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Statistical Deuring–Heilbronn phenomenon

by

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To the memory of Paul Turán

1. Introduction. Let χ_1 be a real primitive character (mod k), and let $\beta_1 = 1-\delta$ be a real zero of the Dirichlet L-function $L(s) = L(s, \chi_1)$. Suppose that β_1 is "exceptional" in the sense that $\delta \leqslant 1/\log k$. According to a theorem of Linnik [3], the existence of an exceptional zero has a certain effect — called by Linnik the Deuring–Heilbronn phenomenon — upon the distribution of the zeros of L-functions. More exactly, there exist calculable constants $c_1 > 0$, $c_2 > 0$ such that if $\varrho = \beta + i\gamma$ is a zero of $L(s, \chi)$ (mod q) and if $\delta\log(qk\tau) \leqslant c_1$, where $\tau = \max(2, |\gamma|)$, then (if the case $\chi = \chi_1$, $\varrho = \beta_1$ is excluded)

$$\beta \leqslant 1 - c_2 \log\left(\frac{c_1 e}{\delta\log(qk\tau)}\right)/\log(qk\tau). \tag{1.1}$$

Linnik's proof of this estimate was very complicated. A much simpler proof, depending on Turán's power sum method, was given by Knapowski [2]. Recently Motohashi [7] and the author [1] have found new proofs of (1.1) on the basis of an idea of A. Selberg.

Our purpose in this paper is to investigate the Deuring–Heilbronn phenomenon from a statistical point of view, considering the distribution of zeros of many L-functions both in the horizontal and in the vertical direction.

Define

$$\varphi(s, \chi) = L(s, \chi)L(s, \chi\chi_1);$$

then for $\sigma > 1$

$$\varphi(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}, \tag{1.2}$$

where

$$a_n = \sum_{d|n} \chi_1(d). \tag{1.3}$$

Denote by $K(Q)$ the set of all primitive characters χ such that $\chi\chi_1$ is also primitive and the conductor of χ lies in the interval $[Q, 2Q]$.

THEOREM. *There exist positive calculable constants a, b and c such that if k exceeds a certain calculable bound, then for*

$$k^{\log\log k} \leqslant Q \leqslant \exp(\delta^{-a}) \tag{1.4}$$

(we suppose that

$$\delta \leqslant (\log k \log\log k)^{-1/a} \tag{1.5}$$

in order that the interval (1.4) be non-empty) all functions $\varphi(s, \chi)$ with $\chi \in K(Q)$ except possibly $Q^2\delta^b$ functions at most, satisfy the following conditions:

(i) *If $\varrho = \beta + i\gamma$ is a (non-trivial) zero of $\varphi(s, \chi)$ with $|\gamma| \leqslant \delta^{-c}$, then ϱ is simple and $\beta = \frac{1}{2}$.*

(ii) *If $\varrho_j = \frac{1}{2} + i\gamma_j$ are two distinct zeros of $\varphi(s, \chi)$ with $|\gamma_j| \leqslant \delta^{-c}$, $j = 1, 2$, then*

$$\gamma_1 - \gamma_2 = \frac{2\pi m}{\log(Q^2k)}\left(1 + O\left(\frac{\log k}{\log Q}\right)\right), \tag{1.6}$$

where m is a non-zero integer.

The assumption that $\chi\chi_1$ be primitive for $\chi \in K(Q)$ was made merely in order to simplify the proof; the theorem holds also if we assume just the primitivity of the character χ itself.

A perhaps interesting feature of the theorem is that already the relatively weak inequality (1.5) for δ implies very restrictive conditions on the zeros of a large number of L-functions. The equation (1.6) says that the ordinates of the zeros of $\varphi(s, \chi)$ lie approximately in an arithmetic progression, and recalling the Riemann–von Mangoldt formula we see that the distance between two adjacent zeros of $\varphi(s, \chi)$ in the strip $|t| \leqslant \delta^{-c}$ is almost always approximately $2\pi/\log(Q^2k)$. This well-distribution of the zeros has arithmetic consequences which might be tested against our existing knowledge. Another possibility, which has also been pointed out by Montgomery and Weinberger (see [5], [6]), is to examine the statistics of the differences of the zeros of L-functions, hoping to find a contradiction with (1.6).

In the proof of the theorem we use techniques that are familiar from zero-density problems. For instance, mean value estimates for Dirichlet polynomials and the functional equation of L-functions play important roles in our argument. The key result is the approximate functional equation for $\varphi(s, \chi)$, which we derive in Lemmas 4 and 5.

2. Preliminary lemmas. The first lemma is the hybrid large sieve inequality (see [4], Theorem 7.5).

LEMMA 1. *Let for each primitive character χ of modulus $\leqslant Q$ a set of points*

$$s_j(\chi) = \sigma_j(\chi) + it_j(\chi), \quad j = 1, \ldots, r_\chi$$

with

$$\sigma_j(\chi) \geqslant \alpha, \quad |t_j(\chi)| \leqslant T - \Delta/2,$$
$$|t_j(\chi) - t_k(\chi)| \geqslant \Delta \quad \text{for} \quad j \neq k$$

be given. Let $a(n)$, $n = 1, \ldots, N$, be any complex numbers. Then

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q}^{*} \sum_{j=1}^{r_\chi} \Big| \sum_{n=1}^{N} a(n)\chi(n) n^{-s_j(\chi)} \Big|^2$$
$$\ll (Q^2T + N)(\Delta^{-1} + \log N)\log\log(3N) \sum_{n=1}^{N} |a(n)|^2 n^{-2\alpha}$$

where $\sum^*$ denotes a sum over primitive characters.

For the next lemma, let λ be a constant such that

$$L(\tfrac{1}{2} + it) \ll k^\lambda \tau, \tag{2.1}$$

where $\tau = \max(2, |t|)$. By a deep theorem of Burgess, we could choose $\lambda = 3/16 + \varepsilon$ for any $\varepsilon > 0$. Actually we need much less, say that $\lambda < 1/3$. The numbers a_n in the following lemma were defined in (1.3).

LEMMA 2. *For $x \geqslant k^{2\lambda}$, $y \geqslant 2x$, we have*

$$\sum_{x \leqslant n \leqslant y} a_n^2 n^{-1} \ll \big(L(1)\log(y/x) + x^{-1/2}k^\lambda\big)^{1/2}\log^4 y, \tag{2.2}$$

and for $x \geqslant 2$

$$\sum_{n \leqslant x} a_n^2 n^{-1/2} \ll x^{1/2}\big(L(1) + x^{-1/2}k^\lambda\big)^{1/2}\log^4 x. \tag{2.3}$$

Proof. We have

$$\sum_{x \leqslant n \leqslant y} a_n n^{-1} \ll \sum_{n=1}^{\infty} a_n n^{-1}(e^{-n/y} - e^{-n/x})$$
$$= \frac{1}{2\pi i}\int_{(1)} \zeta(s+1)L(s+1)(y^s - x^s)\Gamma(s)\,ds$$
$$= L(1)\log(y/x) + \frac{1}{2\pi i}\int_{(-1)} (\ldots).$$

Estimating the last contour integral by (2.1), we get

(2.4) $$\sum_{x\leqslant n\leqslant y} a_n n^{-1} \ll L(1)\log(y/x)+x^{-1/2}k^{\lambda}.$$

Let $M>1$. Since $0\leqslant a_n\leqslant \tau(n)$, the contribution to (2.2) of the numbers n such that $\tau(n)\leqslant M$ is by (2.4)

$$\ll M\big(L(1)\log(y/x)+x^{-1/2}k^{\lambda}\big).$$

On the other hand, the numbers for which $\tau(n)>M$ contribute at most

$$M^{-1}\sum_{x\leqslant n\leqslant y}\tau^3(n)n^{-1} \ll M^{-1}\log^8 y,$$

where we used the estimate

(2.5) $$\sum_{n\leqslant x}\tau^h(n) \ll_h x(\log x)^{2^h-1}.$$

Choosing M optimally, we get (2.2). The proof of (2.3) is similar.

Define

(2.6) $$b_n = \sum_{d|n}\mu(d)\mu(n/d)\chi_1(d),$$

where $\mu(n)$ is the Möbius function. For $\sigma>1$ we have

$$\frac{1}{\varphi(s,\chi)} = \sum_{n=1}^{\infty} b_n\chi(n)n^{-s}.$$

Define further

(2.7) $$F(s,\chi) = \sum_{n=1}^{k} a_n\chi(n)n^{-s},$$

(2.8) $$M(s,\chi) = \sum_{n=1}^{k} b_n\chi(n)n^{-s},$$

(2.9) $$H(s,\chi) = M(s,\chi)F(s,\chi) = \sum_{n=1}^{k^2} d_n\chi(n)n^{-s};$$

here $d_1=1$, $d_n=0$ for $n=2,\ldots,k$, and for $n=k+1,\ldots,k^2$

(2.10) $$d_n = \sum_{\substack{d|n\\ n/k\leqslant d\leqslant k}} a_{n/d}b_d = -\sum_{\substack{d|n\\ \max(d,n/d)>k}} a_{n/d}b_d.$$

LEMMA 3. *We have* $b_n=0$ *unless* $n=n_1^2n_2$, *where* $(n_1,n_2)=1$, $|\mu(n_1)| = |\mu(n_2)|=1$, *in which case*

(2.11) $$b_n = \chi_1(n_1)\mu(n_2)a_{n_2}.$$

Further,

(2.12) $$\sum_{n=k+1}^{k^2} d_n^2 n^{-1} \ll \big(L(1)\log k+k^{\lambda-1/2}\big)^{1/4}\log^{42}k.$$

Proof. The first assertions follow easily from (2.6).

For the proof of (2.12) we use the second part of (2.10), which gives

(2.13) $$|d_n| \leqslant \sum_{\substack{d|n\\ d>k}} a_{n/d}|b_d| + \sum_{\substack{d|n\\ d>k}} a_d|b_{n/d}| = e_n+f_n,$$

say. Since $e_n\leqslant\tau^2(n)$, we have

$$e_n^2 \leqslant \tau^3(n)\sum_{\substack{d|n\\ d>k}}|b_d|.$$

Hence by (2.5) and Schwarz's inequality

(2.14) $$\sum_{n=k+1}^{k^2} e_n^2 n^{-1} \leqslant \sum_{d=k+1}^{k^2}|b_d|d^{-1}\sum_{m\leqslant k^2/d}\tau^3(dm)m^{-1}$$
$$\ll \log^8 k\sum_{d=k+1}^{k^2}|b_d|\tau^3(d)d^{-1}$$
$$\ll \log^{40}k\Big(\sum_{d=k+1}^{k^2}|b_d|^2d^{-1}\Big)^{1/2}.$$

Here only the numbers $d=d_1^2d_2$ with $(d_1,d_2)=1$ contribute something, and by (2.11) $|b_d|\leqslant a_{d_2}$. Hence by Lemma 2 the last d-sum in (2.14) is

$$\leqslant \sum_{k<d_1^2d_2\leqslant k^2} a_{d_2}^2(d_1^2d_2)^{-1}$$
$$\ll \sum_{d_2=1}^{k} a_{d_2}^2 d_2^{-1}(kd_2^{-1})^{-1/2} + \sum_{d_2=k+1}^{k^2} a_{d_2}^2 d_2^{-1}$$
$$\ll \big(L(1)\log k+k^{\lambda-1/2}\big)^{1/2}\log^4 k.$$

Together with (2.14), this implies that

$$\sum_{n=k+1}^{k^2} e_n^2 n^{-1} \ll \big(L(1)\log k+k^{\lambda-1/2}\big)^{1/4}\log^{42}k.$$

The same estimate is obtained (in a simpler way) for the sum where f_n stands instead of e_n. Hence (2.12) follows by (2.13).

3. Formulas for $\varphi(s, \chi)$. The functional equation of L-functions implies a functional equation for $\varphi(s, \chi)$ if both χ and $\chi\chi_1$ are primitive:

$$\varphi(s, \chi) = \psi(s, \chi)\varphi(1-s, \overline{\chi}),$$

where

$$\psi(s, \chi) = \left(\frac{\pi^2}{kq^2}\right)^{s-1/2} \frac{\Gamma\left(\frac{1}{2}(1-s+a')\right)\Gamma\left(\frac{1}{2}(1-s+a'')\right)}{\Gamma\left(\frac{1}{2}(s+a')\right)\Gamma\left(\frac{1}{2}(s+a'')\right)}\,\varepsilon(\chi), \tag{3.1}$$

$|\varepsilon(\chi)| = 1$, and $a', a'' = 0$ or 1.

In the next lemma the function $\varphi(s, \chi)$ is expressed by a formula which plays the role of the approximate functional equation; the underlying idea is due to Ramachandra [8].

LEMMA 4. *If χ and $\chi\chi_1$ are primitive non-principal characters and $Y \geqslant 1$, then we have for $0 < \sigma \leqslant 3/4$*

$$\varphi(s, \chi) = \sum_{n=1}^{\infty} a_n\chi(n)e^{-n/Y}n^{-s} + \psi(s, \chi)F(1-s, \overline{\chi}) + I_1(s, \chi) + I_2(s, \chi), \tag{3.2}$$

where

$$I_1(s, \chi) = -\frac{1}{2\pi i}\int\limits_{(-\sigma)} \psi(s+w, \chi)\Big(\sum_{n>k} a_n\overline{\chi}(n)n^{s+w-1}\Big)Y^w\Gamma(w)\,dw, \tag{3.3}$$

$$I_2(s, \chi) = -\frac{1}{2\pi i}\int\limits_{(4/5-\sigma)} \psi(s+w, \chi)F(1-s-w, \overline{\chi})\,Y^w\Gamma(w)\,dw. \tag{3.4}$$

Proof. Applying a Mellin transform in the Dirichlet series (1.2), we get

$$\sum_{n=1}^{\infty} a_n\chi(n)e^{-n/Y}n^{-s} = \frac{1}{2\pi i}\int\limits_{(2)} \varphi(s+w, \chi)\,Y^w\Gamma(w)\,dw.$$

Here we move the integration to the line $\mathrm{Re}\,w = -\sigma$ (the pole of the integrand at $w = 0$ gives the residue $\varphi(s, \chi)$), use the functional equation, cut the Dirichlet series of $\varphi(1-s-w, \overline{\chi})$ into two parts, corresponding to $n \leqslant k$ and $n > k$, and finally move the integral involving the sum $F(1-s-w, \overline{\chi})$ to the line $\mathrm{Re}\,w = 4/5-\sigma$ (the pole $w = 0$ gives this time the residue $\psi(s, \chi)F(1-s, \overline{\chi})$).

In order to simplify the formulations of the next few lemmas, let us introduce two conventions. The phrase "almost all $\chi \in K(Q)$" means: all $\chi \in K(Q)$, save at most $\ll Q^2\delta^b$ characters, where $b > 0$ is a constant. Also, we do not repeat the assumption that the numbers a, b and c be sufficiently small, but the assertions should be understood in this sense.

We fix

$$Y = Q^2, \tag{3.5}$$

and derive a more practical version of (3.2).

LEMMA 5. *Throughout the rectangle*

$$\tfrac{1}{2} - \log\log Q/\log Q \leqslant \sigma \leqslant 3/4, \quad |t| \leqslant \delta^{-c} \tag{3.6}$$

we have, for almost all $\chi \in K(Q)$,

$$\varphi(s, \chi) = F(s, \chi) + \psi(s, \chi)F(1-s, \overline{\chi}) + E(s, \chi) \tag{3.7}$$

with

$$E(s, \chi) \ll \delta^\eta/\log Q, \tag{3.8}$$

where $\eta > 0$ is a numerical constant.

Proof. By (3.2) and (3.7)

$$E(s, \chi) = \sum_{n=1}^{k} a_n\chi(n)(e^{-n/Y}-1)n^{-s} + \sum_{n>k} a_n\chi(n)e^{-n/Y}n^{-s} + I_1(s, \chi) + I_2(s, \chi).$$

By (3.5) the first sum on the right is trivially $\ll Q^{-1}$, say.

The series above can be cut at $2Y\log Y$ with a negligible error. Consider now the truncated sum. If for each $\chi \in K(Q)$ a point s_χ in the rectangle (3.6) is given, then by Lemmas 1 and 2 (taking $\lambda = 1/4$) we have

$$\sum_{\chi\in K(Q)} \Big|\sum_{k<n\leqslant 2Y\log Y} a_n\chi(n)e^{-n/Y}n^{-s_\chi}\Big|^2$$
$$\ll Q^2(\delta^{-c}+\log Q)\log^5 Q\log\log Q\sum_n a_n^2 n^{-1}$$
$$\ll Q^2(\delta^{-c}+\log Q)\log^{10} Q\big(L(1)\log Q + k^{-1/4}\big)^{1/2}.$$

Since $k^{-1/4} \ll \delta^{1/2}$ and $L(1) \ll \delta\log^2 k$, the last expression is $\ll Q^2\delta^{b+2a+2\eta}$, where $\eta > 0$ is a numerical constant (if a, b and c are sufficiently small). This proves that the modulus of the sum under consideration can exceed $\delta^\eta/\log Q$ in the rectangle (3.6) for at most $\ll Q^2\delta^b$ characters in $K(Q)$.

In a similar way we estimate the sums

$$S_j = \sum_{\chi\in K(Q)} |I_j(s_\chi, \chi)|, \quad j = 1, 2.$$

We will make use of the known estimate

$$(q^2k\tau^2)^{1/2-\sigma} \ll |\psi(s, \chi)| \ll (q^2k\tau^2)^{1/2-\sigma}, \quad -1/2 \leqslant \sigma \leqslant 3/4, \tag{3.9}$$

which follows from (3.1) in view of the estimate

$$\exp\left(-\frac{\pi}{2}|t|\right)\tau^{\sigma-1/2} \ll |\Gamma(s)| \ll \exp\left(-\frac{\pi}{2}|t|\right)\tau^{\sigma-1/2}, \quad 1/4 \leqslant \sigma \leqslant 2. \tag{3.10}$$

Let us consider the sum S_1 first. By the Pólya–Vinogradov character sum estimate it is easily seen that for $\chi \in K(Q)$

$$\sum_{n\leqslant x} a_n\chi(n) \ll x^{1/2}(Qk)^{1/2}\log Q,$$

so that by partial summation for $\sigma = 1$

$$\sum_{n>Q^2} a_n\chi(n)n^{-s} \ll |s|k^{1/2}Q^{-1/2}\log Q.$$

This shows that we may cut the series in the integrand of $I_1(s_\chi, \chi)$ at Q^2 with a negligible error. Further, by Lemmas 1 and 2, for $\operatorname{Re} w = -\operatorname{Re} s_\chi$

$$\sum_{\chi\in K(Q)} \Big|\sum_{k<n\leqslant Q^2} a_n\overline{\chi}(n)n^{s_\chi+w-1}\Big|^2 \ll Q^2\delta^{-c}\log Q\log\log Q\sum_n a_n^2 n^{-2}$$
$$\ll Q^2\delta^{-c}k^{-1}\big(L(1)+k^{-1/4}\big)^{1/2}\log Q\log^5 k.$$

Using also (3.5), (3.9) and (3.10), we get for S_1 an estimate showing that for almost all χ we have $|I_1(s, \chi)| \leqslant \delta^\eta/\log Q$ in the rectangle (3.6).

The sum S_2 can be estimated similarly. So the proof of Lemma 5 is complete.

For the next lemma, define

$$G(s, \chi) = M(s, \chi)F(1-s, \overline{\chi}). \tag{3.11}$$

LEMMA 6. *For almost all $\chi \in K(Q)$ the following estimates hold: for $0 \leqslant \sigma \leqslant 1$, $|t| \leqslant 2\delta^{-c}$*

$$G(s, \chi) \ll (k\log^3 k)^{2|\sigma-1/2|}; \tag{3.12}$$

for $|\sigma-\frac{1}{2}| \leqslant A/\log Q$, $|t| \leqslant \delta^{-c}$ (where $A > 0$ is any fixed number)

$$1/2 \leqslant |G(s, \chi)| \leqslant 3/2, \tag{3.13}$$

$$G'(s, \chi) \ll \log k, \tag{3.14}$$

$$H'(s, \chi) \ll 1, \tag{3.15}$$

$$\frac{d}{ds}\Big\{M(s, \chi)E(s, \chi)\Big\} \ll 1, \tag{3.16}$$

and for $1/2 \leqslant \sigma \leqslant 3/4$, $|t| \leqslant \delta^{-c}$

$$|H(s, \chi)| \geqslant 2/3, \tag{3.17}$$

$$M(s, \chi)E(s, \chi) \ll 1/\log Q. \tag{3.18}$$

Proof. The estimates (3.15) and (3.17) follow from (2.9) by Lemmas 1 and 3. Next note that for $\sigma = \frac{1}{2}$ the functions $G(s, \chi)$ and $H(s, \chi)$ have the same modulus, so that as above we have for almost all χ

$$3/4 \leqslant |G(\tfrac{1}{2}+it, \chi)| \leqslant 5/4 \quad \text{for} \quad |t| \leqslant 3\delta^{-c}. \tag{3.19}$$

In view of the trivial estimate

$$G(s, \chi) \ll k\log^3 k \quad \text{for} \quad \sigma = 0 \text{ or } 1$$

we thus get (3.12) by convexity. Now (3.14) follows from (3.12) by Cauchy's formula, and (3.14) together with (3.19) implies (3.13).

Finally, since we may assume that

$$M(s, \chi) \ll \delta^{-\eta} \quad \text{for} \quad \sigma \geqslant \tfrac{1}{2}-\log\log Q/\log Q, \quad |t| \leqslant 2\delta^{-c},$$

we get (3.16) and (3.18) by (3.8) and Cauchy's formula.

4. Horizontal distribution of the zeros. It is convenient to establish the following preliminary assertion first.

LEMMA 7. *There exists a constant $A > 0$ such that for almost all $\chi \in K(Q)$ the function $\varphi(s, \chi)$ has no zero in the region*

$$\sigma \geqslant \tfrac{1}{2}+A/\log Q, \quad |t| \leqslant \delta^{-c}. \tag{4.1}$$

Proof. By known zero-density estimates we may omit the characters such that $\varphi(s, \chi)$ has a zero in the region $\sigma > 3/4$, $|t| \leqslant \delta^{-c}$. Let $\varrho = \beta+i\gamma$ be a zero of $\varphi(s, \chi)$ in the region (4.1). Multiply both sides of the equation (3.7) by $M(s, \chi)$ and put $s = \varrho$. By (2.9) and (3.11) the resulting equation is

$$0 = H(\varrho, \chi)+G(\varrho, \chi)\psi(\varrho, \chi)+M(\varrho, \chi)E(\varrho, \chi). \tag{4.2}$$

By (3.17), (3.18) and (3.9) this implies that

$$|G(\varrho, \chi)| \geqslant |\psi(\varrho, \chi)|^{-1} \geqslant Q^{2\beta-1},$$

which is impossible for almost all $\chi \in K(Q)$ by (3.12) and (1.4) if A is sufficiently large.

We are now in a position to prove the main result of this section.

LEMMA 8. *For almost all $\chi \in K(Q)$ the function $\varphi(s, \chi)$ has no zero in the region*

$$\sigma > \tfrac{1}{2}, \quad |t| \leqslant \delta^{-c}. \tag{4.3}$$

Proof. Let $\varrho = \beta+i\gamma$ be a zero of $\varphi(s, \chi)$ in the region (4.3). Then also $1-\overline{\varrho} = 1-\beta+i\gamma$ is a zero of $\varphi(s, \chi)$. By Lemma 7 we may assume that $\beta-\frac{1}{2} \leqslant A/\log Q$. We apply (4.2) to ϱ and $1-\overline{\varrho}$, and subtract one equation from the other. By (3.15), (3.16) and (3.14)

$$H(\varrho, \chi)-H(1-\overline{\varrho}, \chi) \ll 2\beta-1,$$

$$M(\varrho, \chi)E(\varrho, \chi)-M(1-\overline{\varrho}, \chi)E(1-\overline{\varrho}, \chi) \ll 2\beta-1,$$

$$G(\varrho, \chi)-G(1-\overline{\varrho}, \chi) \ll (2\beta-1)\log k,$$

whereas by (3.1)

$$|\psi(\varrho, \chi)-\psi(1-\overline{\varrho}, \chi)| \geqslant (2\beta-1)\log Q.$$

Hence, noting also (3.13), we get the inequality

$$\begin{aligned} &2\beta-1\\ &\geqslant |\psi(\varrho,\chi)G(\varrho,\chi)-\psi(1-\bar\varrho,\chi)G(1-\bar\varrho,\chi)|\\ &\geqslant |\psi(\varrho,\chi)-\psi(1-\bar\varrho,\chi)|\,|G(1-\bar\varrho,\chi)|-|\psi(\varrho,\chi)|\,|G(\varrho,\chi)-G(1-\bar\varrho,\chi)|\\ &\geqslant (2\beta-1)\big(\log Q+O(\log k)\big),\end{aligned}$$

a contradiction.

5. Vertical distribution of the zeros. Let $\varrho_j=\frac{1}{2}+i\gamma_j$, $j=1,2$, with $|\gamma_j|\leqslant\delta^{-c}$ be two zeros of $\varphi(s,\chi)$. We may suppose that $|\gamma_1-\gamma_2|\leqslant 1$, for otherwise the assertion (1.6) to be proved is trivial.

Again we use the equation (4.2), and get after a rearrangement

$$G(\varrho_j,\chi)\psi(\varrho_j,\chi)=-H(\varrho_j,\chi)-M(\varrho_j,\chi)E(\varrho_j,\chi),\quad j=1,2. \tag{5.1}$$

Consider the change of the argument of both sides of this equation when ϱ_1 is replaced by ϱ_2. The change of $\operatorname{Arg}\psi(s,\chi)$ is easy to estimate.

LEMMA 9. *For* $\chi\in K(Q)$ *and* $|t_1-t_2|\leqslant 1$, $|t_j|\leqslant T$ $(\geqslant 2)$, *we have*

$$\operatorname{Arg}\psi(\tfrac{1}{2}+it_1,\chi)-\operatorname{Arg}\psi(\tfrac{1}{2}+it_2,\chi)\equiv(t_2-t_1)\big(\log(Q^2k)+O(\log T)\big)\ (\mathrm{mod}\ 2\pi).$$

Proof. This follows immediately from (3.1) and the formula

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O(|s|^{-1}),$$

valid in the angle $-\pi+\varepsilon<\operatorname{Arg}s<\pi-\varepsilon$ for any fixed $\varepsilon>0$.

Returning to the equation (5.1), we see by Lemma 9 and (3.13), (3.14) that the argument change on the left hand side is

$$(\gamma_2-\gamma_1)\big(\log(Q^2k)+O(\log k)\big)\ (\mathrm{mod}\ 2\pi). \tag{5.2}$$

The argument change on the right hand side can be estimated by considering the integral of the modulus of the derivative; by (3.15) and (3.16) the result is $\ll|\gamma_2-\gamma_1|$. Comparing this with (5.2), we get

$$(\gamma_2-\gamma_1)\big(\log(Q^2k)+O(\log k)\big)=2\pi m+O(|\gamma_2-\gamma_1|),$$

where m is an integer (clearly $m\neq 0$). This proves (1.6).

Finally consider the multiplicity of the zeros of $\varphi(s,\chi)$. If the zero $\varrho=\frac{1}{2}+i\gamma$ is not simple, then

$$\varphi(\varrho,\chi)=\varphi'(\varrho,\chi)=0,$$

and consequently

$$\frac{d}{ds}\big(M(s,\chi)\varphi(s,\chi)\big)_{s=\varrho}=0.$$

On the other hand, this derivative is equal to

$$H'(\varrho,\chi)+G'(\varrho,\chi)\psi(\varrho,\chi)+G(\varrho,\chi)\psi'(\varrho,\chi)+\frac{d}{ds}\big(M(s,\chi)E(s,\chi)\big)_{s=\varrho}.$$

It follows by the estimates of Lemma 6 that (for almost all χ)

$$|G(\varrho,\chi)\psi'(\varrho,\chi)|\ll\log k;$$

this is impossible since (again for almost all χ) the left hand side is $\gg\log Q$.

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