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Some personal reminiscences of the mathematical work of Paul Turán

by

PAUL ERDÖS (Budapest)

Our friendship and collaboration spread over most of our active and scientific life. We first met at the University of Budapest in September 1930 and immediately discovered our common interest in number theory and prime numbers in particular. Our last mathematical discussion 11 days before his death was also about prime numbers. My last letter to him (written from Canada) arrived after he was already gone ...

I have written a fairly extensive and highly personal article about our collaboration, his personality (as I saw it) and some of his mathematical work. (See [1] and also [2].) Here I just want to write a few more personal reminiscences and a few lines on his mathematical work and his influence on my own.

Paul Turán did outstanding and pioneering work in various branches of mathematics; his subjects include number theory, function theory, interpolation and approximation, polynomials, differential equations, numerical algebra, group theory, graph theory and combinatorial set theory.

Probably the most important, most enduring and most original of Turáns results are his power sum method and its applications. I was there when it originated in 1938. Turán mentioned these problems and told me that they were not only interesting in themselves but their positive solution would have many applications.

Their importance first of all is that they lead to interesting deep problems of a completely new type; they have quite unexpectedly surprising consequences in many branches of mathematics — differential equations, numerical algebra, and various branches of function theory.

I will not talk about it here in detail since others more competent than I wrote about it. (See Halász [3].) My main contribution to the subject was that during my stay in Canberra in 1960 I called the attention of Atkinson to one of the unsolved problems. This problem of Turán was and is extremely fascinating:

Let $|z_1| = 1,\ |z_i| \leqslant 1,\ 2 \leqslant i \leqslant n$. Put

$$s_k = \Big|\sum_{i=1}^{n} z_i^k\Big|.$$

Prove that there is an absolute constant c such that $\max\limits_{1\leqslant k\leqslant n} s_k > c$. This he conjectured around 1938 — it was settled by Atkinson in 1960 with $c = 1/6$ and Atkinson later improved this to $1/3$. Turán later conjectured that to every $\varepsilon > 0$ there is an $n_0(\varepsilon)$ so that for every $n \geqslant n_0$ $\max\limits_{1\leqslant k\leqslant n} s_k > 1-\varepsilon$. Certainly a very fascinating and beautiful conjecture. Turán never seriously tackled it since, as he often told me, life is short and our time is limited and the number of problems is limitless; therefore he preferred to tackle those conjectures which have applications.

Turán had the remarkable and rare ability to initiate new and fruitful directions of research in various branches of mathematics, in which he sometimes had only a passing interest. I mention just two such examples. The first one is Turán's problem in set mappings (see p. 37 of [1]). In this case Turán's interest in the problem was hardly more than momentary, but the influence on mathematics of this question was certainly not negligible. The other example is his starting the field of extremal problems of graph theory (see pp. 38–39 of [1] and Simonovits [4]). Here his interest was deeper and he occasionally returned to the subject. Both of these ideas of Turán's greatly influenced my own work; all this is discussed in some detail in [1], [2].

His principal work was undoubtedly number theory, his first and last thoughts dealt with this subject.

Turán as a student was deeply interested in analytic number theory, but his attempts of solving many deep problems on prime numbers at that time were not successful; he finally abandoned his ideas of those days as unsuccessful, but the many hours were not wasted since he acquired a superb analytic technique.

His first important success was in 1934 when he obtained a very simple proof of the theorem of Hardy and Ramanujan according to which almost all integers n have asymptotically $\log\log n$ prime factors. To see this Turán simply proved in a few lines

$$(1)\qquad \sum_{n=1}^{x} (\nu(n) - \log\log x)^2 < cx\log\log x \qquad \Big(\nu(n) = \sum_{p\mid n} 1\Big).$$

(1) immediately implies the theorem of Hardy and Ramanujan. A few months later Turán proved that if $f(x)$ is an irreducible polynomial with integer coefficients then for almost all n

$$\big|\nu(f(n)) - \log\log n\big| < \psi(n)\sqrt{\log\log n}$$

where $\psi(n)$ tends to infinity arbitrarily slowly. I proved, again using Turán's method, that for almost all primes $p < x$

$$|\nu(p-1) - \log\log p| < \psi(p)(\log\log x)^{1/2}$$

and I used the method of Turán a great deal in the theory of additive and multiplicative number theoretic functions. His method has been widely used ever since 1934, and there are various generalizations of it (Turán–Kubilius inequality).

Now I state some of our joint work from the 1930's. Our first joint paper is on a problem of elementary number theory. (See [1].) We also gave a simple proof of the following theorem of Ramanoff: Denote by $l_2(k)$ the exponent of $2 \bmod k$; then

$$\sum \frac{1}{kl_2(k)} < \infty \quad \text{and} \quad \sum \frac{1}{kl_a(k)} < c\log\log a.$$

My last mathematical discussion with Turán was on September 15, 1976 (the next day I left for Canada). We talked about the distribution of prime numbers, which was our oldest and most permanent interest — we talked about it at our first and last meeting. By the way, both Turán and Littlewood were "heretics", i.e. they disbelieved in the Riemann hypothesis! I myself strongly believe that primes are distributed at random unless there is some obvious reason to prevent this (e.g. there are no even primes > 2). By the way, this belief, if properly interpreted, implies the Riemann hypothesis. Perhaps in a distant and more enlightened future mathematicians will smile at our naive belief in the random distribution of primes.

One of Turán's favorite problems was the distribution of primes in arithmetic progressions, both locally and globally. The term "prime number race" was coined by him. This is what he had in mind: Denote by $\pi(x; a, d)$ the number of primes $p < x$, $p \equiv a \bmod d$. Which of the $\varphi(d)$ progressions $a_i \bmod d$ wins?, i.e. which of the numbers $\pi(x; a_i, d)$ is maximal? Turán and I — like every other right thinking person — believed that for every permutation $a_{i_1}, \ldots, a_{i_{\varphi(d)}}$ of $\{a_1, \ldots, a_{\varphi(n)}\}$ there are infinitely many values x for which

$$(2)\qquad \pi(x; a_{i_1}, d) > \ldots > \pi(x; a_{i_{\varphi(d)}}, d).$$

Turán obtained some partial results; the general case is intractable at present. Turán believed that not all the permutations are, in general, equally probable.

In a posthumous paper which has just appeared Turán and Knapowski state a very surprising conjecture. The paper was put in its final form not very long before Turán's death and since Knapowski died more than ten years ago the final formulation and responsibility for the conjecture is Turán's alone. Let $p_1 < \ldots$ be the sequence of consecutive primes. The authors remark that no proof is known that for infinitely many ν

$$p_\nu \equiv p_{\nu+1} \equiv p_{\nu+2} \equiv 1 \pmod 4; \tag{3}$$

but they of course believe that (3) has infinitely many solutions. Put

$$f_2(x) = \sum_{\substack{p_\nu < x \\ p_\nu \equiv p_{\nu+1} \equiv 1 \bmod 4}} 1.$$

They prove that $f_2(x) > (\log x)^B$. Now comes the surprising conjecture:

$$f_2(x) = o\left(\frac{x}{\log x}\right)! \tag{4}$$

I would certainly believe that

$$f_2(x) = \left(\tfrac{1}{4} + o(1)\right) \frac{x}{\log x}.$$

We discussed this question on September 15, 1976. Several mathematicians asked me if I know why Turán believed in (4); in our last conversation we had no time for details. Also owing to his illness he tired quickly and it was very sad to hear him say: "It is too bad that I get tired so easily, I have so many new ideas to work out". Mathematics occupied Turán's mind up to the very end — a few hours before his death his last understandable word was $O(1)$.

Using the Riemann hypothesis, Turán proved that if $p(a, b)$ denotes the smallest prime $\equiv a \bmod b$ then

$$p(a, b) < b(\log b)^{2+\varepsilon}$$

for almost all a, i.e. for all but $o(\varphi(b))$ values of a. Presumably slightly more is true — we discussed this a great deal 10 years ago; probably $p(a, b) < b(\log b)^{1+\varepsilon}$ for almost all a — perhaps for all a. I proved (easily from Brun's method) that $p(a, b) > (1+c_1)\varphi(b)\log b$ for at least $c_2\varphi(b)$ values of a. It seems certain that this holds for every c_1 and perhaps as $c_1 \to \infty$, $c_2 \to 0$, but this seems to be beyond our methods even if one assumes the Riemann hypothesis.

Turán and I investigated some unconventional problems on primes. E. g. putting $p_{n+1} - p_n = d_n$, we proved that both $d_{n+1} > d_n$ and $d_n > d_{n+1}$ have infinitely many solutions; also $p_{n-1}^2 > p_n p_{n-2}$ and $p_{n-1}^2 < p_n p_{n-2}$ both have infinitely many solutions. We never could prove that $d_{n+2} > d_{n+1} > d_n$ has infinitely many solutions, and in fact we could not even prove that at least one of the set of inequalities $d_n > d_{n+1} > d_{n+2}$ or $d_n < d_{n+1} < d_{n+2}$ has infinitely many solutions. In fact, there is no doubt that all the $k!$ orderings between $d_n, \ldots, d_{n+k-1}$ will occur, but this again is probably beyond our powers.

Perhaps I should state three of our favourite conjectures.

1. We started with the following problem: Let $1 \leqslant a_1 < \ldots < a_l \leqslant n$ be a sequence of integers which does not contain an arithmetic progression of k terms. Denote $\max l$ by $r_k(n)$. We conjectured $r_k(n) = o(n)$. This far-reaching generalization of the famous theorem of van der Waerden was finally proved by Szemerédi a few years ago. More recently Furstenberg proved it by using methods of ergodic theory. Much further development can be expected in the not too distant future. Many unsolved problems remain: presumably $r_k(n) = o\left(\frac{n}{(\log n)^s}\right)$ for every s. To obtain any asymptotic formula or even a good two-sided inequality for $r_k(n)$ or even $r_3(n)$ would be an outstanding achievement.

2. Another of our conjectures of more than 40 years ago: Let $1 \leqslant a_1 \leqslant \ldots$ be an infinite sequence of integers, and denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Prove that $f(n) > 0$ for all $n > n_0$ implies $\limsup f(n) = \infty$. A stronger version, but perhaps easier to tackle states: If $a_k < ck^2$, $k = 1, 2, \ldots$ then $\limsup f(n) = \infty$.

3. Let $1 \leqslant a_1 < \ldots < a_k \leqslant n$. Assume that the sums $a_i + a_j$ are all distinct. Prove $\max k = n^{1/2} + O(1)$.

The last two conjectures are still open. For further details see [1].

Turán and I did a great deal of work on uniform distribution of sequences. As is well known, Weyl proved that if (x_n) is an infinite sequence of real numbers and if

$$\frac{1}{N}\left|\sum_{n=1}^{N} e^{2\pi i k x_n}\right| \to 0$$

holds for every k then the sequence is uniformly distributed mod 1. We obtain the following quantitative version of this theorem:

Assume

$$s_k = \left|\sum_{n=1}^{N} e^{ikx_n}\right| \leqslant \psi(k), \quad k = 1, \ldots, m. \tag{5}$$

Then

$$\left|\sum_{a < x_n < \beta \pmod{2\pi}} 1 - \frac{\beta - a}{2\pi} n\right| < c\left(\frac{n}{m+1} + \sum_{k=1}^{m} \frac{\psi(k)}{k}\right).$$

Probably our result is not very far for being the best possible. Our result turned out to be very useful and it has been used a great deal. Turán and I obtained a very much stronger result on the error terms by using interpolatory properties — those papers were very little used and seem to be forgotten. While (5) is often easy to check, i.e. it is easy to apply, our interpolatory conditions are very hard to verify and this is the reason why they have almost never been used.

To finish, I state a few facts about Turán. He himself probably considered his "new method" to be his most important contribution. Once, several years before his death, I found him at his desk deeply absorbed in work. I asked him: "What are you working on?" He answered smiling "I am building my pyramid". Perhaps I should explain the meaning of this to the non-Hungarian reader. They refer to a play of a famous Hungarian writer Madách (the play, "The tragedy of Man", was translated into many languages, but is probably not very well known abroad). In one of the scenes, enacted in ancient Egypt, the Pharaoh to accomplish an immortal achievement is building his pyramid. Thus "building my pyramid" would mean: trying to accomplish an immortal achievement, which will live for ever. In fact, he was writing his book.

Several years later, in July 1976, at the meeting on combinatorics at Orsay in Paris, V. T. Sós (Mrs. Turán) gave me the terrible news (which she had known for 6 years) that Paul had leukemia. She told me that I should visit him as soon as possible and that I should be careful in talking to him because he did not know the true nature of his illness. My first reaction was to say that perhaps he should have been told so that he could "finish his pyramid". She said she felt that Paul loved life too much and with the death sentence hanging over him would not be able to live and work very well. (In fact, he could work very well under adverse conditions. For example, the theory of extremal graphs was started in a labour camp in 1940 in the nazi-fascist era.) Nevertheless, I am now fairly sure that her decision was right, since he clearly never tried to find out the true nature of his illness. In fact, a few days before his death, V. T. Sós and their son George (also mathematician) tried to persuade him to dictate some parts of his book to Halász or Pintz. He refused saying "I will write it when I feel better and stronger". Unfortunately he never had the chance. Fortunately his book was finished by his students G. Halász and J. Pintz and will soon appear.

It is always sad when a great man dies while still mentally in his prime.

References

[1] P. Erdös, *Problems in number theory and combinatorics*, Proc. of the Sixth Manitoba conference on Numerical Mathematics, 1976 Congressus numerantium XVIII, University of Manitoba, Winnipeg, pp. 35–58.

[2] — *Paul Turán, 1910–1976: His work in Graph Theory*, Journal of Graph Theory 1 (1977), pp. 96–101.

[3] G. Halász, *Turán Pál matematikai munkássága. A hatványösszeg módszor (Paul Turán's work. The power sum method)*, Mat. Lapok. (to appear).

[4] M. Simonovits, *On Paul Turán's influence on Graph Theory*, ibid., pp. 102–116.

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The number-theoretic work of Paul Turán

by

G. Halász* (Budapest)

Paul Turán made important contributions to many parts of mathematics but it was number theory that captivated his interest unabatedly throughout his life.

The power sum method. Turán made several attempts to solve the deepest problems of analytic number theory, first and foremost among them being the over 100 year old conjecture of Riemann on the zeros of the zeta function. The following has proved to be the most successful (not exactly for the purpose for which it was originally intended).

For s_0 on the vertical line $\operatorname{Re} s = \sigma_0 > 1$ let $r(s_0)$ denote the radius of the largest zero-free disc around s_0 of the zeta function $\zeta(s)$; Riemann's hypothesis is then equivalent to $r(s_0) \geqslant \sigma_0 - 1/2$. In other words, $r(s_0)$ is the radius of regularity for $\frac{\zeta'}{\zeta}(s)$ around s_0 (provided that $\operatorname{Im} s_0$ is sufficiently large, so that the pole at $s = 1$ does not come into play) and by Cauchy's elementary formula

$$\limsup_{\nu=\infty} \sqrt[\nu]{\frac{1}{\nu!}\left|\left(\frac{\zeta'}{\zeta}(s)\right)^{(\nu)}_{s=s_0}\right|} = \frac{1}{r(s_0)}.$$

By differentiating a classical approximation to $\frac{\zeta'}{\zeta}(s)$, the quantity under the νth root can be replaced by

$$\left|\sum_{\varrho}{}' \frac{1}{(s_0-\varrho)^{\nu+1}}\right|,$$

the summation being extended over a finite number of zeros ϱ of $\zeta(s)$ in the "vicinity" of s_0, and attempts to replace the limsup, impossible to calculate, by finite explicit estimations lead to general inequalities for sums of powers of complex numbers.

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