A reciprocity theorem and a three-term relation for generalized
Dedekind–Rademacher sums

by

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To the memory of Professor Paul Turan

1. Introduction. For real $x$, put

$$((x)) = \begin{cases} x - \lfloor x \rfloor & (x \neq \text{integer}), \\ 0 & (x = \text{integer}). \end{cases}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{\mu \mod k} \left( \left( \frac{h\mu}{k} \right) \left( \frac{\mu}{k} \right) \right),$$

where the summation is over a complete residue system $(\mod k)$. It is well known that $s(h, k)$ satisfies ([11], p. 4) the reciprocity relation

$$12kk \{s(h, k) + s(k, h)\} = h^2 - 3hk + k^2 + 1 \quad ((h, k) = 1).$$

Rademacher, at the 1963 Number Theory Institute in Boulder, Colorado, proved the following generalization of (1.1). Define

$$s(h, k; x, y) = \sum_{\mu \mod k} \left( \left( \frac{h\mu + y}{k} \right) \left( \frac{\mu + x}{k} \right) \right),$$

where $x, y$ are arbitrary real numbers. Then

$$s(h, k; x, y) + s(k, h; y, x) = -\frac{1}{4} \delta(x) \delta(y) + ((x))(y) +$$

$$+ \frac{1}{2h} \left\{ h B_y(y) + \frac{1}{hk} B_x(hy + kx) + \frac{k}{h} B_y(x) \right\},$$

where $(h, k) = 1$,

$$\delta(x) = \begin{cases} 1 & (x = \text{integer}), \\ 0 & (x \neq \text{integer}) \end{cases}$$
and 
\[
\tilde{B}_n(x) = B_n(x - [x]), \quad B_n(x) = x^n - \frac{x}{2}.
\]

For \(x = y = 0\), (1.2) reduces to (1.1). Rademacher's proof of (1.2) appeared in [19]. For a simplified proof see [7].

Let \(B_n(x)\) be the Bernoulli polynomial of degree \(n\) defined by
\[
\frac{ue^{ux}}{e^u - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{u^n}{n!}, \quad B_n = B_n(0),
\]
and let \(\tilde{B}_n(x)\) be the Bernoulli function defined by
\[
\tilde{B}_n(x) = \tilde{B}_n(x - [x]).
\]

Apostol ([1], [2]) introduced the generalized Dedekind sum
\[
s_p(h, k) = \sum_{\nu \equiv (m, h, k)} \tilde{B}_p \left( \frac{h\nu + k\nu}{k} \right) \tilde{B}_1 \left( \frac{\mu + y}{k} \right),
\]
and proved the reciprocity theorem
(1.3) \[(p+1)(hBp_s(h, k) + lbp_s(h, k)) = (hB + kB)^{p+1} + pB_{p+1}, \]
where \((h, k, l) = 1, l \text{ odd}, p > 1\). A proof of a different kind was given by the present writer [3].

Rademacher's definition of \(s(h, k; x, y)\) suggests that we define
\[
s_p(h, k; x, y) = \sum_{\nu \equiv (m, h, k)} \tilde{B}_p \left( \frac{h\nu + k\nu}{k} + x \right) \tilde{B}_1 \left( \frac{\mu + y}{k} \right),
\]
which reduces to \(s_p(h, k)\) when \(x = y = 0\). Since \(\tilde{B}_n(x+1) = \tilde{B}_n(x)\), there is no loss in generality in assuming that
(1.4) \[0 \leq x < 1, \quad 0 \leq y < 1.
\]

The writer ([4], [5]) has proved the following

**THEOREM 1.** Let \((h, k) = 1\) and assume that \(x, y \) satisfy (1.4). Then
(1.5) \[(p+1)(hBp_s(h, k; x, y) + lbp_s(h, k; y, x)) = (hB + kB + hy + ky)^{p+1} + pB_{p+1}(hy + ky)
\]
for all \(p \geq 0\).

We may replace (1.5) by the following equivalent formulation in which (1.4) is not assumed:
(1.6) \[(p+1)(hBp_s(h, k; x, y) + lbp_s(h, k; y, x)) = (hB(y) + kB(x))^{p+1} + pB_{p+1}(hy + ky).
\]

It is to be understood that
\[
(hB(y) + kB(x))^{p+1} = \sum_{r=0}^{p+1} \binom{p+1}{r} h^{p+1-r}B_r(y)B_{p+1-r}(x).
\]

Both of the earlier proofs of (1.5) require considerable computation. In the present paper we give a simplified proof that makes use of the following

**LEMMA 1.** Let \((h, k) = 1, \quad hh > 1, 0 \leq x < h + k\). Put \(z = x - [x]\), the fractional part of \(x\). Then we have the identity
(1.7) \[
\sum_{r=0}^{p+1} \binom{p+1}{r} h^{p+1-r}B_r(y)B_{p+1-r}(x) = \frac{\lambda^2 - 1 - \lambda^{2p+1}}{(1 - \lambda^2)(1 - \lambda^{p+1})} - \frac{\lambda^{2p+1}}{1 - \lambda^2},
\]
where the summation is over all \(r, s\) such that \(0 \leq r < k, 0 \leq s < h, \quad h^r + ks + s < hh\).

In the next place, let \(a, b, c\) be three positive integers that satisfy
(1.8) \[(h, c) = (b, c) = (a, b) = 1.
\]

Rademacher [9] has proved the following three-term relation:
(1.9) \[s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12} \left( \frac{a}{bc'} + \frac{b}{ca'} + \frac{c}{ab'} \right),
\]
where \(a', b', c'\) are defined by
\[aa' \equiv 1 \text{ (mod } bc), \quad bb' \equiv 1 \text{ (mod } ca), \quad cc' \equiv 1 \text{ (mod } ab).
\]

The present writer, in extending (1.9) to \(s(h, k; x, y)\), defined the sum [6]
\[
\sum_{\nu \equiv (m, h, k)} \tilde{B}_p \left( \frac{h\nu + k\nu}{k} + x \right) \tilde{B}_1 \left( \frac{\mu + y}{k} \right),
\]
which reduces to \(s_p(h, k)\) when \(x = y = 0\). Since \(\tilde{B}_n(x+1) = \tilde{B}_n(x)\), there is no loss in generality in assuming that
(1.4) \[0 \leq x < 1, \quad 0 \leq y < 1.
\]

Despite the presence of the additional parameters, \(s(h, k; x, y, z)\) is really no more general than \(s(h, k; x, y)\). It was proved that
(1.10) \[s(h, k; x, y, z) + s(b, c; a, y, x) + s(c, a; b, x, y) = \delta \left( \frac{a}{2bc} \tilde{B}_1(ey - bz) - \frac{b}{2ca} \tilde{B}_1(aw - cx) - \frac{c}{2ab} \tilde{B}_1(bx - ay) \right),
\]
where \(\delta = 1\) if integers \(r, s, t\) exist such that
\[
\frac{r + x}{a} = \frac{s + y}{b} = \frac{t + z}{c};
\]
\(\delta = 0\) otherwise.
Mordell ([8]; [11], p. 39) has proved the following result analogous to (1.9):

\[ s(b, c, a) = s(\alpha, b, a) + s(\alpha, a, b) + s(\alpha, c, a) \]

\[ = \frac{1}{6} abc + \frac{1}{4} (bc + ca + ab) + \frac{1}{4} (a + b + c) + \frac{1}{12} \left( \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right) + \frac{1}{12abc} - 2N_2(a, b, c), \]

where \( N_2(a, b, c) \) denotes the number of lattice points in the tetrahedron.

(1.12) \[ 0 < \frac{r}{a} < b, \quad 0 < \frac{s}{b} < a, \quad 0 < \frac{t}{c} < a. \]

We shall prove the following more general theorem.

**Theorem 2.** Let \( a, b, c \) be three positive integers that are relatively prime in pairs and let \( p \) be an arbitrary positive integer. Let \( \alpha, \beta, \gamma \) be real numbers, \( 0 < \alpha < 1, \ 0 < \beta < 1, \ 0 < \gamma < 1 \). Then we have

\[ \{abc\} [a^{p-1}b^{p-1}c^{p-1}(bc + ca + ab)] + \frac{1}{p + 1} \left( \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) - \frac{1}{(p + 1)abc} \sum_{\sigma < 1} (\sigma - 1)^{p-1}, \]

where

\[ w = bos + cas + ab, \quad \omega = \omega - [w], \quad \sigma = \frac{r + \omega}{a} + \frac{s + \omega}{b} + \frac{t + \omega}{c}, \]

and the final summation is over all \( r, s, t \) such that \( 0 < r < a, \ 0 < s < b, \ 0 < t < c, \ 0 < \omega < 1 \).

The proof of Theorem 2 makes use of the following

**Lemma 2.** Let \( a, b, c \) be three positive integers that are relatively prime in pairs and let \( w \) be a real number, \( 0 \leq w < bc + ca + ab \). Put

\[ S_1 = \sum_{bc + ca + ab + w < abc} \frac{a^{bc + ca + ab + w}}{abc}, \]

\[ S_2 = \sum_{bc + ca + ab + w < 2abc} \frac{a^{bc + ca + ab + w}}{abc}, \]

where it is understood that \( 0 \leq r < a, \ 0 \leq s < b, \ 0 \leq t < c \).

Then

\[ \frac{a^{bc}S_1 + S_2}{S_1} = \frac{b^{bc}(1 - a^{bc})}{(1 - a^{bc})(1 - b^{bc})(1 - c^{bc})} \frac{a^{bc} + \omega}{1 - \omega}, \]

where \( \omega = \omega - [w] \), the fractional part of \( w \).

Some special cases of Theorem 2 are discussed in the last section of the paper. See in particular Theorems 3 and 4.

While Theorem 2 generalizes (1.11), it does not of course generalize (1.16). Thus a generalization of (1.16) remains an open question.

**2. Proof of (1.7).** Put

\[ S = \sum_{\lambda_{r+s} = \lambda_k} \lambda_k \]

where the summation is over all \( r, s \) satisfying

\[ 0 \leq r < \ell, \quad 0 \leq s < \ell, \quad hr + ks + s < \ell \kappa. \]

If we divide both sides of (1.7) by \( \lambda_k \), it is clear that we may, without loss in generality assume that \( \lambda \) is an integer, \( 0 \leq \lambda < \ell + k \). Also since (1.7) is symmetric in \( \ell, \kappa \) we may assume that \( \lambda < \kappa \). It follows that, if \( 0 \leq \lambda < \kappa \), the inequality \( hr + ks + s < \ell \kappa \) can be satisfied for all \( r, 0 \leq s < \ell ; \) however, if \( h \leq \lambda < \ell + k \), the value \( r = e - 1 \) must be deleted.

Hence, for \( s \) an integer, \( 0 \leq s < \kappa \), we have

\[ S = \sum_{r=0}^{k-1} \lambda_r \sum_{s=0}^{r-1} \lambda_s = \sum_{r=0}^{k-1} \lambda_r \sum_{s=0}^{r-1} \lambda_s - \sum_{r=0}^{k-1} \frac{1}{k(k-1)} \lambda_k \]

where

\[ \lambda = \frac{a^{bc} + \omega}{1 - \omega}, \]

and the final summation is over all \( r, s \) such that \( 0 \leq r < a, \ 0 \leq s < b, \ 0 \leq t < c, \ 0 < \omega < 1 \).

The exponent on the extreme right is evidently the remainder obtained in dividing \( hr + s \) by \( k \). Hence the set of numbers

\[ \{hr + s - k[(hr + s)/k], 0 \leq r \leq k - 1\} \]

is identical, except for order, with the set \{0, 1, 2, \ldots, k\}. It follows
that
\[ \sum_{r=0}^{h-1} \frac{1 - \lambda^{hr+s-k[(hr+s)/k]}}{1 - \lambda} = \frac{1 - \lambda^k}{1 - \lambda} \]
and therefore
\[ S = \lambda^x \frac{1 - \lambda^{hk}}{(1 - \lambda^x)(1 - \lambda^k)} - \frac{\lambda^{hk}}{1 - \lambda} (0 \leq x < h). \]  

Now let \( h \leq s < h + k \). Then, excluding the value \( r = k-1 \), we have
\[
S = \sum_{r=0}^{h-2} \lambda^{hr+s} - \sum_{r=0}^{h-2} \lambda^{s} \lambda^{hr+s-k[(hr+s)/k]} = \sum_{r=0}^{h-2} \lambda^{hr+s} \frac{1 - \lambda^{hk-k[(hr+s)/k]}}{1 - \lambda^k} 
\]
\[
= \frac{\lambda^x}{1 - \lambda^k} \frac{1 - \lambda^{k(h-2)}}{1 - \lambda} - \frac{\lambda^{hk}}{1 - \lambda} \sum_{r=0}^{h-2} \lambda^{hr+s-k[(hr+s)/k]}.
\]
The set of numbers
\[ \{hr+s-k[(hr+s)/k] : 0 \leq r < k-2\} \]
excludes the number \( s-k \) from the set \( \{0, 1, 2, \ldots, k-1\} \).

It follows that
\[ S = \lambda^x \frac{1 - \lambda^{k(h-k)}}{(1 - \lambda^x)(1 - \lambda^k)} - \frac{\lambda^{hk}}{1 - \lambda} \frac{1 - \lambda^k}{1 - \lambda} \frac{1 - \lambda^{k(h-k)}}{1 - \lambda} \]
so that (2.3) holds in this case also.

This completes the proof of (1.7).

3. Proof of Theorem 1. Let
\[ S_p = hh^{k+\sigma_p}(k, \frac{y}{k}, \alpha) + hh^{k+\sigma_p}(k, \frac{y}{k}, \alpha). \]

Then exactly as in [5], § 2, we have
\[ S_p = (hk)^p \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} \left[ B_1 \left( \frac{y}{h}, \alpha \right) + B_1 \left( \frac{y}{h} \right) \right] B_p \left( \frac{\mu}{k} + \frac{\nu}{k} + \frac{y}{k}, \frac{\alpha}{k} \right). \]

We may assume, with no loss in generality, that
\[ 0 \leq \sigma < 1, \quad 0 \leq \nu < 1. \]

Put
\[ \sigma = \frac{\mu}{k} + \frac{\nu}{k} + \frac{y}{k} + \frac{\alpha}{k}, \]
so that
\[ (3.5) \quad 0 \leq \sigma < 2 \quad (0 \leq \mu < k, 0 \leq \nu < k). \]

Since
\[ B_1(x) = x - \frac{1}{2} \quad (0 \leq x < 1), \]
it follows from (3.3) that
\[ B_1(x) + B_1(y) = B_1(x+y) + \frac{1}{2} f(x+y), \]
where
\[ f(x) = \frac{-1}{1 + x} \quad (0 \leq x < 1), \]
\[ f(x) = \frac{-1}{1 + x} \quad (1 \leq x < 2). \]

Thus (3.2) becomes
\[ S_p = (hk)^p \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} \left[ B_1(\sigma) + \frac{1}{2} f(\sigma) \right] B_p(\sigma) \]
and therefore
\[ (3.8) \quad S_p = (hk)^p \left\{ \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} B_1(\sigma) B_p(\sigma) + \frac{1}{2} \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} B_p(\sigma) + \frac{1}{2} \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} B_p(\sigma) \right\}. \]

It follows from the multiplication theorem
[\[ B_n(kx) = k^{-n} \sum_{\mu(\mod h)} B_n \left( \frac{w + \mu}{k} \right) \]
that
\[ \sum_{\sigma=0}^{h-1} \sum_{\alpha=0}^{h-1} B_p(\sigma) = (hk)^{1-p} B_p(hy + k\sigma) \]
Thus (3.8) becomes
\[ (3.9) \quad S_p = (hk)^p T_p - (hk)^p U_p + \frac{1}{2} k h B_p(hy + k\sigma), \]
where
\[ T_p = \sum_{\sigma=0}^{k-1} \sum_{\alpha=0}^{h-1} B_1(\sigma) B_p(\sigma) \]
and
\[ U_p = \sum_{\sigma=0}^{k-1} \sum_{\alpha=0}^{h-1} B_1(\sigma) B_p(\sigma). \]

It is proved in [5] that
\[ (3.12) \quad (hk)^p T_p = \frac{p}{p+1} B_{p+1}(\xi) + \frac{1}{p+1} (Bhk + B + \xi)^{p+1} + \frac{1}{2} hkB_p(\xi), \]
where

\[ z = \hbar y + \zeta, \quad \zeta = z - [z], \]

so that \( \zeta \) is the fractional part of \( z \).

To evaluate \( U_p \), we consider

\[
\sum_{p = 0}^{\infty} U_p \frac{(hk)^p}{p!} = \sum_{\mu = 0}^{\infty} \sum_{\nu = 0}^{\infty} B_\mu(\sigma) \frac{(hk)^p}{p!} = \sum_{\mu = 0}^{\infty} \sum_{\nu = 0}^{\infty} \frac{h^\mu k^\nu}{\sigma^{h^\mu k^\nu}}.
\]

Since by (3.4) and (3.13)

\[ h^\mu k^\nu = h^\mu k^\nu + \zeta, \]

we get

\[
\sum_{p = 0}^{\infty} U_p \frac{(hk)^p}{p!} = \frac{h^\mu k^\nu}{\sigma^{h^\mu k^\nu}} - \sum_{\mu = 0}^{\infty} \sum_{\nu = 0}^{\infty} \frac{\sigma^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}}.
\]

By (1.7) the double sum on the right is equal to

\[ \frac{\sigma^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}} - \frac{\sigma^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}}. \]

Thus (3.14) becomes

\[
\sum_{p = 0}^{\infty} U_p \frac{(hk)^p}{p!} = \frac{h^\mu k^\nu}{\sigma^{h^\mu k^\nu}} - \sum_{\mu = 0}^{\infty} \sum_{\nu = 0}^{\infty} \frac{\sigma^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}}.
\]

Now multiply both sides by \( u \) and we have

\[
\sum_{p = 0}^{\infty} U_p \frac{(hk)^p}{p!} u^p = \frac{h^\mu k^\nu u^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}} - \frac{h^\mu k^\nu u^{h^\mu k^\nu}}{(h^\mu k^\nu)^{h^\mu k^\nu}}.
\]

Thus, equating coefficients, we get

\[
(p + 1)(hk)^p U_p = (hkB + h^\mu k^\nu + \zeta)^{p+1} - (hk + B + \zeta)^{p+1}.
\]

We now substitute from (3.12) and (3.15) in (3.9) and get (1.5). This completes the proof of Theorem 1.

4. Proof of (1.14). Let \( a, b, c \) be three positive integers that are relatively prime in pairs:

\[
(b, c) = (a, a) = (a, b) = 1.
\]

Without loss of generality we may assume that \( w \) is also an integer, \( 0 \leq w < bc + ca + ab \).

Put

\[
S_1 = \sum_{bc + ca + ab + w < abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w},
\]

and

\[
S_2 = \sum_{bc + ca + ab + w < 2abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w},
\]

where it is understood in such sums that

\[
0 \leq r < a, \quad 0 \leq s < b, \quad 0 \leq t < a.
\]

By (4.2) we have

\[
S_1 = \sum_{bc + ca + ab + w < abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w} = \sum_{1 \leq w < 0} \frac{\sigma^{h^b k^c + h^c k^a + h^a k^b + w}}{1 - \sigma^{h^b k^c + h^c k^a + h^a k^b + w}}.
\]

where \( R(m/ab) \) denotes the remainder obtained in dividing \( m \) by \( ab \).

Put

\[
U = \{ u \mid u = a(bk + ca) + w, \quad a(bk + ca) + w < abc, \}
\]

\[
V = \{ v \mid v = a(bk + ca) + w, \quad a(bk + ca) + w > abc, \}
\]

Thus

\[
S_2 = \sum_{bc + ca + ab + w < 2abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w} = \sum_{bc + ca + ab + w < 2abc} \frac{\sigma^{h^b k^c + h^c k^a + h^a k^b + w}}{1 - \sigma^{h^b k^c + h^c k^a + h^a k^b + w}}.
\]

In the next place we take \( S_2 = S_2' + S_2'' \), where

\[
S_2' = \sum_{bc + ca + ab + w < 2abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w},
\]

\[
S_2'' = \sum_{bc + ca + ab + w < 2abc} \sigma^{h^b k^c + h^c k^a + h^a k^b + w}.
\]

Clearly

\[
S_2' = \sum_{bc + ca + ab + w < 2abc} \sum_{1 \leq w < 0} \frac{\sigma^{h^b k^c + h^c k^a + h^a k^b + w}}{1 - \sigma^{h^b k^c + h^c k^a + h^a k^b + w}}.
\]
As for $S'_2$, we have

$$
S'_2 = \sum_{br+aw+w=ab} a^{r+aw+w(1)} \sum_{t<c<r} \frac{1}{1-a^{br+aw+w}} a^{ab}
$$

$$
= \sum_{t \leq r} \sum_{c \leq r} \frac{a^{ab}}{1-\frac{1}{a^{br+aw+w}}} = \sum_{t \leq r} \frac{1}{1-a^{br+aw+w}} a^{ab}
$$

$$
= \frac{1}{1-a^{ab}} \sum_{t \leq r} \frac{a^{ab}}{1-a^{br+aw+w}} a^{ab}.
$$

It follows from (4.6) and (4.8) that

$$
a^{abc}S_1 + S'_2 = \frac{a^{abc}}{1-a^{ab}} \sum_{t \leq r} a^t + \frac{1}{1-a^{ab}} \sum_{t \leq r} a^t - \frac{a^{abc}}{1-a^{ab}} \left( \sum_{u \leq v} a^{b(u+ab)} + \sum_{u \leq v} a^{R(v, ab)} \right).
$$

Since

$$
\sum_{u \leq v} a^{b(u+ab)} + \sum_{u \leq v} a^{R(v, ab)} = \sum_{m=0}^{d-1} a^m = \frac{1-a^{ab}}{1-a},
$$

we have

$$
a^{abc}S_1 + S'_2 = \frac{a^{abc}}{1-a^{ab}} \sum_{t \leq r} a^t + \frac{1}{1-a^{ab}} \sum_{t \leq r} a^t - \frac{a^{abc}}{1-a^{ab}} \cdot \frac{1-a^{ab}}{1-a}.
$$

Hence, by (4.7) and (4.9),

$$
a^{abc}S_1 + S'_2 + S''_2 = \left( \frac{1-a^{abc}}{1-a^{ab}} + \frac{a^{abc}}{1-a^{ab}} \right) \sum_{t \leq r} a^t + \frac{1}{1-a^{ab}} \sum_{t \leq r} a^t - \frac{a^{abc}}{1-a^{ab}} \cdot \frac{1-a^{ab}}{1-a}.
$$

$$
= \frac{1}{1-a^{ab}} \left( \sum_{t \leq r} a^t + \sum_{t \leq r} a^t \right) - \frac{a^{abc}}{1-a}.
$$

$$
= \frac{1}{1-a^{ab}} \sum_{t \leq r} \sum_{s=0}^{r-1} a^{br+as+w} - \frac{a^{abc}}{1-a}.
$$

Thus

$$
a^{abc}S_1 + S'_2 = \frac{a^{abc}(1-a^{abc})}{(1-a^{ab})(1-a^{ab})(1-a^{ab})} - \frac{a^{abc}}{1-a}.
$$

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5. Proof of Theorem 2. Put

$$
S_p = a^{p-1}s_p(b, a; cy+az, w) + b^{p-1}s_p(a, b; ay+aw, y) + c^{p-1}s_p(ab, c; by+ay, z).
$$

Then by (1.11) and the multiplication theorem for $B_p(a)$ we have

$$
S_p = (abc)^{p-1} \sum_{r=0}^{p-1} \binom{p-1}{r} \sum_{s=0}^{p-1} \binom{p-1}{s} a^{r+s} \left( \frac{r+x+s+y}{a} + \frac{z}{c} \right) \times
$$

$$
\left( \frac{r+x+s+y}{a} + \frac{z}{c} \right).
$$

If, for brevity, we put

$$
\xi = \frac{r+x+s+y}{a}, \quad \eta = \frac{s+y}{b}, \quad \zeta = \frac{t+z}{c},
$$

(5.2) may be written compactly in the form

$$
S_p = (abc)^{p-1} \sum_{\xi, \eta, \zeta} B_p(\xi + \eta + \zeta)B_1(\xi) + B_1(\eta) + B_1(\zeta).
$$

We assume in what follows that $a, b, c, x, y, z$ satisfy the inequalities

$$
0 \leq a < 1, \quad 0 \leq b < 1, \quad 0 \leq c < 1,
$$

so that

$$
0 \leq x < 1, \quad 0 \leq y < 1, \quad 0 \leq z < 1.
$$

It follows from (5.6) and the definition of $B_1(\omega)$ that

$$
B_1(\xi) + B_1(\eta) + B_1(\zeta) = B_1(\sigma) + \epsilon,
$$

where $\sigma = \xi + \eta + \zeta$ and

$$
\epsilon = \epsilon(\sigma) = \begin{cases} -1 & (0 \leq \sigma < 1), \\ 0 & (1 \leq \sigma < 2), \\ 1 & (2 \leq \sigma < 3). \end{cases}
$$

Thus (5.4) becomes

$$
(abc)^{p-1} S_p
$$

$$
= \sum_{\xi, \eta, \zeta} B_p(\sigma)(B_1(\sigma) + \epsilon(\sigma))
$$

$$
= \sum_{\xi, \eta, \zeta} B_p(\sigma)(B_1(\sigma) - 1) + \sum_{\xi, \eta, \zeta} B_p(\sigma)B_1(\sigma) + \sum_{\xi, \eta, \zeta} B_p(\sigma)B_1(\sigma) + 1
$$

$$
= \sum_{\xi, \eta, \zeta} B_p(\sigma)B_1(\sigma) + \sum_{\xi, \eta, \zeta} B_p(\sigma)\left\{ 2 \sum_{\xi, \eta, \zeta} B_p(\sigma) + \sum_{\xi, \eta, \zeta} B_p(\sigma) \right\}
$$

$$
= R_p + T_p - U_p, \quad \text{say}.
$$
Clearly
\[ R_p = \sum_{\xi, \eta, \zeta} B_p(\xi + \eta + \zeta) B_1(\xi + \eta + \zeta) \]
\[ = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} B_p \left( \frac{r + x}{a} + \frac{s + y}{b} + \frac{t + z}{c} \right) B_1 \left( \frac{r + x}{a} + \frac{s + y}{b} + \frac{t + z}{c} \right) \]
\[ = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} B_p \left( \frac{e^{\omega}}{abc} \right) B_1 \left( \frac{e^{\omega}}{abc} \right) \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{o=1}^{\infty} \frac{e^{\omega}}{abc} \left( \frac{m}{abc} + \frac{n}{abc} + \frac{o}{abc} \right) \]
where
\[ (5.10) \quad w = bcw + caw + abs, \quad \omega = \omega - [\omega]. \]

Then, exactly as in the proof of (3.12), we get, first,
\[ (ab)^p U_p = \frac{p}{p+1} B_{p+1}(\omega) = \frac{1}{p+1} (bca + B + \omega)^{p+1} + \frac{1}{2} abc B_p(\omega). \]

As for \( T_p \), by the multiplication theorem for \( B_p(x) \), we have
\[ T_p = \sum_{\xi, \eta, \zeta} B_p(\xi + \eta + \zeta) = (ab)^{p-1} B_1(bcw + caw + abs). \]

Thus, by (5.10),
\[ (ab)^p T_p = abc B_p(\omega) = abc B_p(\omega). \]

To evaluate \( U_p \), we take
\[ (5.13) \quad \sum_{p=1}^{\infty} \frac{U_p (abu)^p}{p!} = \sum_{p=1}^{\infty} \frac{(abu)^p}{p!} \left( 2 \sum_{\sigma=1}^{\infty} B_\sigma(\omega) + \sum_{1 < \sigma < 2} B_\sigma(\sigma - 1) \right) \]
\[ = \sum_{p=1}^{\infty} \frac{(abu)^p}{p!} \left( 2 \sum_{\sigma=1}^{\infty} B_\sigma(\omega) + \sum_{1 < \sigma < 2} B_\sigma(\sigma - 1) \right) \]
\[ = 2 \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} \left( 2 \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} + \sum_{1 < \sigma < 2} \frac{abu}{abu - 1} e^{abu(\sigma - 1)u} \right) \]
\[ = \frac{abu}{abu - 1} \left( 2 \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} + \sum_{1 < \sigma < 2} \frac{abu}{abu - 1} e^{abu(\sigma - 1)u} \right) \]
\[ = \frac{abu}{abu - 1} \left( (abu - 1) \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} + \right. \]

By (4.10), with
\[ abu = ab(\xi + \eta + \zeta) = bcw + caw + abu, \quad w = bcw + caw + abu, \]
we have
\[ \frac{abu}{abu - 1} \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} = \frac{abu}{abu - 1} = \frac{abu^{abu+1}}{(abu-1)(abu-1)(abu-1)}. \]

Then it follows from (5.13) that
\[ \frac{abu}{abu - 1} \sum_{\sigma=1}^{\infty} \frac{abu}{abu - 1} = \frac{abu^{abu+1}}{(abu-1)(abu-1)} - \frac{abu^{abu+1}}{(abu-1)(abu-1)(abu-1)}. \]

Equating coefficients of \( u^p/p! \), we get
\[ (5.14) \quad (ab)^p U_p = p(ab)^p \sum_{\sigma=1}^{\infty} (abu)^{\sigma-1} \frac{abu}{abu - 1} + \frac{1}{p+1} (abu(\sigma - 1)^{p-1} + \frac{1}{p+1} (bcw + caw + abs)^{p-1} + \]
\[ + \frac{1}{p+1} (bcw + caw + abs)^{p-1} + \]
\[ + \frac{1}{p+1} (bcw + caw + abs)^{p-1} + \]
\[ + \frac{1}{p+1} (bcw + caw + abs)^{p-1} + \]

Thus, by (5.9), (5.11), (5.12) and (5.14), we have
\[ (5.15) \quad abu S_p = \frac{p}{p+1} B_{p+1}(\omega) + \frac{3}{2} abc B_p(\omega) - p(ab)^p \sum_{\sigma=1}^{\infty} (abu)^{\sigma-1} + \]
\[ + \frac{1}{p+1} (abu(\sigma - 1)^{p-1} + \frac{1}{p+1} (abu(\sigma - 1)^{p-1} + \]
\[ + \frac{1}{p+1} (abu(\sigma - 1)^{p-1} + \]

This completes the proof of Theorem 2.
6. Some special cases. To begin with we take \( p = 0 \) in Theorem 2. We assume that
\[
0 < x < 1, \quad 0 < y < 1, \quad 0 < z < 1.
\]
Then by (5.1) we find that
\[
abcs_4 = bex + ay + abc - \frac{1}{2}(bc + ca + ab).
\]
As for (5.15), we have
\[
abcs_4 = \left( -\frac{1}{2}abc - \frac{1}{3} \right) + \frac{1}{2}abc + \left( \frac{1}{2}abc + \frac{1}{3} - abc - 3 \right) - \frac{1}{2abc} \left( -bc - ca - ab - 3w \right) \right)
\]
which reduces to the right hand side of (6.2).

The special case \( p = 1 \) takes more computation. By (5.15) we have
\[
abcs_1 = \frac{1}{2}B_1(\omega) + \frac{1}{2}(abcB + B + \omega)^2 - \frac{1}{6abc} \left( bc + ca + ab \right)^2 - \frac{1}{6abc} \left( bc + ca + ab \right)^2 - \frac{1}{6abc} \left( bc + ca + ab \right)^2.
\]
Clearly, the sum
\[
\sum_{c_{\mathbb{C}}} 1 = N_s(a, b, c) + 1
\]
is equal to the number of lattice points in the tetrahedron
\[
\begin{align*}
0 < r < a, \quad 0 < s < b, \quad 0 < t < c, \quad 0 < \frac{r}{a} + \frac{s}{b} + \frac{t}{c} < 1.
\end{align*}
\]

We now specialize further by taking \( x = y = z = 0, \omega = w = 0 \). Thus (6.3) reduces to
\[
abcs_1 = \frac{1}{12} + \frac{1}{2}(abcB + B + \omega)^2 - \frac{1}{2abc} \left( bc + ca + ab \right)^2 - \frac{1}{6abc} \left( bc + ca + ab \right)^2 - \frac{1}{6abc} \left( bc + ca + ab \right)^2.
\]
Simplifying, we get
\[
s_1(ba, a) + s_1(ca, b) + s_1(ab, c)
\]
\[
= -\frac{5}{4} + \frac{1}{12abc} + \frac{1}{6} \left( \frac{a + b + c}{a + b + c} \right) + \frac{1}{4} \left( \frac{bc + ca + ab}{a + b + c} \right) + \frac{1}{4} \left( \frac{bc + ca + ab}{a + b + c} \right) + N_s(a, b, c).
\]
Since \( s_1(h, k) = s(h, k) + \frac{1}{4} \), it is evident that (6.6) is identical with Mordell's theorem (11), thus furnishing a partial check on Theorem 2. Moreover (6.3) yields the following direct generalization of Mordell's theorem.

Theorem 3. Let \( a, b, c \) be positive integers that are relatively prime in pairs. Let \( x, y, z \) be real numbers, \( 0 < x < 1, 0 < y < 1, 0 < z < 1 \). Let
\[
w = bex + ay + abz, \quad \omega = w - [w].
\]
Then we have
\[
s_1(ba, a) + s_1(ca, b) + s_1(ab, c) = \frac{3}{2} B_1(\omega) + \frac{1}{2abc} B_1(\omega) - N_3(a, b, c) - 1 +
\]
\[
+ \frac{1}{2abc} \left( \left( abcB + B + \omega \right)^2 - (abcB + B + ab + c + \omega)^2 \right) +
\]
\[
+ \frac{1}{6abc} \left( \left( bc + ca + ab \right)^2 - (bc + ca + ab + w - abc)^2 \right).
\]
Finally we state the special case of Theorem 2 with \( x = y = z = 0 \). Theorem 4. Let \( a, b, c \) satisfy the usual requirements and \( p \geq 0 \). Then we have
\[
s_{p-1}(ba, a) + s_{p-1}(ca, b) + s_{p-1}(ab, c)
\]
\[
= \frac{3}{2} B_p \left( \frac{p}{p+1} \right) B_{p+1} +
\]
\[
+ \frac{1}{p+1(abc)} \left( abcB + B + \omega \right)^{p+1} - (abcB + B + abc)^{p+1} +
\]
\[
+ \frac{1}{(p+1)(p+2)(abc)} \left( bc + ca + ab \right)^{p+2} - (bc + ca + ab + ab - abc)^{p+2} - p(abc)^{p-1} \sum_{c_{\mathbb{C}}} \left( \frac{r}{a} + \frac{s}{b} + \frac{t}{c} - 1 \right)^{p-1},
\]
where the summation is over all \( r, s, t \) satisfying
\[
0 < r < a, \quad 0 < s < b, \quad 0 < t < c, \quad 0 < \frac{r}{a} + \frac{s}{b} + \frac{t}{c} < 1.
\]

References


Quadratic diophantine equations with parameters

by

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To the memory of Paul Turán

1. In an earlier paper [3] written in collaboration with the late Harold Davenport we proved:

THEOREM A. Let \( a(t), b(t) \) be polynomials with integral coefficients. Suppose that every arithmetical progression contains an integer \( v \) such that the equation \( a(v)x^2 + b(v)y^2 = z^2 \) has a solution in integers \( x, y, z \) not all 0. Then there exist polynomials \( x(t), y(t), z(t) \) in \( \mathbb{Z}[t] \), not all identically 0, such that \( a(t)x(t)^2 + b(t)y(t)^2 = z(t)^2 \) identically in \( t \).

From this result we derived:

THEOREM B. Let \( F(x, y, t) \) be a polynomial with integral coefficients which is of degree at most 2 in \( x \) and \( y \). Suppose that every arithmetical progression contains an integer \( t \) such that the equation \( F(x, y, t) = 0 \) is soluble in rational numbers for \( x \) and \( y \). Then there exist rational functions \( x(t), y(t) \) in \( \mathbb{Q}(t) \) such that \( F(x(t), y(t), t) \equiv 0 \) identically in \( t \).

Earlier, one of us asked [6] whether a result similar to Theorem B holds if \( F(x, y, t) \) is replaced by any polynomial \( F(x, y, t_1, \ldots, t_r) \) and the stronger assumption is made that for all integral \( r \)-tuples \( t_1, \ldots, t_r \), the equation \( F(x, y, t_1, \ldots, t_r) = 0 \) is soluble in the rational numbers for \( x \) and \( y \). The stronger assumption is needed since the hypothesis analogous to the one of Theorem B involving arithmetical progressions is not sufficient already for \( F(x, y, t) = x^2 - y^2 - t \). We shall show here that if \( F \) is of degree at most 2 in \( x \) and \( y \) a hypothesis analogous to the one of Theorem B suffices for any number of parameters \( t \). We shall also indicate an equation of an elliptic curve over \( \mathbb{Q}(t) \) for which the stronger assumption involving all integers \( t \) does not seem to suffice.

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