by our choice of r. Thus from (20), and since $|c_q| \leqslant I$ and $|S(-q)| = |S(q)|$, we have

$$\sum_{q=1}^{K} |S(q)| \gg N^s.$$

Hence there is a q_0, $1 \leqslant q_0 \leqslant K \leqslant I^{-1}N^{s/10}$ with

$$|S(q_0)|^2 \gg N^{2s}K^{-2}. \tag{21}$$

Now let h be odd with $1 \leqslant h \leqslant (s+5)/3$. Then by the lower bound for I in the hypothesis of Lemma 2, the inequality (13) holds for

$$Z = N^{1-(s/(4(s-h+1)))} I^{2/(s-h+1)}.$$

With this choice of Z we have

$$Z^{s-h+1}N^{s+h-1}(\log N)^s \leqslant N^{2s-(s/4)}I^2N^{s/20} = N^{2s-(s/5)}I^2 \leqslant N^{2s}K^{-2},$$

by the definition of K, if N is large enough. Hence by (21) the second alternative of Lemma 4 holds with $k = q_0$. This implies the second alternative of Lemma 2 with $m = 2q_0 = 2k$ for N large enough.

References

[1] R. J. Cook, *On the fractional parts of an additive form*, Proc. Cambridge. Phil. Soc. 72 (1972), pp. 209–212.

[2] I. Danicic, *An extension of a theorem of Heilbronn*, Mathematika 5 (1958), pp. 30–37.

[3] H. Davenport, *Indefinite quadratic forms in many variables (II)*, Proc. London Math. Soc. (3) 8 (1958), pp. 109–126.

[4] H. Heilbronn, *On the distribution of the sequence θn^2* (mod 1), Quart. J. Math. 19 (1948), pp. 249–256.

[5] H. P. Schlickewei, *On indefinite diagonal forms in many variables* (to appear).

[6] W. M. Schmidt, *Small fractional parts of polynomials*, A. M. S. Regional Conference Series in Math. (to appear).

[7] B. Segre, *Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane*, Ann. Mat. Pura Appl. (4) 64 (1964), pp. 1–76.

[8] I. M. Vinogradov, *The method of trigonometric sums in the theory of numbers* (English transl.), Wiley-Interscience, 1954.

POLISH ACADEMY OF SCIENCES, Warsaw
UNIVERSITY OF COLORADO AND MATH. INST. UNIV. FREIBURG
UNIVERSITY OF COLORADO, Boulder

Received on 21. 10. 1977 (992)

ACTA ARITHMETICA
XXXVII (1980)

On the differences of additive functions, II

by

P.D.T.A. Elliott* (Boulder, Colo.)

In memory of Professor P. Turán

An arithmetic function $f(n)$ is said to be additive if it satisfies the relation $f(ab) = f(a)+f(b)$ whenever a and b are coprime (positive) integers. Concerning real-valued additive functions we have:

THEOREM 1. *Let a be a positive integer. Then the following three conditions are equivalent:*

(i) *There is a constant B so that the inequality*

$$\sum_{n\leqslant x} |f(n+a)-f(n)|^2 \leqslant Bx$$

holds for all $x \geqslant 2$;

(ii) *There is a constant C so that the inequality*

$$\sum_{n\leqslant x} n^{-1}|f(n+a)-f(n)|^2 \leqslant C\log x$$

holds for all $x \geqslant 2$;

(iii) *There is a constant A so that the series*

$$\sum_{p}\sum_{k=1}^{\infty} p^{-k}|f(p^k)-A\log p^k|^2$$

converges.

As a companion to this result we have

THEOREM 2. *Let λ and μ be real numbers, $\lambda+\mu \neq 0$. Let a and b be integers. Then the following three conditions are equivalent:*

(i) *There is a constant B so that the inequality*

$$\sum_{n\leqslant x} |\lambda f(n+a)+\mu f(n+b)|^2 \leqslant Bx$$

holds for all $x \geqslant 2$;

* Supported by N. S. F. contract number MCS 75-08233.

(ii) *There is a constant C so that the inequality*

$$\sum_{n\leqslant x} n^{-1}|\lambda f(n+a)+\mu f(n+b)|^2 \leqslant C\log x$$

holds for all $x\geqslant 2$;

(iii) *The series*

$$\sum_{p}\sum_{k=1}^{\infty} p^{-k}|f(p^k)|^2$$

converges, and the sums

$$\sum_{p^k\leqslant x} p^{-k}f(p^k)$$

are uniformly bounded for all $x\geqslant 2$.

When $a=1$ and the function $f(n)$ is strongly additive, a proof of the equivalence of the validity of propositions (i) and (iii) of Theorem 1 was given in a recent paper of the author, with a title similar to that of the present paper, and to appear in "Mathematika". Most of the proof given there can be modified to deal with the more general situation of Theorem 1, making use of (ii) in place of (i). However, some remarks should be made concerning the first step of the argument, which would now amount to the following:

Assuming the validity of (ii) (*in the statement of Theorem* 1) *we shall prove that*

$$\sum_{n\leqslant x}|f(n)|^2 \leqslant Dx(\log x)^2, \tag{1}$$

for some constant D and all $x\geqslant 2$.

Suppose first that both n and a are odd. Then at least one of the numbers $n-a$ and $n-3a$ is exactly divisible by 2. Otherwise we should have $n-a\equiv n-3a \pmod 4$, and so $2a\equiv 0 \pmod 4$, which is impossible. Thus if $\varepsilon_0=1$ or 3, as the case may be, $n-\varepsilon_0 a=2m_1$ for some odd integer m_1, and

$$f(n)=f(n)-f(n-\varepsilon_0 a)+f(2)+f(m_1). \tag{2}$$

We can now repeat this procedure with m_1 in place of n, to obtain $m_1-\varepsilon_1 a=2m_2$, and so on. After k steps (say) the process terminates, and we have a representation

$$f(n)=f(n)-f(n-\varepsilon_0 a)+kf(2)+f(m_k)+\sum_{j=1}^{k-1}\big(f(m_j)-f(m_j-\varepsilon_j a)\big).$$

Since $k\leqslant(\log n)/\log 2$, an application of the Cauchy–Schwarz inequality shows that

$$|f(n)|^2\leqslant E\Big(|f(n)-f(n-\varepsilon_0 a)|^2+\sum_{j=1}^{k-1}|f(m_j)-f(m_j-\varepsilon_j a)|^2+\log n\Big)\log n$$

for some constant E.

Writing m_0 for n we sum over the odd integers n not exceeding x to obtain

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}}}|f(n)|^2 \leqslant E\log x\sum_{n}\Big(\log x+\sum_{\substack{m_j\leqslant x\\ \varepsilon_j=1}}|f(m_j)-f(m_j-a)|^2+\sum_{\substack{m_j\leqslant x\\ \varepsilon_j=3}}|f(m)-f(m-3a)|^2\Big).$$

If an integer n gives rise to a particular m_j ($=m$ say) in the "decomposition" associated with the representation (2), then we may recover n from m by a sequence of operations

$$m\to 2m\to(2m+\eta_1 a)\to 2(2m+\eta_1 a)\to 2(2m+\eta_1 a)+\eta_2 a\to,$$

etc., where each $\eta_1,\eta_2,\ldots,$ has a value 1 or 3. Since n does not exceed x, the number of η_i in such a sequence of operations, say l, is restricted by $2^l m\leqslant x$. Moreover, the totality of all integers n ($\leqslant x$) which can give rise to the integer m is included in those obtained by choosing each η_i to be 1 or 3 "at random". In this way not more than $2^l\leqslant x/m$ distinct values of n may be reached, so that (for example)

$$\sum_{n}\sum_{\substack{m_j\leqslant x\\ \varepsilon_j=3}}|f(m_j)-f(m_j-3a)|^2\leqslant x\sum_{3a<m\leqslant x} m^{-1}|f(m)-f(m-3a)|^2.$$

Since

$$|f(m)-f(m-3a)| \leqslant |f(m)-f(m-a)|+|f(m-a)-f(m-2a)|+|f(m-2a)-f(m-3a)|$$

we obtain from (ii) the upper bound $\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}}}|f(n)|^2\leqslant Fx(\log x)^2$ for some constant F and all $x\geqslant 2$.

Moreover, if n is even then $n-a$ is odd, hence

$$\sum_{\substack{a<n\leqslant x\\ n\ \mathrm{even}}}|f(n)|^2\leqslant \sum_{\substack{a<n\leqslant x\\ n\ \mathrm{even}}}|f(n)-f(n-a)+f(n-a)|^2$$

$$\leqslant 2\sum_{a<n\leqslant x}|f(n)-f(n-a)|^2+2\sum_{\substack{t\leqslant x-a\\ t\ \mathrm{odd}}}|f(t)|^2\leqslant Gx(\log x)^2.$$

Altogether, the bound (1) is established for odd values of a.

If a is even we argue as follows. Let $a=2^r b$, where $2\nmid b$. Then as in the above argument we obtain

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}}}|f(n)|^2\leqslant Hx(\log x)^2+Jx\sum_{\substack{b<m\leqslant x\\ m\ \mathrm{odd}\\ 2\|(m-b)}} m^{-1}|f(m)-f(m-b)|^2+ \\ +Jx\sum_{\substack{3b<m\leqslant x\\ m\ \mathrm{odd}\\ 2\|(m-3b)}} m^{-1}|f(m)-f(m-3b)|^2.$$

If (say) $2\|(m-b)$, where m is odd, then

$$f(m)-f(m-b) = f(2^r m)-f\big(2^r(m-b)\big)-f(2^r)+f(2^{r+1})-f(2),$$

and

$$\sum_{\substack{b<m\leqslant x\\ m\ \mathrm{odd}\\ 2\|(m-b)}} m^{-1}|f(m)-f(m-b)|^2 \leqslant K \sum_{b<w\leqslant 2^r x} w^{-1}\big(|f(w)-f(w-a)|^2+1\big) \leqslant L\log x.$$

The sum involving $f(m)-f(m-3b)$ may be similarly treated, and once again we obtain a bound

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}}} |f(n)|^2 \leqslant Mx(\log x)^2.$$

Suppose now that n is even; then at least one of the integer $n+a$, $n+2a$ is not divisible by 2^{r+1}. Let $n+2a = 2^j s$, where $1\leqslant j\leqslant r$, $2\nmid s$, for example. Then

$$f(n) = f(n)-f(n+2a)+f(2^j)+f(s)$$

and

$$|f(n)|^2 \leqslant 3|f(n+2a)-f(n)|^2+3|f(2^j)|^2+3|f(s)|^2.$$

Amongst those even integers n which do not exceed x, at most $2r$ can give rise to a particular integer s in the above manner, so that

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{even}}} |f(n)|^2 \leqslant N\Big(\sum_{n\leqslant x}|f(n+2a)-f(n)|^2+\sum_{\substack{s\leqslant x\\ s\ \mathrm{odd}}}|f(s)|^2\Big)+$$
$$+N\Big(\sum_{n\leqslant x}|f(n+a)-f(n)|^2+\sum_{\substack{s\leqslant x\\ s\ \mathrm{odd}}}|f(s)|^2\Big)+Nx$$
$$\leqslant Rx(\log x)^2$$

for some constants N, R, and all $x\geqslant 2$.

The desired inequality (1) has now been obtained in every case where a is non-zero.

This completes our remarks concerning the proof of Theorem 1.

Proof of Theorem 2. Suppose, for the time being, that $|\lambda|>|\mu|$. Let $\varrho = \mu/\lambda$. Then we can rewrite the hypothesis (ii) of Theorem 2 in the form

(3) $$\sum_{n\leqslant x} n^{-1}|f(n)+\varrho f(n+d)|^2 \leqslant D\log x,\qquad x\geqslant 2,$$

where d is a (positive, negative or zero) integer, and D is a constant.

We first prove that

(4) $$f(n) = O\big((n\log n)^{1/2}\big).$$

From (3),

$$f(n) = -\varrho f(n+d)+O\big((n\log n)^{1/2}\big).$$

Arguing inductively we obtain

$$f(n) = (-\varrho)^j f(n+jd)+O\big((n\log n)^{1/2}\big)$$

the implied constant now depending upon the positive integer j. Let q ($\geqslant 3$) be a prime which does not divide d. Then there is a value for j in the range $1\leqslant j\leqslant q^2$ so that $n+jd\equiv q\ (\mathrm{mod}\, q^2)$, and so

$$|f(n)| \leqslant |\varrho|\,|f(n+jd)|+c_1(n\log n)^{1/2} \leqslant |\varrho|\,|f(m)+c_2(n\log n)^{1/2}_j|$$

where $n+jd = qm$, $q\nmid m$, and the integer m is bounded above by $m\leqslant (n+q^2d)/q\leqslant n/2$ provided only that n exceeds some fixed number n_0, say.

Arguing inductively (once again) we obtain the bound

$$|f(n)| \leqslant c_2\Big\{(n\log n)^{1/2}+|\varrho|\Big(\frac{n}{2}\log\frac{n}{2}\Big)^{1/2}+\ldots\Big\} \leqslant c_3(n\log n)^{1/2},$$

as was asserted in (4).

Our next step is to prove that

(5) $$S = \sum_{n\leqslant x} n^{-1}|f(n)|^2 \leqslant c_4\log x\qquad (x\geqslant 2).$$

In fact for each integer n

$$f(n) = f(n)+\varrho f(n+d)-\varrho f(n+d)$$

so that

$$S \leqslant \sum_{n\leqslant x} n^{-1}\{|f(n)+\varrho f(n+d)|^2+\varrho^2|f(n+d)|^2+$$
$$+2|f(n)+\varrho f(n+d)|\,|\varrho f(n+d)|\}$$
$$\leqslant E\log x+\varrho^2\sum_{n\leqslant x-d} n^{-1}|f(n+d)|^2+$$
$$+2|\varrho|\Big\{\sum_{n\leqslant x} n^{-1}|f(n)+\varrho f(n+d)|^2\sum_{n\leqslant x} n^{-1}|f(n+d)|^2\Big\}^{1/2}$$

this last step by means of (4) and the Cauchy–Schwarz inequality. Since

$$\sum_{n\leqslant x-d} n^{-1}|f(n+d)|^2 = \sum_{n\leqslant x-d}(n+d)^{-1}|f(n+d)|^2+\sum_{n\leqslant x-d}\frac{d}{n(n+d)}|f(n+d)|^2$$

we obtain the inequality

$$S \leqslant \varrho^2 S+c_5(S\log x)^{1/2}+c_6\log x$$

for certain constants c_5 and c_6. Hence

$$(1-\varrho^2)S \leqslant \begin{cases}(c_5+c_6)\log x & \text{if} \quad S \leqslant \log x,\\ (c_5+c_6)(S\log x)^{1/2} & \text{if} \quad S > \log x\end{cases}$$

and (5) is (in either case) established.

For a sufficiently large constant A, and all large enough x,

$$\sum_{\substack{\sqrt{x}<n\leqslant x\\ |f(n)|>A}} \frac{1}{n} \leqslant \frac{c_4}{A^2}\log x < 10^{-2}\log x,$$

so that

$$\sum_{\substack{\sqrt{x}<n\leqslant x\\ |f(n)|\leqslant A}} \frac{\mu^2(n)}{n} > \left(\frac{6}{\pi^2} - 10^{-2}\right)\log x + O(1) > \frac{1}{4}\log x.$$

Define

$$F(y) = \sum_{\substack{\sqrt{x}<n\leqslant y\\ |f(n)|\leqslant A}} \mu^2(n), \qquad \beta = \sup_{\sqrt{x}<y\leqslant x} y^{-2}F(y).$$

Then

$$\frac{1}{4}\log x < \int_{\sqrt{x}}^{x} \frac{1}{y}\,dF(y) \leqslant \int_{\sqrt{x}}^{x} \frac{F(y)}{y^2}\,dy + 1 \leqslant \beta \int_{\sqrt{x}}^{x} \frac{dy}{y} + 1$$

where we have applied an integration by parts. For all sufficiently large x we see that $\beta > 1/3$.

In the terminology of Erdös (see also Ryavec [3]) we have proved that the additive function $f(n)$ is finitely distributed. Hence there is a constant C so that the series

$$\sum_p p^{-1}|f(p) - C\log p|^2$$

converges. Since $F(y)$ counts *square free* integers n for which $|f(n)| \leqslant A$, we can readily show that the constant C has the value zero, and hence that the series

$$\sum p^{-1}f(p)^2$$

converges.

We now appeal to the dual of the Turán–Kubilius inequality, in the form

$$(6) \qquad \sum_{p^k\leqslant x} p^k \Bigg|\sum_{\substack{n\leqslant x\\ p^k\| n}} \frac{a_n}{n} - p^{-k}(1-p^{-1})\sum_{n\leqslant x}\frac{a_n}{n}\Bigg|^2 \leqslant c_7 \log x \sum_{n\leqslant x} \frac{|a_n|^2}{n};$$

(set $\sigma = 1+(\log x)^{-1}$ in Lemma 6 of the author's paper [1]). Here the contribution of the summands involving p^k with $k \geqslant 2$ is all that we need.

Set $a_n = f(n)$. Then (typically)

$$\sum_{\substack{n\leqslant x\\ p^k\| n}} \frac{a_n}{n} = \sum_{\substack{m\leqslant x/p^k\\ p\nmid m}} \frac{1}{mp^k}\big(f(p^k)+f(m)\big) = \frac{f(p^k)}{p^k} \sum_{\substack{m\leqslant x/p^k\\ p\nmid m}} \frac{1}{m} + O(p^{-k}\log x),$$

and we deduce from (6) that

$$\sum_{p^k\leqslant x,\,k\geqslant 2}\sum \frac{f(p^k)^2}{p^k}\Bigg(\sum_{\substack{m\leqslant x/p^k\\ p\nmid m}} \frac{1}{m}\Bigg)^2 \leqslant c_8(\log x)^2, \qquad x \geqslant 2.$$

Since

$$\sum_{\substack{m\leqslant x/p^k\\ p\nmid m}} \frac{1}{m} \geqslant \sum_{m\leqslant x^{2/3}} \frac{1}{m} - \frac{1}{p}\sum_{r\leqslant x}\frac{1}{r} = \frac{2}{3}\log x - \frac{1}{p}\log x + O\left(\frac{1}{\sqrt{x}}\right) \geqslant \frac{1}{7}\log x$$

if $p^k \leqslant x^{1/3}$ and x is sufficiently large, we deduce that

$$\sum_{p^k\leqslant x^{1/3},\,k\geqslant 2}\sum \frac{f(p^k)^2}{p^k} \leqslant 49\,c_8.$$

This establishes the convergence of the first series in part (iii) of Theorem 2.

An appeal to the Turán–Kubilius inequality in the form

$$\sum_{n\leqslant x} n^{-1}\Big|f(n) - \sum_{p^k\leqslant x} p^{-k}f(p^k)\Big|^2 \leqslant c_9\log x \sum_{n\leqslant x} n^{-1}|f(n)|^2, \qquad x \geqslant 2,$$

allows us to assert that the partial sums

$$\sum_{p^k\leqslant x} \frac{f(p^k)}{p^k}, \qquad x \geqslant 2,$$

are uniformly bounded.

It is now straightforward to complete the proof of Theorem 2 in every case save when $|\lambda| = |\mu|$. In this case the hypothesis (ii) reduces to

$$\sum_{n\leqslant x} n^{-1}|f(n)+f(n+d)^2| = O(\log x)$$

for some integer d and all $x \geqslant 2$. In view of what we have so far proved it is enough to consider the case when d is positive. But then

$$f(n+2d) - f(n) = \{f(n+2d)+f(n+d)\} - \{f(n+d)+f(n)\}$$

so that

$$\sum_{n\leqslant x} n^{-1}|f(n+2d)-f(n)|^2 = O(\log x), \qquad x\geqslant 2.$$

The outstanding case of Theorem 2 may thus be deduced from Theorem 1.

This completes our proof of Theorem 2.

In the latter stages of this proof the influence of Professor Turán's ideas is clearly visible.

References

[1] P.D.T.A. Elliott, *On the differences of additive functions*, Mathematika 24(1977), pp. 153–165.

[2] P. Erdös, *On the distribution function of additive functions*, Annals of Math. 4 (1946), pp. 1–20.

[3] C. Ryavec, *A characterization of finitely distributed additive functions*, J. Number Theory 2(1970), pp. 393–403.

Received on 22. 10. 1977 (993

ACTA ARITHMETICA
XXXVII (1980)

A lower bound for linear forms in logarithms

by

MICHEL WALDSCHMIDT (Paris)

Dedicated to the memory of Paul Turán

We give an explicit lower bound for a non-homogeneous linear form in logarithms of algebraic numbers with algebraic coefficients. We pay a special attention to the dependence on the degree of the algebraic numbers, and on the number of terms in the linear form.

1. The main result. We consider a linear form in logarithms of algebraic numbers

$$\Lambda = \beta_0+\beta_1\log\alpha_1+\ldots+\beta_n\log\alpha_n,$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, and $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers. Our aim is to prove a new lower bound for $|\Lambda|$ assuming that it does not vanish. For a complete history of this subject, we refer to [2].

When K is a number field, we denote by $\mathscr{L}_K$ the set of the logarithms of the elements of K^*:

$$\mathscr{L}_K = \{l\in \boldsymbol{C};\ e^l\in K\}.$$

When $l\in\mathscr{L}_K$ and $\alpha = e^l$, we write $l = \log\alpha$. We use the "absolute logarithmic height" $h(\alpha)$ of Néron and Lang [6] (the definition, and connections with Mahler's measure and with the usual height, are detailed in § 2 below).

Our main result is the following.

THEOREM. *Let K be a number field of degree D over $\boldsymbol{Q}$, $l_1, \ldots, l_n$ be non-zero elements of $\mathscr{L}_K$, and $\beta_0, \ldots, \beta_n$ be elements of K. Define $\alpha_j = e^{l_j}$, $(1\leqslant j\leqslant n)$, and*

$$\Lambda = \beta_0+\beta_1\log\alpha_1+\ldots+\beta_n\log\alpha_n.$$