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On the remainder term of the prime number formula II
On a theorem of Ingham

by

J. PINTZ (Budapest)

To the memory of Paul Turán

1. In this paper we shall investigate the connection between the zerofree region of $\zeta(s)$ and the remainder term of the prime number formula.

The fact that the complex zeros of $\zeta(s)$ "have an influence" on the distribution of primes was discovered by Riemann in 1859. However it was nearly 40 years later that Hadamard's deep function-theoretic work concerning the analytical properties of $\zeta(s)$ made it possible in 1896 to prove the prime number theorem.

Wiener showed that only one property of $\zeta(s)$ is essential, namely

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma = 1. \tag{1.1}$$

De la Vallée Poussin showed in 1899 that the domain

$$\sigma > 1 - \frac{c_1}{\log(|t|+2)} \tag{1.2}$$

contains no zeros of $\zeta(s)$. From this one gets for the remainder terms the estimations:

$$\Delta_i(x) = O\left(x \exp(-c_2\sqrt{\log x})\right) \qquad (1 \leqslant i \leqslant 4) \tag{1.3}$$

where we use the notations

$$\begin{aligned}
\Delta_1(x) &\stackrel{\text{def}}{=} \pi(x) - \operatorname{li} x \stackrel{\text{def}}{=} \sum_{p \leqslant x} 1 - \int_0^x \frac{dt}{\log t}, \\
\Delta_2(x) &\stackrel{\text{def}}{=} \Pi(x) - \operatorname{li} x \stackrel{\text{def}}{=} \sum_{\nu \geqslant 1} \frac{1}{\nu}\pi(x^{1/\nu}) - \operatorname{li} x, \\
\Delta_3(x) &\stackrel{\text{def}}{=} \theta(x) - x \stackrel{\text{def}}{=} \sum_{p \leqslant x} \log p - x, \\
\Delta_4(x &\stackrel{\text{def}}{=} \psi(x) - x \stackrel{\text{def}}{=} \sum_{n \leqslant x} \Lambda(n) - x.
\end{aligned} \tag{1.4}$$

It turned out that larger zerofree domains of $\zeta(s)$ imply smaller remainder terms. A general theorem in this direction was proved by Ingham [8], Theorem 22.

THEOREM. *Suppose that $\zeta(s)$ has no zeros in the domain*

$$\sigma > 1-\eta(|t|) \tag{1.5}$$

where $\eta(t)$ is, for $t \geqslant 0$, a decreasing function, having a continuous derivative $\eta'(t)$ and satisfying the following conditions:

$$0 < \eta(t) \leqslant \tfrac{1}{2}, \tag{1.6}$$

$$\eta'(t) \to 0 \quad as \quad t \to \infty, \tag{1.7}$$

$$\frac{1}{\eta(t)} = O(\log t) \quad as \quad t \to \infty. \tag{1.8}$$

Let ε be a fixed number satisfying $0 < \varepsilon < 1$, and let

$$\omega(x) \overset{\text{def}}{=} \min_{t \geqslant 1} \big(\eta(t)\log x + \log t\big). \tag{1.9}$$

Then we have

$$\frac{\Delta_i(x)}{x} = O\left(\frac{1}{e^{\frac{1}{2}(1-\varepsilon)\omega(x)}}\right) \quad (1 \leqslant i \leqslant 4). \tag{1.10}$$

In the important special case

$$\eta(t) = \frac{c_3}{\log^{\beta}(t+2)} \quad (\beta > 0) \tag{1.11}$$

Ingham's theorem says: if

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1-\frac{c_3}{\log^{\beta}(|t|+2)}, \tag{1.12}$$

then one has

$$\frac{\Delta_i(x)}{x} = O\Big(\exp\big(-c_4(\log x)^{1/(1+\beta)}\big)\Big). \tag{1.13}$$

From the results of Korobov–Vinogradov we know that for $\beta = \frac{2}{3}+\varepsilon$ the assumption (1.12) is true, and so we get

$$\frac{\Delta_i(x)}{x} = O\Big(\exp\big(-(\log x)^{3/5-\varepsilon}\big)\Big). \tag{1.14}$$

2. Concerning the theorem of Ingham in the special case (1.11) one can ask whether finer analytic methods can lead to a better estimate for $\Delta_i(x)$ than that given by (1.13). In other words: is it possible to prove the converse of the theorem of Ingham, i.e. to show that (1.13) implies (1.12)? Another equivalent formulation of the problem would be whether supposing that there are (perhaps infinitely many) zeros in the domain given by (1.12), it is possible the estimation in (1.13) to hold for $\Delta_i(x)$.

In the affirmative case it would mean that the distribution of the primes also has an "influence" on the zeros of $\zeta(s)$, i.e. it would clarify somewhat the connection between the distribution of primes and the ζ-roots. Answering the other formulation of the question in the affirmative would mean the satisfactory fact, that our methods of deducing the remainder term of the prime number formula from zero-free regions are in some sense optimal (i.e. at least in the special case (1.11) and apart from the constants).

For a long time the only possibility to prove the prime number theorem and to estimate the remainder term was through the zero-free region of $\zeta(s)$. Considering Wiener's theorem this situation was quite understandable, as an elementary proof of the prime number theorem is also an elementary proof for (1.1).

However, in 1949, the ingenious elementary methods of Erdös [6] and Selberg [13] led to a proof of the prime number theorem. Later also a remainder term of the form

$$\frac{\Delta_i(x)}{x/\log x} = O\left(\frac{1}{\log^{B} x}\right) \quad (i = 1, 2) \tag{2.1}$$

was proved with $B \approx 1/200$ by van der Corput [4] in 1955, somewhat later with $B \approx 1/10$ by Kuhn [9]. In 1960 Breusch [2] proved this with $B = 1/6$, and in the same year Bombieri [1] with an arbitrary B. In 1970 Diamond and Steinig [5], using a refinement of a method of Wirsing [16], proved the sharper estimate where they replaced the right side of (2.1) by

$$O\Big(\exp\big(-(\log x)^{1/7}(\log\log x)^{-2}\big)\Big).$$

Now another interesting problem is whether the mentioned estimates for the remainder term imply (larger and larger) zero-free regions of $\zeta(s)$. The affirmative answer would mean a new elementary method (even though weaker than the older analytic ones) for finding zero-free regions of $\zeta(s)$.

The arising questions were (partially) answered by Turán [15] who applying the powersum method proved in 1950 the converse of Ingham's theorem in the important special case given by (1.11). His theorem sounds as follows.

THEOREM (Turán). *If for a β with $0 < \beta < 1$ we have*

$$\frac{\Delta_i(x)}{x} = O\Big(\exp\big(-c_5(\log x)^{1/(1+\beta)}\big)\Big), \tag{2.2}$$

then $\zeta(s)$ does not vanish in the domain

$$\sigma > 1 - \frac{c_6}{\log^{\beta}|t|}, \qquad |t| \geqslant c_7(\beta). \tag{2.3}$$

A partial converse of Ingham's theorem (see (1.6)–(1.10)) was proved by W. Staś [14] in 1961, using again Turán's method:

THEOREM (Staś). *Let $0 < \varepsilon < 1$ be fixed. Let $\eta(t)$ be a continuous decreasing function for $t \geqslant 0$ with the properties:*

$$0 < \eta(t) \leqslant \tfrac{1}{2}, \tag{2.4}$$

$$\eta(t) < \left(4(1-\varepsilon)\log\frac{3\cdot 10^3}{1-\varepsilon}\right)^{-1}, \qquad t > c_8(\varepsilon). \tag{2.5}$$

If

$$\frac{1}{\eta(t)} = O(\log t), \tag{2.6}$$

then

$$(1-\varepsilon)\eta(t)\log\frac{1}{\eta(t^2)} \leqslant 1, \qquad t \geqslant c_9(\varepsilon). \tag{2.7}$$

Let us suppose that

$$\left|\frac{\Delta_4(x)}{x}\right| \leqslant c_{10}\exp\left(-\tfrac{1}{2}(1-\varepsilon)\omega(x)\right) \tag{2.8}$$

where

$$\omega(x) \overset{\text{def}}{=} \min_{t\geqslant 1}\left(\eta(t)\log x + \log t\right). \tag{2.9}$$

Then $\zeta(s) \neq 0$ in the domain

$$\sigma > 1 - \frac{1-\varepsilon}{1600}\eta(t^{4/(1-\varepsilon)}), \qquad t > \max\left(c_{11},\, c_{12}(\varepsilon)\right). \tag{2.10}$$

This theorem already gives an answer to the second question, namely that the elementary estimate (2.1) implies the non-vanishing of $\zeta(s)$ in the region

$$\sigma > 1 - \frac{c_{13}\log t}{t^{2/B}}, \qquad t > c_{14}. \tag{2.11}$$

3. To prove an almost complete converse of Ingham's theorem it is necessary that Ingham's theorem and also the converse be optimal. However, a short reflection on the theorem tells us that this is not the case. Namely, if the least upper bound of the real parts of the zeros is θ, then one can take

$$\eta(t) = 1 - \theta \tag{3.1}$$

and we get

$$\frac{\Delta_i(x)}{x} = O\left(\frac{1}{x^{\frac{1}{2}(1-\varepsilon)(1-\theta)}}\right). \tag{3.2}$$

On the other hand, it is well known (see e.g. Ingham [8], Theorem 30) that (3.1) implies

$$\Delta_i(x) = O(x^{\theta+\varepsilon}), \tag{3.3}$$

i.e.

$$\frac{\Delta_i(x)}{x} = O\left(\frac{1}{x^{(1-\varepsilon')(1-\theta)}}\right). \tag{3.4}$$

Here the consequence (3.3) is optimal regarding the exponent as we have by a theorem of E. Schmidt [12] for $\theta > \frac{1}{2}$

$$\Delta_i(x) = \Omega_{\pm}(x^{\theta-\varepsilon}) \qquad (1 \leqslant i \leqslant 4), \tag{3.5}$$

and the same holds also by a theorem of Littlewood [10] for $\theta = \frac{1}{2}$.

This suggests that Ingham's theorem gives an error estimate which is about the square root of the true error, i.e. one may conjecture that in formula (1.10) the factor $\frac{1}{2}(1-\varepsilon)$ can be replaced by $1-\varepsilon$.

This is really true, namely we have the

THEOREM 1. *Suppose that $\zeta(s)$ has no zero in the domain*

$$\sigma > 1 - \eta(|t|) \qquad (0 < \eta(t) \leqslant \tfrac{1}{2}) \tag{3.6}$$

where $\eta(t)$ is for $t \geqslant 0$, a continuous decreasing function. Let $0 < \varepsilon < 1$ be fixed; further, let

$$\omega(x) \overset{\text{def}}{=} \min_{t\geqslant 1}\left(\eta(t)\log x + \log t\right). \tag{3.7}$$

Then we have

$$\bar{\Delta}_i(x) = O\left(\frac{1}{e^{(1-\varepsilon)\omega(x)}}\right) \tag{3.8}$$

where

$$\bar{\Delta}_i(x) = \begin{cases} \dfrac{\Delta_i(x)}{x} & \text{for } i = 3, 4, \\[2mm] \dfrac{\Delta_i(x)}{x/\log x} & \text{for } i = 1, 2. \end{cases} \tag{3.9}$$

One can see that we do not make use of conditions (1.7)–(1.8) for $\eta(t)$, and at the same time we get a better result than that in (1.10). However the proof is deeper.

4. For the proof of Theorem 1 we first consider the case

$$\lim_{t\to\infty}\eta(t) \stackrel{\text{def}}{=} A \neq 0. \tag{4.1}$$

Then we have

$$\eta(t) < A + A\varepsilon/3 \quad \text{for} \quad t > t_0(\varepsilon) \tag{4.2}$$

and thus for $x > x_0(\varepsilon) = \big(t_0(\varepsilon)\big)^{3/\varepsilon A}$ one gets

$$\omega(x) \leqslant \eta(x^{A\varepsilon/3})\log x + \log(x^{A\varepsilon/3}) \tag{4.3}$$

$$\leqslant \left(A + \frac{A\varepsilon}{3}\right)\log x + \frac{A\varepsilon}{3}\log x = A\left(1 + \frac{2\varepsilon}{3}\right)\log x.$$

On the other hand, (4.1) says that the halfplane $\sigma > 1 - A$ is zero-free and so, as already mentioned in (3.4), we have

$$\Delta_i(x) = O\left(\frac{1}{x^{(1-\varepsilon/3)A}}\right) \tag{4.4}$$

and thus we get the assertion of the theorem using (4.3), because

$$\left(1 - \frac{\varepsilon}{3}\right)A\log x > (1-\varepsilon)\left(1 + \frac{2\varepsilon}{3}\right)A\log x \geqslant (1-\varepsilon)\,\omega(x). \tag{4.5}$$

The case

$$\lim_{t\to\infty}\eta(t) = 0 \tag{4.6}$$

is much deeper. Here we must use the density theorem of Carlson [3], i.e. the fact that for $0 < \varepsilon < \frac{1}{2}$ the number of zeros with $\beta \geqslant 1-\varepsilon$, $0 < \gamma \leqslant T$

$$N(1-\varepsilon, T) \leqslant c_{13} T^{4\varepsilon(1-\varepsilon)}\log^6 T < c(\varepsilon) T^{4\varepsilon}, \tag{4.7}$$

further the "shortened" explicit prime number formula

$$\Delta_4(x) = -\sum_{|\gamma|\leqslant x^2} \frac{x^{\varrho-1}}{\varrho} + O\left(\frac{\log^2 x}{x}\right). \tag{4.8}$$

(See e.g. Ingham [8], Theorem 29.)

We note that it is enough to prove our theorem for $i = 4$; for $i = 2$ it follows from this by partial summation, and from this it trivially follows for the cases $i = 1, 3$ (since $\omega(x) = o(\log x) < \frac{1}{2}\log x$).

Let $\varepsilon = 5\varepsilon'$. Then the contribution of the zeros with $\beta \leqslant 1 - \varepsilon'$ is

$$O\left(\log^2 x \cdot \frac{1}{x^{\varepsilon'}}\right) = O\left(\frac{1}{e^{\frac{\varepsilon'}{2}\log x}}\right) = O\left(\frac{1}{e^{\omega(x)}}\right) \tag{4.9}$$

owing to the well-known relation (see e.g. Ingham [8], Theorem 25b)

$$\sum_{|\gamma|\leqslant T} \frac{1}{|\gamma|} = O(\log^2 T) \tag{4.10}$$

and since by (4.6)

$$\omega(x) \leqslant \eta(x^{\varepsilon'/4})\log x + \log x^{\varepsilon'/4} < \frac{\varepsilon'}{2}\log x \quad \text{for} \quad x > x_0(\varepsilon'). \tag{4.11}$$

Let

$$u \stackrel{\text{def}}{=} \log t, \qquad g(u) \stackrel{\text{def}}{=} \eta(t), \qquad r \stackrel{\text{def}}{=} \log x; \tag{4.12}$$

further let $u_0 = u_0(r)$ denote the least number u for which

$$f_r(u) = g(u)r + u$$

is minimal for a given r. Thus

$$g(u)r + u \geqslant g(u_0)r + u_0 \stackrel{\text{def}}{=} \tilde{\omega}(r) = \omega(x). \tag{4.13}$$

Owing to (4.7), the total number of zeros in a domain of the type

$$\sigma > 1 - \varepsilon', \qquad T/e \leqslant t \leqslant T \tag{4.14}$$

is $< c(\varepsilon')T^{4\varepsilon'}$, and thus for the contribution of zeros with

$$\beta > 1 - \varepsilon', \qquad |\gamma| \leqslant x^2 \tag{4.15}$$

to the right side of (4.8) we get the upper bound (with the notations in (4.12) and (4.13)):

$$O\left(\int_0^{2r} \frac{c(\varepsilon')e^{4\varepsilon' u}}{x^{\eta(t)}\cdot t}\,du\right) \tag{4.16}$$

$$= O\left(\int_0^{2r} \frac{e^{4\varepsilon' u}}{e^{g(u)r+u}}\,du\right) = O\left(\int_0^{2r} \frac{e^{-\varepsilon' u}}{e^{g(u)r+u(1-5\varepsilon')}}\,du\right) = O\left(\int_0^{\infty} \frac{e^{-\varepsilon' u}}{e^{(g(u)r+u)(1-5\varepsilon')}}\,du\right)$$

$$= O\left(\frac{1}{e^{\tilde{\omega}(r)(1-5\varepsilon')}}\cdot\frac{1}{\varepsilon'}\right) = O\left(\frac{1}{e^{\omega(x)(1-\varepsilon)}}\right)$$

which also proves the theorem in the case $A = 0$.

Further we note that, by the density theorem of Halász and Turán [7] in case $A = 0$, the number ε can be replaced by

$$c\left(\frac{\omega(x)}{\log x}\right)^{3/2}\log^3\left(\frac{\log x}{\omega(x)}\right).$$

Finally, we note that in the case $A = 0$ we could allow even finitely many zeros in the domain (3.6), since for a small enough ε' these zeros are in the halfplane $\sigma \leqslant 1-\varepsilon'$, and there we did not make use of our condition (3.6).

If $A > 0$, and there are finitely many zeros within the domain (3.6) but these zeros are in the halfplane $\sigma \leqslant 1-A$, the statement of the theorem and the proof obviously remains unchanged.

However, if $A > 0$, and there are finitely many zeros in the domain (3.6) among them at least one zero in the halfplane $\sigma > 1-A$, then let us denote by $\varrho_0 = \beta_0 + i\gamma_0$ the zero in question with the maximal real part. Then the statement of the theorem is already not true, but on the contrary we have by Theorem 1 or Theorem 2 of [11] (for the second theorem see Section 5)

$$\Delta_i(x) = \Omega_\pm\left(\frac{1}{x^{1-\beta_0}}\right) \tag{4.17}$$

whereas obviously for a small enough ε

$$(1-\varepsilon)\,\omega(x) \geqslant (1-\varepsilon)A\log x > (1-\beta_0)\log x. \tag{4.18}$$

5. Now it is possible already to prove an almost complete converse of Theorem 1 applying Theorem 2 of part I [11], which sounds:

THEOREM 2′. *Let $0 < \varepsilon \leqslant 1/50$ and let us assume the existence of a $\varrho_0 = \beta_0 + i\gamma_0$ zero of $\zeta(s)$ with*

$$\beta_0 = \tfrac{1}{2} + \delta_0 > \tfrac{1}{2} + \varepsilon$$

and

$$\gamma_0 > \exp\exp(10^{12}/\varepsilon^3). \tag{5.1}$$

Then, for $1 \leqslant i \leqslant 4$, for every H satisfying

$$H^{\varepsilon^4/10^7} > \max(\gamma_0, c), \tag{5.2}$$

we have in the interval

$$I = [H, H^{1+\varepsilon}] \tag{5.3}$$

an x_i' and $x_i'' \in I$ for which

$$\Delta_i(x_i') > \frac{1}{\gamma_0^{1+\varepsilon}(x_i')^{1-\beta_0}}, \quad \Delta_i(x_i'') < -\frac{1}{\gamma_0^{1+\varepsilon}(x_i'')^{1-\beta_0}} \tag{5.4}$$

hold.

Using this theorem, we can show that if a domain (3.6) contains an infinity of zeros, then (3.8) cannot hold if we replace $1-\varepsilon$ by $1+\varepsilon$; even $\Delta_i(x)$ has large positive and negative values, which hurt (3.8).

However here we must have somewhat stronger restrictions for the corresponding function $\eta(t)$.

6. So we can state

THEOREM 2. *Suppose that $\zeta(s)$ has an infinity of zeros in the domain*

$$\sigma \geqslant 1 - g(\log|t|), \quad 0 < g(u) \leqslant \tfrac{1}{2} \tag{6.1}$$

where $g(u)$ is for $u \geqslant 0$ a continuous decreasing function,

$$g'(u) \nearrow 0 \quad \text{for} \quad u \to \infty \tag{6.2}$$

(by which we now mean that $g'(u)$ tends to 0 monotonically increasing for $u \geqslant c$ and if $\lim_{u\to\infty} g(u) = 0$ then $g'(u)$ tends to 0 strictly monotonically increasing for $u \geqslant c$).

Let ε be a fixed real number with $0 < \varepsilon < 1$. Further, let

$$\omega(x) \overset{\text{def}}{=} \min_{u \geqslant 0}\big(g(u)\log x + u\big). \tag{6.3}$$

Then we have

$$\Delta_i(x) = \Omega_\pm\left(\frac{1}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{6.4}$$

First we remark that the relatively natural conditions for $g(u)$ are satisfied in the most interesting special case where

$$g(u) = \frac{\log^B u\,(\log\log u)^C}{u^A} \quad \text{for} \quad u > u_0 = u_0(A, B, C)$$

and

$$g(u) = g(u_0) \quad \text{for} \quad 0 \leqslant u < u_0$$

where $A > 0$, B, C are arbitrary or $A = 0$, $B < 0$, C is arbitrary, or $A = B = 0$ and $C < 0$.

This theorem gives that the elementary estimate (2.1) for the remainder term implies the zero-free region

$$\sigma > 1 - \frac{c(\varepsilon)}{t^{\frac{1}{B}(1+\varepsilon)}}$$

which is better than that given in (2.11).

For the proof we shall consider first the case

$$\lim_{u\to\infty} g(u) = 0.$$

We shall use the notations (4.12), (4.13). Let $r > c' = c'(g)$. Then for $u < c$ (given in (6.2))

$$g(u)r + u \geqslant g(c)r > g(\sqrt{r})r + \sqrt{r} \;\; (= o(r));$$

thus $f_r(u) = g(u)r+u$ takes his minimal value for $u \geqslant c$. $f_r(u)$ cannot take its minimal value at two places, because by (6.2) there is at most one $u \geqslant c$ for which

$$\frac{df_r(u)}{du} = g'(u)r+1 = 0 \Leftrightarrow r = -\frac{1}{g'(u)}. \tag{6.5}$$

Thus (6.5) has a unique solution $u = u_0(r)$, and so by (6.2) to any $u \geqslant c''$ $= c''(g)$ there is an $r = \tilde{r}(u)$ given by the second equality of (6.5) for which $u = u_0(r)$ (for the definition see (4.12) and (4.13)).

Further for $u \geqslant dr$ (d any given positive constant) we have

$$g(u)r+u \geqslant dr > g(\sqrt{r})r+\sqrt{r} \quad (= o(r)); \tag{6.6}$$

thus we get

$$u_0(r) = o(r), \tag{6.7}$$

i.e.

$$\lim_{u\to\infty} \frac{\tilde{r}(u)}{u} = \infty. \tag{6.8}$$

Now let us consider a number u' with $g(u') < \frac{1}{2}$. Further let

$$\varrho_n = \beta_n+i\gamma_n = \beta_n+ie^{u_n} \quad (\beta_n > \tfrac{1}{2},\ \gamma_n > 0) \tag{6.9}$$

be the nth zero in (6.1) satisfying $u_n > u'$ ordered according to the increasing u_n-values.

Now we apply Theorem 2′. Let us choose ε small enough ($\varepsilon < \frac{1}{2}-g(u')$). Then for $n > n_0(\varepsilon)$, (5.1), $\beta_n > \frac{1}{2}+\varepsilon$ and using (6.8), inequality (5.2) is satisfied with the choice

$$H_n = e^{\tilde{r}(u_n)}. \tag{6.10}$$

Thus we get the existence of an $x_i' \in [H_n, H_n^{1+\varepsilon}]$, i.e. an x_i', with

$$\tilde{r}(u_n) \leqslant \log x_i' \leqslant (1+\varepsilon)\tilde{r}(u_n) \tag{6.11}$$

for which, by (5.4),

$$\begin{aligned} \Delta_i(x_i') &> \frac{1}{(x_i')^{1-\beta_n}\gamma_n^{1+\varepsilon}} = \frac{1}{e^{(1-\beta_n)\log x_i'+(1+\varepsilon)u_n}} \\ &\geqslant \frac{1}{e^{g(u_n)(1+\varepsilon)\tilde{r}(u_n)+(1+\varepsilon)u_n}} = \frac{1}{e^{\omega(H)(1+\varepsilon)}} \\ &> \frac{1}{e^{\omega(x_i')(1+\varepsilon)}}, \end{aligned} \tag{6.12}$$

as, by definition (6.3), $\omega(x)$ is trivially an increasing function of x.

The inequality in the other direction can be proved *mutatis mutandis*.

In the case $\lim_{u\to\infty} g(u) = A > 0$ we have $\theta \geqslant 1-A$ and so we get from (3.5)

$$\Delta_i(x) = \Omega_\pm\left(\frac{1}{x^{(1-\theta)(1+\varepsilon)}}\right) = \Omega_\pm\left(\frac{1}{x^{A(1+\varepsilon)}}\right) = \Omega_\pm\left(\frac{1}{e^{\omega(x)(1+\varepsilon)}}\right), \tag{6.13}$$

because $\omega(x) = g(u_0)\log x+u_0 \geqslant g(u_0)\log x \geqslant A\log x$.

So the proof of Theorem 2 is completed.

Thus by Theorem 1 (with the remarks at the end of Section 4 concerning the case of finitely many zeros in (3.6)) and Theorem 2, if we give a domain of the type (6.1) (with a function $g(u)$ having the properties described in Theorem 2) then the behaviour of the remainder terms is clear in the following cases

(a) the domain is zero-free,
(b) it contains finitely many zeros,
(c) it contains infinitely many zeros.

7. Finally we can formulate an interesting corollary of Theorems 1 and 2. To state the corollary in a transparent form we introduce the following

DEFINITION. Let C be the class of the real functions $\omega(x)$ ($x \geqslant 1$) for which there exists a continuous decreasing function $g(u)$ ($u \geqslant 0$) with $0 < g(u) \leqslant \frac{1}{2}$ and with $g'(u) \nearrow 0$ as $u \to \infty$ (see (6.2)) satisfying

$$\omega(x) = \min_{u\geqslant 0}\left(g(u)\log x+u\right). \tag{7.1}$$

With the above definition and with the notation (3.8) we have the

COROLLARY. *Let $0 < \varepsilon < 1$ be fixed; $\omega(x) \in C$. If there is an i ($1 \leqslant i \leqslant 4$) for which*

$$\Delta_i(x) = \Omega\left(\frac{1}{e^{(1-\varepsilon)\omega(x)}}\right), \tag{7.2}$$

then for all j ($1 \leqslant j \leqslant 4$) we have

$$\Delta_j(x) = \Omega_\pm\left(\frac{1}{e^{(1+\varepsilon)\omega(x)}}\right). \tag{7.3}$$

The corollary states that various forms of the remainder term have about the same oscillation.

But even more remarkable is the phenomenon concerning the distribution of primes, asserted by the corollary, namely the fact that the remainder term is somewhat "symmetrical": if it assumes large positive values, then it must assume negative values with an absolute value about the same order of magnitude.

It is interesting to note that the corollary is an assertion concerning exclusively primes; ζ-zeros occur only in the proof. Therefore it would be very interesting (but it seems to be nearly hopeless) to prove it directly, avoiding the theory of the ζ function.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
Budapest, Hungary

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Statistical Deuring–Heilbronn phenomenon

by

MATTI JUTILA (Turku)

To the memory of Paul Turán

1. Introduction. Let χ_1 be a real primitive character (mod k), and let $\beta_1 = 1-\delta$ be a real zero of the Dirichlet L-function $L(s) = L(s, \chi_1)$. Suppose that β_1 is "exceptional" in the sense that $\delta \leqslant 1/\log k$. According to a theorem of Linnik [3], the existence of an exceptional zero has a certain effect — called by Linnik the Deuring–Heilbronn phenomenon — upon the distribution of the zeros of L-functions. More exactly, there exist calculable constants $c_1 > 0$, $c_2 > 0$ such that if $\varrho = \beta + i\gamma$ is a zero of $L(s, \chi)$ (mod q) and if $\delta \log(qk\tau) \leqslant c_1$, where $\tau = \max(2, |\gamma|)$, then (if the case $\chi = \chi_1$, $\varrho = \beta_1$ is excluded)

$$\beta \leqslant 1 - c_2 \log\left(\frac{c_1 e}{\delta \log(qk\tau)}\right) / \log(qk\tau). \tag{1.1}$$

Linnik's proof of this estimate was very complicated. A much simpler proof, depending on Turán's power sum method, was given by Knapowski [2]. Recently Motohashi [7] and the author [1] have found new proofs of (1.1) on the basis of an idea of A. Selberg.

Our purpose in this paper is to investigate the Deuring–Heilbronn phenomenon from a statistical point of view, considering the distribution of zeros of many L-functions both in the horizontal and in the vertical direction.

Define

$$\varphi(s, \chi) = L(s, \chi) L(s, \chi\chi_1);$$

then for $\sigma > 1$

$$\varphi(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}, \tag{1.2}$$

where

$$a_n = \sum_{d|n} \chi_1(d). \tag{1.3}$$