalities use (3.2), (3.7), (3.8), and a little computation.

$$T \leqslant e^{(1+2n^{-1})W^*}. \tag{3.9}$$

$$E_1qL_n \leqslant (V_{n-1}^*)^{1+n^{-1}}. \tag{3.10}$$

$$(L_{-1}+1)(L_0+1)\ldots(L_n+1)D^2U \leqslant \exp(U/2^{22}q^nD). \tag{3.11}$$

$$(L_{-1}+1)\operatorname{Log}V_{n-1}^* \leqslant \left((38+n^{-1})2^{-23}\right)\frac{U}{q^nD}. \tag{3.12}$$

$$(L_1+L_n) \leqslant e^{(1+n^{-1})W^*}. \tag{3.13}$$

$$12q^nL_nE_1S \leqslant (L_{-1}+1)(V_{n-1}^*)^{1+2n^{-1}}. \tag{3.14}$$

We deduce from (3.13)

$$T\operatorname{Log}(L_1+L_ne^{DW^*}) \leqslant (2+n^{-1})U/c_1c_3q^nD. \tag{3.15}$$

We define $f_1, f_2, f_3, f_3', f_4, f_5, f_6$ and f_7 by

$$f_1 = (1+2n^{-1})(c_1c_3)^{-1} + (1+2n^{-1})(c_1c_4)^{-1} + \left(19+79(2n)^{-1}\right)2^{-22}.$$

$$f_2 = \left((8n)^{-1} + \frac{11}{10}(\operatorname{Log}4)^{-n-1}\right)(c_1c_3)^{-1} + (2+2n^{-1})(c_1c_4)^{-1} + (38+40n^{-1})2^{-22}.$$

$$f_3 = (c_0-1)^{-1}\{f_1+f_2+(2c_1c_2)^{-1}+(2+n^{-1})(c_1c_3)^{-1}+(2+n^{-1})2^{-22}\} + n^{-1}2^{-24}.$$

$$f_3' = f_3 + n^{-1}2^{-22}.$$

$$f_4 = 2^{-n}(f_1+f_3') + (2c_1c_2)^{-1} + (2+n^{-1})2^{-n}(c_1c_3)^{-1} + 2^{-n-22}.$$

$$f_5 = f_1+f_2+f_3+(2c_1c_2)^{-1}+(2+n^{-1})(c_1c_3)^{-1}+(1+n^{-1})2^{-22}.$$

$$f_6 = f_5 + (2c_1c_2)^{-1}.$$

$$f_7 = (4c_1)^{-1} - \{\tfrac{1}{2}(f_1+f_3') + 4(c_1c_2)^{-1} + (\tfrac{5}{2}+n^{-1})(2c_1c_3)^{-1} + (c_3n^{-2}+2+n^{-1})2^{-24}\}.$$

Therefore (3.1) implies

$$f_7 \geqslant f_6 + 2^{-24}n^{-1}. \tag{3.16}$$

We observe also that this inequality implies $f_4 \leqslant 1/2$.

We consider the functions

$$\Delta(z+\lambda_{-1};L_{-1}+1;\lambda_0+1;0)\qquad(0\leqslant\lambda_{-1}\leqslant L_{-1},\ 0\leqslant\lambda_0\leqslant L_0).$$

From Lemma 2.4 we see that these polynomials are linearly independent and satisfy

$$|\Delta(z+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ t)|\leqslant\exp(f_1U/q^nD)\tag{3.17}$$

for $0\leqslant t\leqslant T$ and $|z|\leqslant 2q^nL_nE_1S$ (use (3.9), (3.12) and (3.14)). Moreover, for all rational integers Q, s with $1\leqslant Q\leqslant qL_n$, $s\geqslant 1$, the numbers

$$\Delta(Q^{-1}s+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ t)$$
$$(0\leqslant\lambda_{-1}\leqslant L_{-1},\ 0\leqslant\lambda_0\leqslant L_0,\ \ 0\leqslant t\leqslant T)$$

are rational numbers with a common denominator which is at most (by Lemma 2.4 and by (3.10) and (3.12))

$$(qL_n)^{2(L_{-1}+1)(L_0+1)}3^{(L_{-1}+1)T}\leqslant\exp(f_2U/q^nD).\tag{3.18}$$

To each $(2n+3)$-tuple $(\lambda_{-1},\ldots,\lambda_n,\ \tau_0,\ldots,\tau_{n-1},\ J)$ of non-negative integers satisfying $0\leqslant\lambda_j\leqslant L_j$ $(-1\leqslant j\leqslant n)$, we associate a polynomial

$$\Lambda_J(z,\tau)=\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}(\lambda_n\beta_0q^J)^{\tau_0'}\Delta(q^{-J}z+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ \tau_0-\tau_0')\times$$
$$\times\prod_{r=1}^{n-1}(\lambda_r+\lambda_n\beta_r)^{\tau_r}.$$

We write λ for $(\lambda_{-1},\ldots,\lambda_n)$, τ for $(\tau_0,\ldots,\tau_{n-1})$, and $|\tau|\leqslant T$ for $\tau_0+\ldots+\tau_{n-1}\leqslant T$.

Finally we use a remark from [8], § 4.2: the set

$$\{\beta_0^{m_0}\ldots\beta_{n-1}^{m_{n-1}}\cdot\alpha_1^{k_1}\ldots\alpha_n^{k_n};\ m_r,k_j\in\boldsymbol{N},\ m_0+\ldots+m_{n-1}+k_1+\ldots+$$
$$+k_n\leqslant D-1\}$$

is a set of generators of K over $\boldsymbol{Q}$; we choose a subset of free generators $\xi_1,\ldots,\xi_D$, and we write

$$\xi_d=\Big(\prod_{r=0}^{n-1}\beta_r^{m_{r,d}}\Big)\Big(\prod_{j=1}^{n}\alpha_j^{k_{j,d}}\Big),\qquad 1\leqslant d\leqslant D.$$

3.3. Construction of the rational integers $p_d(\lambda)$.

LEMMA 3.2. *There exist rational integers $p_d(\lambda)$ $(1\leqslant d\leqslant D,\ 0\leqslant\lambda_j\leqslant L_j,\ -1\leqslant j\leqslant n)$, not all of which are zero, bounded in absolute value by*

$$\exp(f_3U/q^nD),$$

such that for all $(n+1)$-tuples $(\tau_0,\ldots,\tau_{n-1},s)\in\boldsymbol{N}^{n+1}$ satisfying $|\tau|\leqslant T$, $0\leqslant s\leqslant S$ and $(s,q)=1$, the following equation holds:

$$\sum_\lambda\sum_{d=1}^{D}p_d(\lambda)\,\xi_d\Lambda_0(s,\tau)\,\alpha_1^{\lambda_1s}\ldots\alpha_n^{\lambda_ns}=0.$$

Proof of Lemma 3.2. For $0\leqslant t\leqslant T$ and $0\leqslant s\leqslant S$, let $d_{t,s}$ be a common denominator of the rational numbers

$$\Delta(s+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ t')\qquad(0\leqslant\lambda_{-1}\leqslant L_{-1},\ 0\leqslant\lambda_0\leqslant L_0,$$
$$0\leqslant t'\leqslant t);$$

therefore, by (3.18),

$$d_{t,s}\leqslant\exp(f_2U/q^nD).\tag{3.19}$$

We consider the following system of $\binom{T+n}{n}\Big(1-\dfrac{1}{q}\Big)S$ equations:

$$\sum_\lambda\sum_{d=1}^{D}p_d(\lambda)\sum_{\tau'}d_{\tau_0,s}\Delta(s+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ \tau_0-\tau_0')\Big(\prod_{i=1}^{n-1}\lambda_i^{\tau_i-\tau_i'}\Big)\times$$
$$\times\Big(\prod_{r=0}^{n-1}\binom{\tau_r}{\tau_r'}\lambda_n^{\tau_r'}\beta_r^{\tau_r'+m_{r,d}}\Big)\cdot\Big(\prod_{j=1}^{n}\alpha_j^{\lambda_js+k_{j,d}}\Big)=0.$$

Here, τ' stands for $(\tau_0',\ldots,\tau_{n-1}')$, with $0\leqslant\tau_r'\leqslant\tau_r$ $(0\leqslant r\leqslant n-1)$. We have $(L_{-1}+1)(L_0+1)\ldots(L_n+1)$ unknowns $p_d(\lambda)$ in $\boldsymbol{Z}$. We use Lemma 2.1 with

$$X=\max_{\tau,s}\Big\{\sum_\lambda\sum_{d=1}^{D}\sum_{\tau'}d_{\tau_0,s}|\Delta(s+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ \tau_0-\tau_0')|\Big(\prod_{i=1}^{n-1}\lambda_i^{\tau_i-\tau_i'}\Big)\times$$
$$\times\Big(\prod_{j=0}^{n-1}\binom{\tau_j}{\tau_j'}\lambda_n^{\tau_j'}\Big)\cdot\exp\Big\{\sum_{r=0}^{n-1}(\tau_r+D)h(\beta_r)+\sum_{j=1}^{n}(L_js+D)h(\alpha_j)\Big\}$$
$$\leqslant(L_{-1}+1)\ldots(L_n+1)D(L_1+L_n)^T\cdot\exp\Big\{(f_1+f_2)U/q^nD+(T+nD)W+$$
$$+S\sum_{j=1}^{n}L_jV_j+nDV_n\Big\}$$
$$\leqslant\exp\{(c_0-1)(f_3-n^{-1}2^{-24})U/q^nD\},$$

thanks to (3.4), (3.11), (3.13), (3.17) and (3.19). Using (3.6) we easily deduce Lemma 3.2.

3.4. The main inductive argument. Let J be a non-negative integer, with

$$J\leqslant\Big[\frac{\operatorname{Log}L_n}{\operatorname{Log}q}\Big]+1.$$

The following claim will be referred in the sequel as "the main inductive argument".

There exist rational integers $p_d^{(J)}(\lambda)$ $(1 \leqslant d \leqslant D,\ 0 \leqslant \lambda_j \leqslant L_j^{(J)},\ -1 \leqslant j \leqslant n)$, *not all of which are zero, bounded in absolute value by*

$$\exp(f_3 U/q^n D),$$

such that for all $\tau \in N^n$ *with* $|\tau| \leqslant q^{-J}T$, *the function*

$$\varphi_{J,\tau}(z) = \sum_\lambda \sum_{d=1}^D p_d^{(J)}(\lambda)\,\xi_d \Delta_J(z,\tau)\,\alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z}$$

satisfies

$$\varphi_{J,\tau}(s) = 0$$

for $0 \leqslant s \leqslant q^J S$ *and* $(s, q) = 1$. *Moreover,*

$$L_{-1}^{(J)} = L_{-1}, \quad L_0^{(J)} = L_0, \quad \textit{and} \quad L_j^{(J)} \leqslant q^{-J}L_j \quad (1 \leqslant j \leqslant n).$$

This assertion holds for $J = 0$, with $p_d^{(0)}(\lambda) = p_d(\lambda)$, by Lemma 3.2. Throughout the present section, we assume that the main inductive argument is correct for some non-negative integer J, with $q^J \leqslant L_n$, and we manage to prove it for $J+1$.

For each $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ with $0 \leqslant \lambda_j \leqslant L_j^{(J)}$ $(-1 \leqslant j \leqslant n)$, we define

$$p^{(J)}(\lambda) = \sum_{d=1}^D p_d^{(J)}(\lambda)\,\xi_d;$$

we remark that

(3.20) $$\operatorname{Log}|p^{(J)}(\lambda)| \leqslant f_3' U/q^n D.$$

We consider the functions

$$f_{J,\tau}(z) = \sum_\lambda p^{(J)}(\lambda)\,\Delta_J(z,\tau)\,e^{\lambda_n \beta_0 z}\cdot \alpha_1^{\gamma_1 z} \ldots \alpha_{n-1}^{\gamma_{n-1} z},$$

where $\gamma_j = \lambda_j + \lambda_n \beta_j$ $(1 \leqslant j \leqslant n-1)$.

We need three lemmas. The first one provides a relation between $f_{J,\tau}$ and $\varphi_{J,\tau}$.

LEMMA 3.3. *For* $|\tau| \leqslant T$ *and* $|z| \leqslant q^{J+n}S$,

$$|f_{J,\tau}(z) - \varphi_{J,\tau}(z)| \leqslant e^{-U/2}.$$

Proof of Lemma 3.3. We first remark that

$$f_{J,\tau}(z) - \varphi_{J,\tau}(z) = \sum_\lambda p^{(J)}(\lambda)\,\Delta_J(z,\tau)\,\alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z}(e^{\lambda_n \Lambda z} - 1).$$

By the inequality

$$|e^w - 1| \leqslant |w|e^{|w|} \quad \text{for all } w \in \boldsymbol{C},$$

we have

$$|e^{\lambda_n \Lambda z} - 1| \leqslant q^{-J}L_n|\Lambda|q^{n+J}S\exp(q^n L_n|\Lambda|S) \leqslant D^2 U|\Lambda|.$$

Hence, using (3.4), (3.11), (3.15), (3.17) and (3.20), we obtain

$$|f_{J,\tau}(z) - \varphi_{J,\tau}(z)| \leqslant D^2 U|\Lambda|(L_{-1}+1)\ldots(L_n+1)(L_1 + L_n e^{DW})^T \times$$
$$\times \exp\Big\{(f_1 + f_3')U/q^n D + q^n SD\sum_{j=1}^n L_j V_j\Big\}$$
$$\leqslant |\Lambda|\exp(f_4 U).$$

Finally we use the fact that (3.16) implies $f_4 \leqslant 1/2$. This establishes the result.

The following lemma gives lower bounds for the algebraic numbers $\varphi_{J,\tau}(s)$ and $\varphi_{J,\tau}(s/q)$ with $s \in \boldsymbol{Z}$, when they do not vanish.

LEMMA 3.4. *Let* $\tau \in N^n$ *satisfy* $|\tau| \leqslant T$.

1. *For* k *and* s *integers with* $0 \leqslant k \leqslant n$ *and* $0 \leqslant s \leqslant q^{J+k}S$, *either* $\varphi_{J,\tau}(s) = 0$ *or*

(3.21) $$\operatorname{Log}|\varphi_{J,\tau}(s)| > -f_5 U/q^{n-k}.$$

2. *For* $0 \leqslant s \leqslant q^{J+1}S$, *either* $\varphi_{J,\tau}(s/q) = 0$ *or*

(3.22) $$\operatorname{Log}|\varphi_{J,\tau}(s/q)| > -f_6 U.$$

Proof of Lemma 3.4. For the proof of (3.21) (resp. of (3.22)), we define $\sigma = s$ (resp. $\sigma = s/q$). The number $\varphi_{J,\tau}(\sigma)$ is in K (resp. in K_1). Let $d_{J+1,\tau,\sigma}$ be a common denominator of all the numbers

$$\Delta(q^{-J}\sigma + \lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ t), \quad (0 \leqslant \lambda_{-1} \leqslant L_{-1},\ 0 \leqslant \lambda_0 \leqslant L_0,$$
$$0 \leqslant t \leqslant \tau_0).$$

Thus from Lemma 2.4 and (3.18) we deduce

(3.23) $$d_{J+1,\tau,\sigma} \leqslant \exp(f_2 U/q^n D).$$

The number $d_{J+1,\tau,\sigma}\varphi_{J,\tau}(\sigma)$ is a polynomial in $\beta_0, \beta_1, \ldots, \beta_{n-1}$, and $\alpha_1, \ldots, \alpha_n$ (resp. and $\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}$):

$$d_{J+1,\tau,\sigma}\varphi_{J,\tau}(\sigma)$$
$$= \sum_\lambda \sum_{d=1}^D \sum_{\tau'} p_d^{(J)}(\lambda)\,q^{\tau_0' J} d_{J+1,\tau,\sigma}\Delta(q^{-J}\sigma + \lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ \tau_0 - \tau_0') \times$$
$$\times \Big(\prod_{i=1}^{n-1}\lambda_i^{\tau_i - \tau_i'}\Big)\Big(\prod_{r=0}^{n-1}\binom{\tau_r}{\tau_r'}\lambda_n^{\tau_r'}\beta_r^{\tau_r' + m_{r,d}}\Big)\Big(\prod_{j=1}^n \alpha_j^{\lambda_j + k_{j,d}}\Big).$$

The length of this polynomial is at most

$$(L_{-1}+1)\ldots(L_n+1)D(L_1+L_n)^T\cdot\exp\{(f_1+f_2+f_3)U/q^n D\}.$$

We now use Lemma 1.2. In the case $\sigma = s$, we get, from (3.4), (3.8), (3.11), (3.13), (3.17) and (3.23):

$$-\operatorname{Log}|d_{J+1,\tau,s}\varphi_{J,\tau}(s)| \leqslant (D-1)\{\operatorname{Log}((L_{-1}+1)\dots(L_n+1)D)+ + T\operatorname{Log}(L_1+L_n)+(f_1+f_2+f_3)\,U/q^nD\}+ + DTW+U/c_1c_2q^{n-k-1}+nD^2W+nD^2V_n;$$

using (3.23), once more, we obtain (3.21).

Similarly, in the case $\sigma = s/q$, we get

$$-\operatorname{Log}|d_{J+1,\tau,s/q}\varphi_{J,\tau}(s/q)| \leqslant (q^nD-1)\{\operatorname{Log}((L_{-1}+1)\dots(L_n+1)D)+T\operatorname{Log}(L_1+L_n)+ +(f_1+f_2+f_3)\,U/q^nD\}+q^nDTW+U/c_1c_2+nq^nD^2(W+V_n).$$

It is now clear that (3.22) holds. This completes the proof of Lemma 3.4.

The next lemma is an application of the interpolation formula (Lemma 2.3).

LEMMA 3.5. *Assume that k is an integer with $0 \leqslant k \leqslant n-1$ such that*

$$|f_{J,\tau}(s)| \leqslant e^{-U/2}$$

for all $(n+1)$-tuples $(\tau_0, \dots, \tau_{n-1}, s) \in N^{n+1}$ satisfying

$$|\tau| \leqslant \left(1-\frac{k}{2n}\right)q^{-J}T, \quad 0 \leqslant s \leqslant q^{J+k}S \quad \text{and} \quad (s,q)=1.$$

Then for all n-tuples $(\tau_0, \dots, \tau_{n-1}) \in N^n$ with

$$|\tau| \leqslant \left(1-\frac{k+1}{2n}\right)q^{-J}T,$$

we have

$$\sup_{|z|\leqslant q^{J+k+1}S}|f_{J,\tau}(z)| \leqslant \exp(-f_7U/q^{n-k-1}).$$

Proof of Lemma 3.5. Using the differential equations

$$\frac{d^m}{dz^m}f_{J,\tau} = \sum_{|\mu|=m}\frac{m!}{\mu_0!\dots\mu_{n-1}!}\Big(\prod_{j=1}^{n-1}(\log\alpha_j)^{\mu_j}\Big)f_{J,\tau+\mu}$$

together with the inequality

$$\max_{0\leqslant m\leqslant T}(1+|\log\alpha_1|+\dots+|\log\alpha_{n-1}|)^m \leqslant \exp(T\operatorname{Log}(1+nDV_{n-1})) \leqslant e^{U/6},$$

we deduce from the hypotheses that

$$\left|\frac{d^m}{dz^m}f_{J,\tau}(s)\right| \leqslant e^{-U/3}$$

for

$$|\tau| \leqslant \left(1-\frac{k+1}{2n}\right)q^{-J}T, \quad 0 \leqslant m \leqslant q^{-J}T/2n, \quad 0 \leqslant s \leqslant q^{J+k}S, \quad (s,q)=1.$$

We now apply Lemma 2.3 to each function

$$f_{J,\tau}, \quad |\tau| \leqslant \left(1-\frac{k+1}{2n}\right)q^{-J}T,$$

with

$$r = \tfrac{1}{2}q^{J+k+1}S, \quad R = 4E_1r, \quad \mathscr{E} = \{s \in N;\ 0 \leqslant s \leqslant q^{J+k}S,\ (s,q)=1\},$$

$$t = [T/2q^Jn]+1, \quad l = q^{J+k}\left(1-\frac{1}{q}\right)S.$$

From (3.5), we clearly have

$$R\sum_{j=1}^{n}L_j^{(J)}|\log\alpha_j| \leqslant 2q^{k+1}E_1S\sum_{j=1}^{n}L_j|\log\alpha_j| \leqslant 4q^{k+1}U/c_1c_2q^n.$$

Therefore

$$\operatorname{Log}|f_{J,\tau}|_R \leqslant \operatorname{Log}((L_{-1}+1)\dots(L_n+1)D)+(f_1+f_3')\,U/q^nD+ + T\operatorname{Log}(L_1+L_ne^{DW})+4q^{k+1}U/c_1c_2q^n \leqslant \frac{U}{q^{n-k-1}}\left(f_1/2+f_3'/2+(2+n^{-1})(2c_1c_3)^{-1}+4(c_1c_2)^{-1}+2^{-23}\right).$$

Further, since

$$l > q^{J+k}(q-1)\left(c_3nDW^*(\operatorname{Log}E_1)^{-1}-1\right)$$

and

$$t > U/2c_1c_3nq^{n+J}DW^*-1/2nq^J,$$

we see that

$$lt\operatorname{Log}E_1 > (q-1)\frac{U}{2q^{n-k}}\left(c_1^{-1}-(2c_1c_3)^{-1}-c_3(2c_5)^{-1}\right).$$

We conclude that

$$2|f|_R\left(\frac{4r}{R}\right)^{lt} \leqslant 2\exp\Big\{-U/4c_1q^{n-k-1}+ +\frac{U}{q^{n-k-1}}\left(\tfrac{1}{2}(f_1+f_3')+4(c_1c_2)^{-1}+(5+2n^{-1})(4c_1c_3)^{-1}+2^{-23}+c_3n^{-2}2^{-24}\right)\Big\}.$$

On the other hand we have $r/l \leqslant q^2/2(q-1)$ and

$$lt\operatorname{Log}\left(\frac{18r}{\delta l}\right) \leqslant \left(1-\frac{1}{q}\right)q^k\frac{1}{2n}ST\operatorname{Log}(9q^2/(q-1))+q^{J+k}(1-1/q)S$$

$$\leqslant \frac{q-1}{2q}\cdot\frac{\operatorname{Log}\left(\frac{9q^2}{q-1}\right)}{\operatorname{Log}(2q)}\cdot\frac{U}{c_1}+\frac{q-1}{q^3}\cdot\frac{U}{c_1c_2nDV_n}$$

$$\leqslant U/6-\operatorname{Log}5$$

thanks to (3.3). Thus

$$5\cdot\left(\frac{18r}{\delta l}\right)^{u}\max_{\substack{s\in\mathscr{S}\\0\leqslant m\leqslant l}}\frac{1}{m!}\left|\frac{d^m}{dz^m}f_{J,\tau}(s)\right|\leqslant e^{-U/6}.$$

Lemma 3.5 easily follows.

We shall now use the three preceeding lemmas, first to perform an extrapolation which will extend the range of s (at the cost of diminishing the range of τ) for which $\varphi_{J,\tau}(s)=0$, and then for the interpolation procedure on the multiples of $1/q$.

Here is the extrapolation.

LEMMA 3.6. *Let k be any integer with $0\leqslant k\leqslant n-1$. For all $(n+1)$-tuples $(\tau_0,\ldots,\tau_{n-1},s)\in N^{n+1}$ such that*

$$|\tau|\leqslant(1-k/2n)q^{-J}T,\quad 0\leqslant s\leqslant q^{J+k}S,\quad (s,q)=1,$$

we have

$$\varphi_{J,\tau}(s)=0.$$

Proof of Lemma 3.6. For $k=0$, this assertion is our main inductive hypothesis. We assume that Lemma 3.6 holds for some integer k, with $0\leqslant k\leqslant n-2$, and we prove it for $k+1$. We first use Lemma 3.3, and then Lemma 3.5; we get

$$|f_{J,\tau}(s)|\leqslant\exp(-f_7U/q^{n-k-1})$$

for $|\tau|\leqslant\big(1-(k+1)/2n\big)q^{-J}T$ and $0\leqslant s\leqslant q^{J+k+1}S$. We now use Lemma 3.3:

$$|\varphi_{J,\tau}(s)|\leqslant e^{-U/2}+\exp(-f_7U/q^{n-k-1}),$$

for the same values of τ, s; finally, Lemma 3.4 and (3.16), (3.21) enable us to conclude that the considered numbers $\varphi_{J,\tau}(s)$ vanish. This proves the assertion of Lemma 3.6 for $k+1$.

We now interpolate our auxiliary functions on the points s/q, $s\in Z$.

LEMMA 3.7. *For all $(n+1)$-tuples $(\tau_0,\ldots,\tau_{n-1},s)\in N^{n+1}$ with*

$$|\tau|\leqslant q^{-J-1}T\quad\text{and}\quad 0\leqslant s\leqslant q^{J+1}S,\quad (s,q)=1,$$

we have

$$\varphi_{J,\tau}(s/q)=0.$$

Proof of Lemma 3.7. By Lemmas 3.3 and 3.6, the hypotheses of Lemma 3.5 hold for $k=n-1$; hence

$$|f_{J,\tau}(z)|\leqslant\exp(-f_7U)$$

for $|z|\leqslant q^{J+n}S$ and $|\tau|\leqslant q^{-J-1}T$. Consequently

$$|f_{J,\tau}(s/q)|\leqslant\exp(-f_7U)$$

for $0\leqslant s\leqslant q^{J+1}S$ and $|\tau|\leqslant q^{-J-1}T$. Using Lemma 3.3 and (3.16), (3.22) once more, we deduce Lemma 3.7.

We now prove the main induction hypothesis for $J+1$ (the following arguments are taken from [4]; see also [2] and [6]).

Using our assumption

$$[K(\alpha_1^{1/q},\ldots,\alpha_n^{1/q}):K]=q^n,$$

we express the numbers $\varphi_{J,\tau}(s/q)$, given by Lemma 2.6, on the basis

$$\{(\alpha_1^{l_1/q}\ldots\alpha_n^{l_n/q});\ (l_1,\ldots,l_n)\in\{0,1,\ldots,q-1\}^n\}.$$

We choose $(\lambda_1^0,\ldots,\lambda_n^0)$, with $0\leqslant\lambda_j^0\leqslant q-1$ $(1\leqslant j\leqslant n)$, in such a way that at least one of the numbers

$$p^{(J)}(\lambda_{-1},\lambda_0,\lambda_1^0+\lambda_1q,\ldots,\lambda_n^0+\lambda_nq)\qquad(0\leqslant\lambda_j\leqslant L_j^{(J+1)},\ -1\leqslant j\leqslant n),$$

with

$$L_{-1}^{(J+1)}=L_{-1},\quad L_0^{(J+1)}=L_0,\quad\text{and}\quad L_j^{(J+1)}=\left[\frac{1}{q}(L_j^{(J)}-\lambda_j^0)\right]\quad(1\leqslant j\leqslant n),$$

is non zero. We denote by $p^{(J+1)}(\lambda_{-1},\lambda_0,\ldots,\lambda_n)$ the thus obtained numbers, and by $p_d^{(J+1)}(\lambda)$ the rational integers defined by

$$p^{(J+1)}(\lambda)=\sum_{d=1}^{D}p_d^{(J+1)}(\lambda)\,\xi_d.$$

Hence the numbers

$$\sum_{(\lambda)}p^{(J+1)}(\lambda)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\left(\frac{\lambda_n^0}{q}+\lambda_n\right)^{\tau_0'}\cdot\beta_0^{\tau_0'}\cdot q^{(J+1)\tau}\times$$

$$\times\Delta(q^{-J-1}s+\lambda_{-1};\ L_{-1}+1;\ \lambda_0+1;\ \tau_0-\tau_0')\prod_{r=1}^{n-1}\big(\lambda_r^0+\lambda_rq+(\lambda_n^0+\lambda_nq)\beta_r\big)^{\tau_r}\prod_{j=1}^{n}\alpha_j^{\lambda_js}$$

are zero for $|\tau|<q^{-J-1}T$, and $0\leqslant s<q^{J+1}S$, with $(s,q)=1$.

We use first Lemma 1.5 to see that the same holds with $\left(\frac{\lambda_n^0}{q}+\lambda_n\right)^{\tau_0}$ replaced by $\lambda_n^{\tau_0}$, and then the relations

$$\lambda_r+\lambda_n\beta_r=\frac{1}{q}\big(\lambda_r^0+\lambda_rq+(\lambda_n^0+\lambda_nq)\beta_r-(\lambda_r^0+\lambda_n^0\beta_r)\big)$$

to see that the same holds with $\lambda_1^0+\lambda_1q,\ldots,\lambda_n^0+\lambda_nq$ replaced by $\lambda_1,\ldots,\lambda_n$.

This completes the main induction argument.

3.5. The contradiction. Following the arguments of [2], we show that our hypothesis $|\Lambda| \leqslant \exp\{-U\}$ leads to a contradiction.

We write the main inductive argument with $J_0 = \left[\dfrac{\operatorname{Log} L_n}{\operatorname{Log} q}\right] + 1$; clearly we have $q^{J_0} > L_n$, hence $L_n^{(J_0)} = 0$, and we get

$$\sum_{\lambda_{n-1}}^{L_{n-1}^{(J_0)}} \Big(\sum_{\lambda_{-1}}^{L_{-1}^{(J_0)}} \cdots \sum_{\lambda_{n-2}}^{L_{n-2}^{(J_0)}} p^{(J_0)}(\lambda) \Delta(q^{-J_0}s + \lambda_{-1}; L_{-1}+1; \lambda_0+1; \tau_0) \times \\ \times \alpha_1^{\lambda_1 s} \ldots \alpha_{n-1}^{\lambda_{n-1} s} \cdot \lambda_1^{\tau_1} \ldots \lambda_{n-2}^{\tau_{n-2}} \Big) \lambda_{n-1}^{\tau_{n-1}} = 0$$

for all $(n+1)$-tuples $(\tau_0, \ldots, \tau_{n-1}, s)$ with $|\tau| \leqslant q^{-J_0}T$, $0 \leqslant s < q^{J_0}S$, with $(s, q) = 1$. In particular these equations hold for

$$0 \leqslant \tau_0 \leqslant \tfrac{1}{2} q^{-J_0} T, \quad 0 \leqslant \tau_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1).$$

Since the Vandermonde determinant

$$|\lambda_{n-1}^{\tau_{n-1}}| \quad (0 \leqslant \lambda_{n-1},\ \tau_{n-1} \leqslant L_{n-1}^{(J_0)})$$

does not vanish, the new sum in parenthesis above is zero. By arguing $n-1$ times in the same way, we obtain

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda_{-1}, \lambda_0, \ldots, \lambda_{n-1}, 0) \Delta(q^{-J_0}s + \lambda_{-1}; L_{-1}+1; \lambda_0+1; t) = 0$$

for $0 \leqslant \lambda_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-1)$, and $0 \leqslant t \leqslant \frac{1}{2} q^{-J_0}T$, $0 \leqslant s \leqslant q^{J_0}S$, with $(s, q) = 1$. This implies that each polynomial

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda) \Delta(z + \lambda_{-1}; L_{-1}+1; \lambda_0+1; 0) \qquad (0 \leqslant \lambda_j \leqslant L_j^{(J_0)},\ -1 \leqslant j \leqslant n)$$

has at least $\frac{1}{2} q^{-J_0}T \cdot \frac{1}{2} q^{J_0}S$ zeros; but since these polynomials have degree at most $L_{-1}L_0$, it follows that

$$p^{(J_0)}(\lambda_{-1}, \ldots, \lambda_n) = 0 \quad \text{for} \quad 0 \leqslant \lambda_j \leqslant L_j^{(J_0)},\ -1 \leqslant j \leqslant n,$$

contrary to construction. This establishes the desired contradiction, and completes the proof of Proposition 3.1.

3.6. Conclusion.

PROPOSITION 3.8. *With the notations of the theorem, we assume that there exists a prime number q such that the field $K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$ has degree q^n over K. Then*

$$|\Lambda| > \exp\{-C_1(n, q) D^{n+2} V_1 \ldots V_n (W + \operatorname{Log}(DV_n^+ E)) (\operatorname{Log}(EDV_{n-1}^+)) \times \\ \times (\operatorname{Log} E)^{-n-1}\},$$

where $C_1(n, q) \leqslant 2^{6n+24} n^{n+4} q^{3n+2}$.

Proof of Proposition 3.8. We first remark that the lower bound is weaker when D is large; therefore there is no loss of generality to assume $K = Q(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n)$. In the case $\beta_n = -1$, we can write

$$\max\{W^*, V_n^+, W^* V_n^+ (\operatorname{Log} E_1)^{-1}, \operatorname{Log} E_1\} \\ \leqslant (W^* + \operatorname{Log} E_1)(V_n^+ + \operatorname{Log} E_1)(\operatorname{Log} E_1)^{-1} \\ \leqslant q^{n-1}(q+1) D^n V_1 \ldots V_n (W^* + \operatorname{Log} E_1)(\operatorname{Log} E_1)^{-n};$$

consequently, if $W^* \geqslant \operatorname{Log} E_1$ and $2^{22} \leqslant e^2 c_0 c_1 c_2 c_3 c_4$, then $U_1 \leqslant U_2$. In Proposition 3.1, we bound W^* by $(n \operatorname{Log}(2^{11} q^2 n))(W + \operatorname{Log}(EDV_n^+))$ and $\operatorname{Log} V_{n-1}^*$ by $(n \operatorname{Log}(2^{13} q^2 n)) \operatorname{Log}(EDV_{n-1}^+)$.

Now we remove the hypothesis $\beta_n = -1$: there is no loss of generality to assume $\beta_n \neq 0$, therefore we can multiply through by β_n^{-1}, and use the following consequence of Lemma 2.2:

$$|\beta_n^{-1}| \geqslant \exp(-Dh(\beta_n)) \geqslant \exp(-DW).$$

Since $h(\beta_r \beta_n^{-1}) \leqslant 2W$, we get the same bound for W^*. Therefore, we can assume $\beta_n = -1$, provided that we replace c_0' by, say, $c_0'' = c_0' + 2^{-9} = c_0 + 2^{-8}$. Consequently we get

(3.24) $$C_1(n, q) \leqslant c_0'' c_1 c_2^n c_3 c_4 q^{3n} (q-1) n^{2n+3} (n!)^{-1} (\operatorname{Log}(2^{11} n q^2))(\operatorname{Log}(2^{13} n q^2)).$$

In the case $n = 1$, we choose $c_0 = 2$, $c_2 = 2^6$, $c_3 = c_4 = 2^8$, and we bound $(1 + 2^{-9})(1.075)(q-1)(\operatorname{Log}(2^{13} q^2))^2$ by $2^5 q^2$. This yields

$$C_1(1, q) \leqslant 2^{30} q^5.$$

In the case $n = 2$, we choose $c_0 = 31/10$, $c_2 = 2^5$, $c_3 = c_4 = 2^8$, and we bound $(3.1 + 2^{-8})(1.075)(q-1)(\operatorname{Log}(2^6 q))(\operatorname{Log}(2^7 q))$ by $2^5 q^2$. Therefore

$$C_1(2, q) \leqslant 2^{41} q^8.$$

In the case $n = 3$, we choose $c_0 = 3$, $c_2 = 2^5$, $c_3 = c_4 = 2^8$, and we bound $(3 + 2^{-8})(1.075)(q-1) 3^8 (\operatorname{Log}(2^{13} 3 q^2))^2$ by $2^{20} q^2$. Hence

$$C_1(3, q) \leqslant 2^{52} q^{11}.$$

Finally, in the case $n \geqslant 4$, we choose $c_0 = 5$, $c_2 = 2^6 e^{-1}$, $c_3 = c_4 = 2^9$, and we bound $(5 + 2^{-8})(1.075)(q-1) n^n (n!)^{-1} e^{-n} (\operatorname{Log}(2^{13} n q^2))^2$ by $2^4 n q^2$.

This completes the proof of Proposition 3.8.

The preceding proof can be used to show that there exists an effectively computable absolute constant $n_1 \geqslant 1$ such that

(3.25) $$C_1(n, 2) \leqslant (435n)^n \quad \text{for all } n \geqslant n_1.$$

For the proof of (3.25), we remark that $20e2^3 < 435$, and we choose $c_2 = 10 + 435/16e$; we take c_0, c_3, c_4 sufficiently large. Since (3.2) is not satisfied, we replace the numbers 2^{13}, 2^{11}, 2^{22} in the definition of V^*_{n-1}, W^*, U_1 respectively, by very large constants.

4. Linear dependence of logarithms of algebraic numbers. In this section, we prove two auxiliary results which are needed for the final descent. We first prove that if $l_1, \ldots, l_m$ are logarithms of algebraic numbers, and are Q-linearly dependent, then they satisfy a linear dependence relation with rather small coefficients. This result, which improves Theorem 1 of [10] and is proved in a very similar way, shows that the proof of our theorem can be reduced to the case of linearly independent logarithms.

LEMMA 4.1. *Let K be a number field of degree D over Q, and $l_1, \ldots, l_m$ linearly dependent elements of $\mathscr{L}_K$. Define $\alpha_j = \exp(l_j)$ $(1 \leqslant j \leqslant m)$.*

Then there exist rational integers $t_1, \ldots, t_m$, not all zero, such that

$$t_1 l_1 + \ldots + t_m l_m = 0$$

and

$$|t_k| \leqslant \big(9(m-1)D^3\big)^{m-1} V_1 \ldots V_m / V_k \quad (1 \leqslant k \leqslant m),$$

where

$$V_j = \max\Big\{h(\alpha_j),\ \frac{1}{D}|l_j|\Big\} \quad (1 \leqslant j \leqslant m).$$

Proof of Lemma 4.1. We assume, as we may without loss of generality, that $m \geqslant 2$, and that any $m-1$ of $l_1, \ldots, l_m$ are linearly independent. Thus there exists a unique (up to a factor ± 1) set of relatively prime integers $t_1, \ldots, t_m$ such that

$$t_1 l_1 + \ldots + t_m l_m = 0.$$

Let k be an integer, $1 \leqslant k \leqslant m$. We intend to bound $|t_k|$ (the idea of bounding each $|t_k|$ rather than $\max_{1\leqslant j\leqslant m} |t_j|$ is due to Maurice Mignotte).

We define

$$\varphi_{-1}(D) = \max\{N \geqslant 1;\ \varphi(N) \leqslant D\},$$

where φ is Euler function; therefore

$$\varphi_{-1}(D) \leqslant 2D^2 \quad \text{and} \quad \varphi_{-1}(D) \leqslant 4D \operatorname{Log}\operatorname{Log}(6D).$$

We choose positive real numbers $c_1, \ldots, c_m$ in the following way. For $j \neq k$, let [2]

$$c_j^{-1} = (m-1)\cdot\max\Big\{(\operatorname{Log}2)^{-1}\operatorname{Log}|N_{K/Q}(\alpha_j \operatorname{den}\alpha_j)|,\ C_0(D)\operatorname{Log}\max\big(\overline{|\alpha_j|}, \overline{|\alpha_j^{-1}|}\big),\ \frac{1}{2\pi}\varphi_{-1}(D)|l_j|\Big\},$$

where $C_0(D)$ is the constant of Lemma 2.6. Further let

$$c_k = \prod_{\substack{1\leqslant j\leqslant m\\ j\neq k}} c_j^{-1}.$$

Since $c_1 \ldots c_m = 1$, Minkowski's linear form theorem (e.g. [10], Lemma 5) shows that we can find integers $s_1, \ldots, s_m$, not all zero, such that

$$|s_j - s_k t_j/t_k| < c_j \quad (1 \leqslant j \leqslant m,\ j \neq k),$$

and

$$|s_k| \leqslant c_k.$$

For $1 \leqslant j \leqslant m$, let

$$[\alpha_j] = u_j \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p},j}}$$

be the ideal factorization of α_j, where u_j is a unit in K, $e_{\mathfrak{p},j} \in Z$, and $\mathfrak{p}$ runs nominally through all prime ideals of K. It is readily verified that

$$|e_{\mathfrak{p},j}| \leqslant \big(\operatorname{Log}|N_{K/Q}(\alpha_j \operatorname{den}\alpha_j)|\big)/\operatorname{Log}2,$$

and

$$\Big|\sum_{j=1}^m e_{\mathfrak{p},j}s_j\Big| = \Big|\sum_{j=1}^m e_{\mathfrak{p},j}(s_j - s_k t_j/t_k)\Big| < \sum_{\substack{1\leqslant j\leqslant m\\ j\neq k}} |e_{\mathfrak{p},j}|c_j \leqslant 1$$

for all $\mathfrak{p}$, and therefore the number

$$\alpha = \alpha_1^{s_1} \ldots \alpha_m^{s_m}$$

is a unit in K. Similarly, for all embeddings $\sigma\colon K \to C$,

$$\Big|\sum_{j=1}^m s_j \operatorname{Log}|\alpha_j^\sigma|\Big| < \sum_{\substack{1\leqslant j\leqslant m\\ j\neq k}} c_j\big|\operatorname{Log}|\alpha_j^\sigma|\big| \leqslant 1/C_0(D),$$

so by definition of $C_0(D)$, α is a root of unity. Let N be its order; then $N \leqslant \varphi_{-1}(D)$, and

$$N\sum_{j=1}^m s_j l_j \in 2i\pi Z.$$

We observe that

$$\Big|\sum_{j=1}^m s_j l_j\Big| < \sum_{\substack{1\leqslant j\leqslant m\\ j\neq k}} c_j|l_j| \leqslant 2\pi/\varphi_{-1}(D),$$

thus

$$\sum_{j=1}^m s_j l_j = 0.$$

In particular s_k is a multiple of t_k, and therefore $|t_k| \leqslant c_k$. Since

$$\overline{|\alpha|}\operatorname{den}\alpha \leqslant M(\alpha) \quad \text{and} \quad M(\alpha) = M(\alpha^{-1})$$

(2) As usual, $\overline{|\alpha|}$ denotes the maximum of the absolute values of the conjugates of α.

for all non zero algebraic numbers a, we have

$$c_k \leqslant (m-1)^{m-1} \prod_{\substack{1\leqslant j\leqslant m\\ j\neq k}} \max\{C_0(D)\operatorname{Log} M(a_j),\ \varphi_{-1}(D)|l_j|/2\pi\}.$$

This completes the proof of Lemma 4.1.

We deduce from Lemma 4.1 the following corollary (compare with [8], Lemma C).

LEMMA 4.2. *Let K be a number field of degree D, and let a be a non zero element of K. We choose a non zero determination $\log a$ of the logarithm of a. Let m be a positive integer such that the number $\exp\left\{\frac{1}{m}\log a\right\}$ belongs to K. Then*

$$m \leqslant 9D^3 \max\{h(a),\ |\log a|/D\}.$$

Proof of Lemma 4.2. Let $l_1 = \log a$, $l_2 = l_1/m$. By Lemma 4.1, we have $t_1l_1+t_2l_2 = 0$, with rational integers t_1, t_2 not both zero, and $|t_2|$ satisfies the desired bound. Since $l_2 \neq 0$, t_2 is a multiple of m, and the result follows.

Remark. The preceding proofs show that, more precisely,

(a) if a is a root of unity, then $|m| \leqslant \varphi_{-1}(D)|\log a|/2\pi$;

(b) if a is a unit, but not a root of unity, then $|m| \leqslant C_0(D)\operatorname{Log}\max\{|a|, |a^{-1}|\}$;

(c) if a is not a unit, $|m| \leqslant (\operatorname{Log}2)^{-1}\operatorname{Log}|N_{K/Q}(a\operatorname{den} a)|$.

The following result is a refined version of [4], Proposition 2, and [6], last chapter, Lemma 5.1. The main feature is that we do not assume that l_1 is a rational multiple of $i\pi$, and nevertheless we never have to consider linear forms in $n+1$ logarithms.

PROPOSITION 4.3. *Let K be a number field of degree D, and $l_1, \ldots, l_n$ be linearly independent elements of $\mathscr{L}_K$. Define $a_j = \exp(l_j)$ $(1 \leqslant j \leqslant n)$, and let $V_1 \leqslant \ldots \leqslant V_n$ satisfy*

$$V_j \geqslant \max\{h(a_j),\ |l_j|/D\} \quad (1 \leqslant j \leqslant n).$$

Then there exist elements $l'_1, \ldots, l'_n$ of $\mathscr{L}_K$, together with rational integers $m_{s,j}$ $(1 \leqslant s \leqslant n,\ 0 \leqslant j \leqslant s)$, such that

(a) *For $1 \leqslant s \leqslant n$,*

$$m_{s,0}l_s = \sum_{j=1}^{s} m_{s,j}l'_j \quad \text{and} \quad m_{s,0} > 0.$$

(b) *For each prime p such that K contains the p-th roots of unity, the field $K(\exp(l'_1/p), \ldots, \exp(l'_n/p))$ has degree p^n over K.*

(c) *For $1 \leqslant s \leqslant n$,*

$$\max\{h(e^{l'_s}),\ |l'_s|/D\} \leqslant V_1 + \ldots + V_s.$$

(d) *For $1 \leqslant s \leqslant n$,*

$$\max_{0\leqslant j\leqslant s} |m_{s,j}| \leqslant (9sD^3)^s s!\,(V_s^+)^s.$$

Remark. The inequality (c) explains why we loose $n!$ in the final descent, and, essentially, nothing more. It does not seem straightforward to replace it by $V'_s \leqslant cV_s$ with an absolute constant c.

Proof of Proposition 4.3. Let M be the $\boldsymbol{Z}$-module generated by $l_1, \ldots, l_n$, and let M' be the set of $l \in \mathscr{L}_K$ such that $l, l_1, \ldots, l_n$ are linearly dependent. Let $l \in M'$; Lemma 4.1 shows that there exists a $k \in \boldsymbol{Z}$ such that $kl \in M$ and

$$1 \leqslant k \leqslant (9nD^3)^n V_1 \ldots V_n.$$

Therefore M is of finite index, say u, in M'. For $1 \leqslant s \leqslant n$ there exist integers $k_{s,1}, \ldots, k_{s,s}$, with $k_{s,s} > 0$, such that

$$\sum_{j=1}^{s} k_{s,j}l_j \in uM'.$$

Let $k_{s,s}$ denote the smallest positive integer for which such a relation holds (therefore $1 \leqslant k_{s,s} \leqslant u$). After dividing by u, we may assume $0 \leqslant k_{s,j} \leqslant u-1$ $(1 \leqslant s \leqslant n,\ 1 \leqslant j \leqslant s-1)$. Let us define $l'_1, \ldots, l'_n$ by

$$\sum_{j=1}^{s} k_{s,j}l_j = ul'_s \quad (1 \leqslant s \leqslant n).$$

It is easily checked that $l'_1, \ldots, l'_n$ is a basis of the $\boldsymbol{Z}$-module M'. (It is easier to work additively with logarithms, as here, rather than multiplicatively with K^*, because of the torsion; here, M and M' are free $\boldsymbol{Z}$-modules.)

It is readily verified that

$$|l'_s| \leqslant |l_1| + \ldots + |l_s| \quad (1 \leqslant s \leqslant n).$$

Similarly, using (2.1) and (2.2) we deduce

$$h(a'_s) \leqslant h(a_1) + \ldots + h(a_s) \quad (1 \leqslant s \leqslant n),$$

where $a'_s = \exp(l'_s)$. This proves (c).

Since $l_s, l'_1, \ldots, l'_s$ are linearly dependent, Lemma 4.1 gives

$$m_{s,0}l_s = \sum_{j=1}^{s} m_{s,j}l'_j \quad (1 \leqslant s \leqslant n),$$

with $m_{s,0} > 0$, and $\max\limits_{0\leqslant j\leqslant s}|m_{s,j}|$ satisfying (d).

Let p be a prime such that K contains the pth roots of unity. Assume that the field $K\big((a_1')^{1/p}, \ldots, (a_n')^{1/p}\big)$ has degree $< p^n$ over K. Then by Kummer's theory [6] we have a relation

$$(a_1')^{r_1} \ldots (a_n')^{r_n} = \eta^p$$

for some $\eta \in K^*$ and non negative rational integers $r_1, \ldots, r_n$, with

$$1 \leqslant \max_{1 \leqslant j \leqslant n} r_j \leqslant p-1.$$

Let $\log\eta$ be any determination of the logarithm of η. We find a rational integer r_0 such that

$$\sum_{j=1}^{n} r_j l_j' - p\log\eta + 2i\pi r_0 = 0.$$

Let us consider the number $\lambda = \log\eta - 2i\pi r_0/p$. Since K contains the pth roots of unity, $\lambda \in \mathscr{L}_K$. Moreover $p\lambda \in M'$, hence $\lambda \in M'$. Thus there exist rational integers $a_1, \ldots, a_n$ such that

$$\lambda = a_1 l_1' + \ldots + a_n l_n'.$$

From the relations

$$p\lambda = \sum_{j=1}^{n} r_j l_j' = \sum_{j=1}^{n} p a_j l_j'$$

we deduce $r_j = pa_j$ $(1 \leqslant j \leqslant n)$, which contradicts $1 \leqslant \max_{1 \leqslant j \leqslant n} r_j \leqslant p-1$. This proves (b), and Proposition 4.3 follows.

5. Proof of the theorem. We distinguish three cases.

Case a. *The field $K(a_1, \ldots, a_n)$ has degree 2^n over K.* As already seen in the proof of Proposition 3.8, we deduce the desired result from Proposition 3.1, provided that we replace, firstly $C(n)$ by $C_2(n)$, with

$$C_2(n) \leqslant c_0'' c_1 c_2^n c_3 c_4 2^{3n} n^{2n+4} (n!)^{-1},$$

secondly $W + \mathrm{Log}(EDV_n^+)$ by

$$\max\Big\{W;\ n\,\mathrm{Log}(2^{13} nDV_n^+);\ \frac{2}{n}\,\mathrm{Log}\,E\Big\},$$

and thirdly $\mathrm{Log}(EDV_{n-1}^+)$ by

$$\mathrm{Log}(2^{15} nDV_{n-1}^+ E).$$

Case b. *The numbers $l_1, \ldots, l_n$ are linearly independent (over $\mathbf{Z}$).* We first use Proposition 4.3 with $q = 2$; we get

$$m_{s,0} l_s = \sum_{j=1}^{s} m_{s,j} l_j' \quad (1 \leqslant s \leqslant n),$$

with

$$\max_{\substack{0 \leqslant j \leqslant s\\ 1 \leqslant s \leqslant n}} |m_{s,j}| \leqslant (9nD^3)^n n! (V_n^+)^n,$$

and

$$\max\{h(a_j'),\ |l_j'|/D\} \leqslant V_j' = jV_j \quad (1 \leqslant j \leqslant n).$$

Substituting, we get

$$\Lambda = \beta_0 + \sum_{j=1}^{n} \beta_j' l_j',$$

where

$$\beta_j' = \sum_{s=j}^{n} m_{s,j}\beta_s/m_{s,0} \quad (1 \leqslant j \leqslant n).$$

From Lemma 2.7 we deduce

$$\max_{1 \leqslant j \leqslant n} \{h(\beta_j')\} \leqslant nW + n^2(n+1)\,\mathrm{Log}(D^3 V_n^+) + \varkappa_n,$$

with

$$\varkappa_n = n^2(n+1)\,\mathrm{Log}(9n) + n(n+1)\,\mathrm{Log}\,n! + \mathrm{Log}\,n.$$

Moreover, since

$$|l_j'|/V_j' \leqslant \max_{1 \leqslant t \leqslant j} |l_t|/V_j \leqslant \max_{1 \leqslant t \leqslant j} |l_t|/V_t \leqslant 4D/E,$$

we obtain

$$E \leqslant \min\{e^{DV_1'};\ 4D \min_{1 \leqslant j \leqslant n} V_j'/|l_j'|\}.$$

Therefore, using the case a, we obtain the desired bound, with $C(n)$ replaced by $C_3(n)$,

$$C_3(n) \leqslant C_2(n) n! \big(1 + \mathrm{Log}(2^{15} n^2)\big),$$

provided that we replace $W + \mathrm{Log}(EDV_n^+)$ by

$$\max\Big\{nW + n^2(n+1)\,\mathrm{Log}(D^3 V_n^+) + \varkappa_n;\ n\,\mathrm{Log}(2^{13} n^2 DV_n^+);\ \frac{2}{n}\,\mathrm{Log}\,E\Big\}.$$

Case c. *The numbers $l_1, \ldots, l_n$ are linearly dependent.* There is no loss of generality to assume $n \geqslant 2$. Among $l_1, \ldots, l_n$, we select $l_1^0, \ldots, l_r^0$ which are $\mathbf{Q}$-linearly independent, such that, for $1 \leqslant j \leqslant n$,

$$b_{j,0} l_j = \sum_{s=1}^{r} b_{j,s} l_s^0,$$

with rational integers $b_{j,0} > 0$, $b_{j,1}, \ldots, b_{j,r}$. Moreover, by Lemma 4.1, we can choose

$$\max_{\substack{0 \leqslant s \leqslant r\\ 1 \leqslant j \leqslant n}} |b_{j,s}| \leqslant \big(9(n-1)D^3\big)^{n-1} (V_n^+)^{n-1}.$$

Plainly, we have

$$\Lambda = \beta_0 + \sum_{s=1}^{r} \beta_s^0 l_s^0,$$

where

$$\beta_s^0 = \sum_{j=1}^{n} \beta_j b_{j,s}/b_{j,0} \qquad (1 \leqslant s \leqslant r).$$

Lemma 2.7 shows that

$$\max_{1\leqslant s\leqslant r} h(\beta_s^0) \leqslant nW + n(n^2-1)\mathrm{Log}(D^3 V_n^+) + \varkappa_n',$$

with

$$\varkappa_n' \leqslant n(n^2-1)\mathrm{Log}\big(9(n-1)\big) + \mathrm{Log}\, n.$$

Let us define $V_s^0 = V_{n-r+s}$ $(1 \leqslant s \leqslant r)$. From the case b (with n replaced by $r \leqslant n-1$) we see that the conclusion of the theorem holds with $C(n)$ replaced by $C_4(n)$,

$$C_4(n) \leqslant C_3(n)(8c_2 n^2)^{-1},$$

provided that we replace the factor $W + \mathrm{Log}(EDV_n^+)$ by

$$\max\{(n-1)nW + n(n-1)^2(n+2)\mathrm{Log}(D^3V_n^+) + (n-1)\varkappa_n' + \varkappa_{n-1};$$
$$n\mathrm{Log}(2^{13}n^2DV_n^+);\ \mathrm{Log}\,E\}.$$

Conclusion of the proof. Because of the small value of $C_4(n)$, there is no loss of generality to assume that we are in case b. Since $n\mathrm{Log}(2^{13}n^2) < 3n^2(n+1)$ for $n \geqslant 2$ and $3n^2(n+1) < \varkappa_n$ for $n \geqslant 2$, we plainly have $C(1) \leqslant 13(\mathrm{Log}\,2)C_3(1)$, and $C(n) \leqslant \varkappa_n C_3(n)$ for $n \geqslant 2$. We use the values of c_0, c_2, c_3, c_4 indicated in the proof of Proposition 3.8, and $c_1 = 98/25$. Further we use the following bounds:

$$(1+2^{-9})\frac{98}{100}13(\mathrm{Log}\,2)(1+15\,\mathrm{Log}\,2) < 2^7,$$

$$(24\,\mathrm{Log}\,3 + 19\,\mathrm{Log}\,2)(1+17\,\mathrm{Log}\,2)\left(\frac{31}{10}+2^{-8}\right)\frac{98}{100} < 2^{11},$$

$$(12\,\mathrm{Log}\,2 + 121\,\mathrm{Log}\,3)(1+15\,\mathrm{Log}\,2 + 2\,\mathrm{Log}\,3)(3+2^{-8})\frac{98}{100}3^{10} < 2^{29},$$

and

$$(5.1) \qquad (5+2^{-8})\frac{98}{100}\big(1+\mathrm{Log}(2^{15}n^2)\big)n^4\varkappa_n 2^n < e^n 2^{31}.$$

Therefore

$$C(1) \leqslant 2^{35}, \quad C(2) \leqslant 2^{53}, \quad C(3) \leqslant 2^{71},$$

and

$$C(n) \leqslant 2^{8n+51} n^{2n} \quad \text{for} \quad n \geqslant 4.$$

This completes the proof of the theorem.

Final remark. If we replace (5.1) by

$$(5+2^{-8})\frac{98}{100}\big(1+\mathrm{Log}(2^{15}n^2)\big)n^4\varkappa_n < e^n 2^{19} \qquad (\text{resp.} < e^n 2^{n+13}),$$

we see that for all $n \geqslant 1$,

$$C(n) \leqslant 2^{9n+39}n^{2n} \quad \text{and} \quad C(n) \leqslant 2^{10n+33}n^{2n}.$$

Finally, since $435 < 161e$, we easily deduce from (3.25) that there exists an absolute constant $n_0 \geqslant 1$ such that

$$C(n) \leqslant (13n)^{2n} \quad \text{for all } n \geqslant n_0.$$

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