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An elementary method in prime number theory

by

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1. Introduction. In number theory one often desires to estimate sums of the form

$$\sum f(p) \tag{1}$$

(or equivalently $\sum \Lambda(n)f(n)$ with Λ von Mangoldt's function) where, for example, $f(n)$ is an exponential function $e(F(n)) = e^{2\pi i F(n)}$, or a character $\chi(n)$. The techniques for the estimation of such sums, whether analytic ([1], [3], [6]) or elementary ([7], [8], [9]) invariably relate such sums to bilinear forms of the kind either

$$\sum_m \sum_n a_m f(mn) \tag{I}$$

or

$$\sum_m \sum_n a_m b_n f(mn). \tag{II}$$

Suppose that the range for n in (1) is $[1, X]$, or equivalently that $f(n) = 0$ when $n > X$. Then as a fairly general principle the estimates for (I) are good provided that $m \leqslant M$ with M small compared with X, and those for (II) are good provided that $m \leqslant M$, $n \leqslant N$ with both M and N small compared with X.

One method (I. M. Vinogradov [9]) of carrying out this procedure is to use the sieve of Eratosthenes to write

$$f(1) + \sum_{\sqrt{X} < p \leqslant X} f(p) = \sum_{m|P} \sum_{n \leqslant X/m} \mu(m) f(mn),$$

where P is the product of those prime numbers not exceeding $\sqrt{X}$. The right hand side of this is of type (I), but has the defect of including m that are close to X. In order to treat these m, Vinogradov has to introduce a combinatorial argument, which for the sharpest estimates is quite involved, that allows him to relate this portion of the expression to bilinear forms of type (II).

In [7] and [8] an elementary method was introduced and developed which avoids these combinatorial difficulties. Consider the identity

$$(2)\qquad \sum_{n} g(1,n)+\sum_{m>u}\sum_{n}\Big(\sum_{\substack{d\mid m\\ d\leqslant u}}\mu(d)\Big)g(m,n) = \sum_{d\leqslant u}\sum_{r}\sum_{n}\mu(d)g(dr,n),$$

which holds for any double sequence $g(m,n)$ for which the right hand side converges absolutely, and is an immediate consequence of the relation

$$\sum_{\substack{d\mid m\\ d\leqslant u}}\mu(d) = 0 \qquad (1<m\leqslant u).$$

Let

$$g(m,n) = \begin{cases}\Lambda(n)f(mn) & (v<n\leqslant X/m),\\ 0 & \text{otherwise.}\end{cases}$$

Then, by (2),

$$(3)\qquad \sum_{v<n\leqslant X}\Lambda(n)f(n) = S_1-S_2-S_3,$$

where

$$(4)\qquad S_1 = \sum_{d\leqslant u}\sum_{k\leqslant X/d}\mu(d)(\log k)f(dk),$$

$$(5)\qquad S_2 = \sum_{k\leqslant uv}\sum_{r\leqslant X/k}\Big(\sum_{\substack{d\leqslant u\\ dn=k}}\sum_{n\leqslant v}\mu(d)\Lambda(n)\Big)f(kr),$$

$$(6)\qquad S_3 = \sum_{m>u}\sum_{v<n\leqslant X/m}\Big(\sum_{\substack{d\mid m\\ d\leqslant u}}\mu(d)\Big)\Lambda(n)f(mn).$$

Now

$$(7)\qquad S_1 \leqslant \int_1^X S_1(\alpha)\frac{d\alpha}{\alpha}$$

where

$$(8)\qquad S_1(\alpha) = \sum_{d\leqslant\min(u,X/\alpha)}\ \sum_{\alpha<k\leqslant X/d}\mu(d)f(dk).$$

Clearly both $S_1(\alpha)$ and S_2 are of type (I) above, whilst S_3 is of type (II). Thus suitable choices for u and v will often ensure that the corresponding estimates are good. We further remark that on some occasions the sum S_2 may be more sharply estimated by breaking it into two parts and treating the second part as a type (II) sum.

In [7] and [8] the above method was applied in the case $f(n) = e(\alpha n)$. The purpose here is to show how the method can be applied to give a

proof of the Bombieri–A. I. Vinogradov theorem concerning the average error term in the distribution of prime numbers in arithmetic progressions. Two essential ingredients (see [1], [3] or [6]) in the proof of this theorem are the Siegel–Walfisz theorem and a mean value theorem giving a bound for

$$(9)\qquad T(Y,Q) = \sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{X\leqslant Y}|\psi(X,\chi)|,$$

where $\sum^*$ denotes summation over the primitive characters χ modulo q. The technique described above enables one to give an *elementary* proof of the following theorem.

THEOREM. *Suppose that* $Q\geqslant 1$, $Y\geqslant 2$, $L = \log YQ$. *Then*

$$T(Y,Q) \ll L^4(Y+Y^{5/6}Q+Y^{1/2}Q^2).$$

As an easy consequence of this and the Siegel–Walfisz theorem one has the corollary.

COROLLARY (Bombieri–Vinogradov).

$$\sum_{q\leqslant Q}\max_{\substack{a,X\\ (a,q)=1,X\leqslant Y}}\left|\psi(X,q,a)-\frac{X}{\varphi(q)}\right| \ll_A Y(\log Y)^{-A}+Y^{1/2}QL^4.$$

2. Lemmata. The first lemma is an immediate consequence of the large sieve inequality (see, for instance, Gallagher [6] or (1.4) of [4], the proofs of which are entirely elementary) and Cauchy's inequality.

LEMMA 1. *Suppose that* a_m $(m = 1,\ldots,M)$ *and* b_n $(n = 1,\ldots,N)$ *are complex numbers. Then*

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\left|\sum_{m=1}^{M}\sum_{n=1}^{N}a_mb_n\chi(mn)\right| \ll \Big((M+Q^2)(N+Q^2)\sum_m|a_m|^2\sum_n|b_n|^2\Big)^{1/2}.$$

The proof of the theorem rests on a maximal version of this.

LEMMA 2. *On the premises of Lemma* 1 *we have*

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{X\leqslant Y}\left|\sum_{\substack{m=1\\ mn\leqslant X}}^{M}\sum_{n=1}^{N}a_mb_n\chi(mn)\right| \ll \Big((M+Q^2)(N+Q^2)\sum_m|a_m|^2\sum_n|b_n|^2\Big)^{1/2}\log YMN.$$

Proof. Let

$$C = \int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha,$$

$\gamma > 0$, and $\delta(\beta) = 1$ when $0 \leqslant \beta < \gamma$, $\delta(\beta) = 0$ when $\beta > \gamma$. Then $C > 0$ and it is easily seen that for $A \geqslant 1$, $\beta \geqslant 0$, $\beta \neq \gamma$, we have

$$\delta(\beta) = \int_{-A}^{A} e^{i\beta\alpha} \frac{\sin\gamma\alpha}{C\alpha} d\alpha + O\left(\frac{1}{A|\gamma-\beta|}\right).$$

Let $\gamma = \log([X]+\frac{1}{2})$, $\beta = \log mn$. Thus

$$\sum_{\substack{m\ n\\ mn\leqslant X}}\sum a_m b_n \chi(mn)$$
$$= \int_{-A}^{A} \sum_m \sum_n a_m m^{i\alpha} b_n n^{i\alpha} \chi(mn) \frac{\sin\gamma\alpha}{C\alpha} d\alpha + O\left(\frac{X}{A}\sum_m\sum_n |a_m b_n|\right).$$

The desired conclusion now follows easily from Lemma 1 on taking $A = YMN$.

3. Proof of the theorem. *If* $Q^2 > Y$, then the theorem follows at once from Lemma 2 on taking $M = 1$, $a_1 = 1$, $b_n = \Lambda(n)$. Hence it can be assumed that $Q^2 \leqslant Y$.

Let

(10) $$u = v = \min(Q^2, Y^{1/3}, YQ^{-2}).$$

By applying Lemma 2 as in the case $Q^2 > Y$ it is easily seen that

$$\sum_{q\leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}^{*} \max_{X\leqslant u^2} |\psi(X,\chi)| \ll (u^2Q + uQ^2)L^2.$$

Hence, on writing $f(n) = \chi(n)$ in (3), to prove the theorem it suffices, by (9), to show that for $j = 1, 2, 3$ the sum

$$T_j = \sum_{q\leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}^{*} \max_{u^2<X\leqslant Y} |S_j|$$

satisfies

(11) $$T_j \ll L^4(Y + Y^{5/6}Q + Y^{1/2}Q^2).$$

By (6),

$$T_3 \leqslant \sum_{M\in\mathcal{M}} T_3(M)$$

where

$$\mathcal{M} = \{2^k u:\ k = 0, 1, \ldots,\ k \leqslant (\log(Yu^{-2}))/\log 2\}$$

and

$$T_3(M) = \sum_{q\leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}^{*} \max_{u^2<X\leqslant Y} \Bigg| \sum_{M<m\leqslant 2M} \sum_{\substack{u<n\leqslant Y/M\\ n\leqslant X/m}} \Big(\sum_{\substack{d\mid m\\ d\leqslant u}} \mu(d)\Big) \Lambda(n)\chi(mn)\Bigg|.$$

By Lemma 2,

$$T_3(M) \ll \Big((M+Q^2)(YM^{-1}+Q^2) \sum_{m\leqslant 2M} d(m)^2 \sum_{n\leqslant Y/M} \Lambda(n)^2\Big)^{1/2} \log Y$$
$$\ll (\log Y)^3 (Y + Y^{1/2}M^{1/2}Q + YM^{-1/2}Q + Y^{1/2}Q^2).$$

This easily gives (11) with $j = 3$.

By (5),

$$T_2 \leqslant T_2' + T_2''$$

where

$$T_2' = \sum_{q\leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}^{*} \max_{u^2<X\leqslant Y} \Bigg| \sum_{k\leqslant u} \sum_{r\leqslant X/k} \Big(\sum_{\substack{d\leqslant u\\ dn=k}} \sum_{n\leqslant v} \mu(d)\Lambda(n)\Big) \chi(kr)\Bigg|$$

and T_2'' is the corresponding expression with $k \leqslant u$ replaced by $u < k \leqslant u^2$. The sum T_2'' is treated in the same way as T_3. Thus

(12) $$T_2'' \ll L^3(Y + Y^{5/6}Q + Y^{1/2}Q^2).$$

The sum T_2' is estimated directly *via* the Pólya–Vinogradov inequality (observe that Schur's proof [5] is elementary). Therefore

$$T_2' \ll (Y + Q^{5/2}u)L^2,$$

and with (10) and (12) this implies (11) with $j = 2$.

By (7) and (8) it is easily seen that T_1 can be estimated in the same way as T_2'.

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