§ 1. Introduction.

1. This paper arose from the consideration of the expression

\[ R_n(\eta) := R(\eta \sqrt{2p(1-p)n} + pn) - R(\eta \sqrt{2p(1-p)n} - pn) \]

with

\[ 0 < p < 1, \quad \eta > 0, \quad n \to \infty, \]

where \( R(x) \) denotes generally the fractional part of \( x \), lying in \( [0,1) \). The expression \( (1.1) \) occurs in the Probability Calculus. Namely, as has been shown by Uspensky [7] and Ostrowski [3], the sum

\[ \sum_{n=0}^{\infty} \left( \frac{n}{\eta} \right) p^n(1-p)^n - \left( \frac{n}{\eta} \right) \]

\[ (|w-pn| \leq \eta \sqrt{2np(1-p)}) \]

can be expressed in the form

\[ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = e^{-x^2} \frac{1 - R_n(\eta)}{\sqrt{2np(1-p)n}} + O \left( \frac{1}{n} \right) \quad (n \to \infty) \]

where \( R_n \) is given by \( (1.1) \).

As a matter of fact a similar formula was first given by Laplace. However, the term \( R_n(\eta) \) was missing in Laplace's deduction. The formula as it had been written down by Laplace was repeated used until the first quarter of this century. It was therefore of importance, that \( R_n(\eta) \) does not tend with \( n \to \infty \) to 0 but is everywhere dense in the interval between 0 and 2. This was announced in [3] and proved in [4].

2. Since, however, very often the sequences in such connection are, not only everywhere dense, but also uniformly distributed that is have a constant density in every point in the corresponding interval, it appears...

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to be worth while investigating whether the sequence \( R_n(\eta) \) \((n \to \infty)\) is uniformly distributed in the interval \((0, 2)\).

It turns out that this is not the case and we succeeded even to obtain explicit expression for the distribution function of the \( R_n(\eta) \) \((n \to \infty)\) for irrational values of \( \rho \). This discussion presented peculiar difficulties which are overcome using a theorem about distribution of the expressions \( R_n(\rho \lambda) \) \((n \to \infty)\), saying that for any interval of the length \( R_n(\rho \lambda) \) for natural \( n \) the modulus of the "error term" is \( \ll n \rho \) \((\rho \geqslant 3)\); see also \([2]\).

Our result is contained in a corresponding result about the more general sequences

\[
R(a\sqrt{\nu} + \nu \lambda) + R(a\sqrt{\nu} - \nu \lambda) \quad (\nu \to \infty)
\]

with a fixed positive \( a \). Further, the sequence \( a\sqrt{\nu} \) \((\nu \to \infty)\) can be replaced in this connection with a more general sequence

\[
a_n = R(s_n) \quad (\nu \to \infty)
\]

where

\[
s_n - s_{n-1} \downarrow 0, \quad \nu(s_n - s_{n-1}) \to \infty \quad (\nu \to \infty).
\]

3. For sequences \( a_n \) of this type it follows immediately from a result by L. Fejér, that they are uniformly distributed in \((0, 1)\) \([1]\); see also \([6]\).

But, in order to carry through our discussion we have to restrict the sequence \( s_n \), further imposing on it a condition which will be formulated in (2.10) in Section 7. In any case we show that the sequence

\[
R(a\sqrt{\nu}) \quad (\nu \to \infty), \quad 0 < a < 1, \quad a > 0
\]

can be used as \( a_n \) in our results.

We will therefore consider generally the sequence

\[
\tau_n = R(a_n + \nu \lambda) + R(a_n - \nu \lambda) \quad (\nu \to \infty)
\]

and prove that for any irrational \( \lambda \) the density of this sequence in the interval \( 0 < x \leqslant 1 \) has the value \( x \) and in the interval \( 1 \leqslant x < 2 \) the value \( 2 - x \).

4. As to the corresponding expressions in the case of a rational fraction \( \lambda \), we show that then the \( \tau_n \) are uniformly distributed if and only if \( 2\lambda \) is an integer.

We prove that the distribution function always exists for rational \( \lambda \) too, and derive explicitly the value of this function in a neighborhood of \( 1 \), where it is rather simple. The expression of the distribution function in the whole interval \((0, 2)\) can be also derived, but the expression obtained is very complicated and we omit it, as it appears to present little interest.

The sequence (1.5) is of course rather a special one. However, our discussion offers one of the very few examples of nonuniform distribution where the distribution function could be obtained explicitly.

§ 2. Discussion of the \( a_n \).

5. Consider a sequence of positive \( d_n \) monotonically falling to 0 and such that

\[
d_n \downarrow 0, \quad nd_n \to \infty \quad (\nu \to \infty).
\]

Since, from a \( \nu \) on, \( d_n > 1/n \), it follows that

\[
\sum_{n=1}^{\infty} d_n = \infty \quad (n \to \infty).
\]

Putting

\[
a_n := R(s_n), \quad d_n = s_n - s_{n-1},
\]

it follows obviously from (2.11) that the numbers \( a_n \) infinitely often run monotonically through the interval \((0, 1)\). Consider the values of \( \nu \) for which

\[
a_{n-1} + d_n \geqslant 1,
\]

Denote these values of \( \nu \) ordered monotonically by \( n \), taking \( n_1 := 1 \):

\[
1 = n_1 < n_2 < n_3 < \ldots
\]

Then, for each \( n \) we have the inequality

\[
0 \leqslant a_{n_k} < a_{n_k+1} < \ldots < a_{n_{k+1}} < 1 \quad (k = 1, 2, \ldots).
\]

We denote the sequence of indices in (2.1) by \( P_n \) and call it the \( k \)-th run. \( A_n \) from a \( \nu \) on \( d_n < 1 \) and at each run \( [s_n] \) is increased by \( 1 \), we see that for the \( k \)-th run

\[
es_n = \nu + \nu + a_n, \quad [s_n] = \nu + \nu \quad (\nu \in P_n, \nu \geqslant n_k),
\]

for a constant \( c \). Put further

\[
n_{k+1} - n_k = : N_k, \quad \frac{1}{d_{n_{k+1}}} = : D_k, \quad n_k = \sum_{n=1}^{k-1} N_n + 1.
\]

6. For a fixed \( \nu \), \( 0 < \nu \leqslant 1 \), denote by \( N_\nu(a) \) the number of the \( a \) from the \( k \)-th run, which are \( < \nu \).

\[
N_\nu(a) := N(a < a \leqslant a, \nu \in P_n).
\]

(Obviously \( N_\nu(1) = N_\nu \)). Then

\[
am_{n+1} a_{n+1} = a < a \leqslant a_{n+1} a_{n+1} \leqslant a_{n+1} = a_{n+1} N_\nu(a)
\]

and therefore

\[
\sum_{a = n+1}^{n+1} d_a < \nu - a_n \leqslant \sum_{a = n+1}^{n+1} d_a.
\]
From (2.1) and (2.5) it follows now

\[(N_n(x) - 1) a_{n+1} < x - a_n < N_n(x) a_n\]

and by (2.5)

\[D_n(x - a_n) \leq N_n(x) \leq D_{n+1}(x - a_n) + 1.\]

Taking \(x = 1\), we obtain

\[D_{n+1}(1 - a_n) + 1 > N_n \geq D_n(1 - a_n).\]

Dividing (2.8) by \(D_n\) it follows

\[1 - a_n + d_{n+1} > \frac{N_n}{D_{n+1}} = \frac{N_n}{1 + N_n} (n_{n+1} d_{n+1}).\]

Here, the left side expression tends to 1 while by (2.1) \(n_{n+1} d_{n+1}\) tends to \(\infty\). It follows \(\frac{N_n}{1 + N_n} \to 0\) and therefore

\[N_n/n_n \to 0.\]

7. We make now the supplementary hypothesis about (2.1):

\[D_{n+1}/D_n \to 1.\]

Then, dividing all three terms of (2.8) by \(D_n\) it follows

\[N_n/D_n \to 1, \quad N_n \to \infty.\]

Divide now all three terms of (2.7) by \(N_n\):

\[(x - a_n) \frac{D_n}{N_n} \leq \frac{N_n(x)}{N_n} \leq (x - a_n) \frac{D_{n+1}}{N_n} \frac{1}{D_n} + 1.\]

Using (2.10) and (2.11) it follows for \(x \to \infty\)

\[\frac{N_n(x)}{N_n} \to x.\]

8. Denote, for a natural \(n\), by \(A(n, x)\) the number of all \(a_n\) with \(v < n\) and \(a_n < x\):

\[A(n, x) := N(a_n < x, v < n).\]

To any \(n\) corresponds a \(k\) such that \(a_k \leq n < a_{k+1}\), and here \(k \to \infty\) with \(n \to \infty\). Thence we have

\[A(n, x) = \sum_{k=1}^{n-1} N_k(x) + \theta N_k = A(n_k, x) + \theta N_k,
\]

where \(\theta\), as well as \(\theta\) with different indices, denotes from now on a number of modulus \(\leq 1\), not necessarily the same for different \(\theta\) in a formula.

Similarly,

\[n = \sum_{k=1}^{n-1} N_k + \theta' N_k = n_k + 1 + \theta' N_k.\]

By Cauchy's convergence theorem it follows from (2.12):

\[\frac{\sum_{k=1}^{n-1} N_k(x)}{n} = \frac{A(n_k, x)}{n_k} \to x.\]

We have now

\[\frac{A(n, x)}{n} = \frac{A(n_k, x) + \theta' N_k}{n_k - 1 + \theta' N_k} = \frac{A(n_k, x)}{n_k - 1 + \theta' N_k}.
\]

Dividing on the right the numerator and denominator by \(n_k\), we see that

\[\frac{A(n, x)}{n} \to x \quad (n \to \infty).\]

This follows also, as has been said in Section 3, from a result by L. Féjer.

\section{An example.}

9. Consider, for \(a > 0\), \(b > 0\) and an \(a\) with \(0 < a < 1\), the sequence

\[s_n = (av + b)^n \quad (v = 1, 2, \ldots).\]

Here the \(d_v\) are defined by

\[d_v = (av + b)^{v - (av - a + b)^v}.\]

They are \(> 0\) and monotonically decreasing.

Writing

\[\frac{d_v}{(av)^v} = \left(1 + \frac{b}{av}\right)^v - \left(1 + \frac{b - a}{av}\right)^v\]

and developing we obtain \(a/v + O(v^{-2})\) and

\[d_v = av + b + O(1/v), \quad d_v \downarrow 0, \quad v \to \infty.
\]

The conditions (2.31) are satisfied.

10. From (2.4a) it follows now for \(x \to \infty\):

\[s_n \sim s_k, \quad an_n + b \sim x^k, \quad n_k \sim \frac{1}{a} x^k.\]
From (3.3) we obtain
\begin{equation}
D_n = 1/d_n \sim \frac{n^{1-\epsilon}}{a^2} \sim \frac{1}{a^n} n^{(1-\epsilon)/a}.
\end{equation}
We see that our supplementary condition (2.10) is also satisfied and
\begin{equation}
N_n \sim \frac{1}{a^n} n^{(1-\epsilon)/a}.
\end{equation}

Taking in particular \( a = \frac{1}{2}, b = 0 \) we obtain
\begin{equation}
\sigma = \rm{Var} \, x, \quad n_n \sim 2n^2, \quad N_n \sim D_n \sim \frac{2n}{a} \quad (x \to \infty).
\end{equation}

\section{4. Introduction of \( A'(x), A''(x), B'(x), B''(x) \).}

11. If we use the notation
\begin{equation}
R(ni) = \lambda,
\end{equation}
the expression (1.5) of \( \tau_n \) can be written as
\begin{equation}
\tau_n = R(\lambda + \lambda) + \lambda - \lambda.
\end{equation}

Our aim is to obtain an expression for the number of \( \tau_n < 2n \) if \( n \to \infty \).
\begin{equation}
T(n, \lambda) := N(n; \tau_n < 2n, \lambda < n) \quad (0 < x \leq 1).
\end{equation}

Observe that \( R(\lambda + \lambda) \) has one of the three values \( \lambda + \lambda \) or \( \lambda + \lambda - 1 \) which lies in \( (0, 1) \) that is \( \lambda < 1 - \lambda \) and \( \lambda + \lambda - 1 \) if \( \lambda > 1 - \lambda \). Similarly \( R(\lambda - \lambda) \) is \( \lambda - \lambda \) if \( \lambda < \lambda \) and \( \lambda - \lambda + 1 \) if \( \lambda > \lambda \).

We can therefore have one of the three values \( 2\lambda, 2\lambda + 1, 2\lambda - 1 \) and we have obviously four cases:
\begin{itemize}
  \item[(A')] \( \lambda < \lambda < 1 - \lambda \), \( \tau = 2\lambda + 1 \),
  \item[(B')] \( 1 - \lambda < \lambda < \lambda \), \( \tau = 2\lambda - 1 \),
  \item[(A'')] \( \lambda > \lambda > 1 - \lambda \), \( \tau = 2\lambda \),
  \item[(B'')] \( \lambda < \lambda < \lambda \), \( \lambda < 1 - \lambda \), \( \tau = 2\lambda \).
\end{itemize}

12. We will first discuss the distribution numbers corresponding to a fixed run \( P_n \) assuming that \( n \to \infty \) runs through the whole set \( P_n \). We call then the corresponding distribution numbers for \( \alpha < x \) in the corresponding cases \( A'_n, A''_n, B'_n, B''_n \) defining them as (1)
\begin{align}
A'_n(x) &:= N(n; \alpha < x, \lambda < \lambda < 1 - \lambda) , \\
A''_n(x) &:= N(n; \alpha < x, \lambda > \lambda > 1 - \lambda).
\end{align}

\begin{align}
B'_n(x) &:= N(n; \alpha < x, \lambda < \lambda < 1 - \lambda) , \\
B''_n(x) &:= N(n; \alpha < x, \lambda > \lambda > 1 - \lambda).
\end{align}

Denote by \( T_\alpha(n) \) the number of \( n \) for which \( \tau_n < 2n \),
\begin{equation}
T_\alpha(n) := N(n; \tau_n < 2n, \lambda < n) .
\end{equation}

Then we say that
\begin{equation}
T_\alpha(n) := A'_n(x - \frac{1}{2}) + B'_n(x + \frac{1}{2}) + A''_n(x + B''_n(x)).
\end{equation}

Indeed, in the cases (A') and (B') if \( \tau_n < 2n \) \( \lambda < x \pm \frac{1}{2} \), while in the cases (A'') and (B''), \( 3\lambda < 2n \) and therefore \( \lambda < x \).

On the other hand, denoting by \( N_n(x) \) the number of \( n < x \) from the \( n \)th run, as in (2.6),
\begin{equation}
N_n(x) := N(n; \alpha < x, \lambda < n) ,
\end{equation}
we obtain easily, according as \( x < \frac{1}{2} \) or \( x > \frac{1}{2} \):
\begin{align}
N_n(x) &= A'_n(x) + A''_n(x) + B'_n(x) \quad (x < \frac{1}{2}) , \\
N_n(x) &= A'_n(\frac{1}{2}) + B'_n(x) + B''_n(x) \quad (x > \frac{1}{2}) .
\end{align}

Our next aim is now to obtain suitable expressions for \( A'_n(x) \) and \( B'_n(x) \), keeping \( x \) fixed.

\section{5. Use of the intervals \( J_x \).}

13. We assume from now an until §8 that \( \lambda \) is irrational. Assume a natural number \( s \) and consider the points of \( (0, 1) \):
\begin{equation}
R_\sigma = \lambda \quad (\sigma = 0, 1, 2, \ldots, s).
\end{equation}

If we order these points monotonically,
\begin{equation}
p_0 < p_1 < \ldots < p_s < 1,
\end{equation}
they decompose the interval \( (0, 1) \) into \( s + 1 \) intervals
\begin{equation}
J_\sigma := (p_{\sigma - 1}, p_\sigma) \quad (\sigma = 1, \ldots, s) , \quad J_{s+1} := (p_s, 1).
\end{equation}

Then the length of \( J_\sigma \) is
\begin{equation}
|J_\sigma| := \delta_\sigma = p_\sigma - p_{\sigma - 1} \quad (1 \leq \sigma \leq s) , \quad \delta_{s+1} := 1 - \delta_s.
\end{equation}

Put
\begin{equation}
\delta = \max \delta_\sigma = 1 .
\end{equation}

Denote now by \( N_\sigma^{(x)} \) the number of the \( \sigma \)th from the \( x \)th run, which lie in \( J_\sigma \):
\begin{equation}
N_\sigma^{(x)} := N(n; \alpha < x, \lambda < n) .
\end{equation}
Then, for two consecutive of these \( a \) from \( J \), the difference \( a_{s+1} - a_s \) lies between \( d_n \) and \( d_{n+1} \). Since the number of subintervals of \( J \), obtained by introducing the corresponding \( a \), lies between \( N^{(s)} + 1 \) and \( N^{(s)} - 1 \) we obtain the inequality

\[
(5.7) \quad (1+N^{(s)})d_n \geq A_s \geq (N^{(s)}-1)d_{n+1} \quad (s = 1, \ldots, s+1).
\]

Using (2.5) it follows

\[
(5.8) \quad D_n A_s - 1 \leq N^{(s)} \leq D_{n+1} A_s + 1 \quad (s = 1, \ldots, s+1).
\]

By (2.10) we can write

\[
D_{n+1}/D_n = 1 - \varepsilon_n, \quad \varepsilon_n \geq 0, \quad \varepsilon_n \to 0.
\]

There exists therefore for each \( x \geq 1 \):

\[
(5.9) \quad \eta_x = \max_{r \geq x-1} \varepsilon_r \to 0 \quad (x \to \infty).
\]

We assume \( x \) so large that

\[
(5.8a) \quad \eta_x < 1/20.
\]

We can then write the expression on the right of (5.8) as \( D_n A_s (1 - \varepsilon_n) + 1 \) and obtain finally for \( N^{(s)} \) the formula:

\[
(5.10) \quad N^{(s)} = D_n A_s (1 + \varepsilon_n) + \theta.
\]

14. We will now have to use the following lemma ([5]):

**Lemma 1.** Consider subinterval, \( J \), of the interval \( (0, 1) \), closed from the left, whose length is \( R(n, \lambda) \) for an integer \( n \geq 0 \). Denote for a natural interval \( n \) by \( N(J, n) \) the number of \( \lambda_n = R(n, \lambda) \) with \( \nu < n \), lying in \( J \).

Then

\[
(5.11) \quad N(J, n) = R(n, \lambda)n + \theta n, \quad |\theta| < 1.
\]

From this lemma it follows:

**Lemma 2.** Under the assumptions of Lemma 1 assume \( N \) a natural number and denote by \( N(J, n, N) \) the number of the \( \lambda_n \), lying in \( J \) with \( n < \nu < n + N \). Then, applying (5.11) to the interval \( (mod 1) \) \( J + n\lambda \):

\[
(5.12) \quad N(J, n, N) - J[N, \lambda] = \theta n, \quad |\theta| < 1.
\]

15. If we apply (5.12) to the interval \( (p_s, 1-p_s), p_s = R(s') \lambda < 1/2 \), the length of this interval is

\[
1 - 2p_s = R(-2s') \lambda,
\]

our \( n_0 = 2s' \) will be \( \leq 2s \), and we obtain for the number of corresponding cases for \( a \), from \( J \):

\[
(5.13) \quad N(p; p_s < \lambda < 1 - p_s, a_s \in J_s) = N^{(s)}(1 - 2p_s) + 26s \quad (p_s < 1/2).
\]

Correspondingly, if \( p_s = R(s')' \lambda > 1/2 \), the length of the interval \( (1 - p_s, p_s) \) is

\[
2p_s - 1 = R(2s') \lambda,
\]

our \( |n_0| = 2s' \) is again \( \leq 2s \) and we obtain for the number of corresponding cases with \( a \), from \( J \):

\[
(5.14) \quad N(p; p_s < \lambda < 1 - p_s, a_s \in J_s) = N^{(s)}(2p_s - 1) + 26s \quad (p_s > 1/2).
\]

**§ 6. Expressions for \( A_s(x) \) and \( B_s(x) \).**

16. We now define the numbers corresponding to \( A_s \) and \( B_s \), if we replace the condition \( a_s < x \) by the condition \( a_s \in J_s \):

\[
(6.1) \quad \bar{A}_s = N(p; a_s < \lambda < 1 - a_s, a_s \in J_s, p \in P_x) \quad (p < 1),
\]

\[
(6.2) \quad \bar{B}_s = N(p; 1 - a_s < \lambda < a_s, a_s \in J_s, p \in P_x) \quad (p < 1).
\]

The number \( \bar{A}_s \) is obviously increased if we replace \( a_s \) in the condition for the \( \gamma_s \) intervals by \( p_{s-1} \), and decreased, if \( a_s \) is replaced by \( p_s \). We obtain for the \( \gamma_s \) from \( p_s \)

\[
N(p; p_s < \lambda < 1 - p_s, a_s \in J_s) \leq \bar{A}_s \leq N(p; p_{s-1} < \lambda < 1 - p_{s+1}, a_s \in J_s).
\]

Applying now (5.13) with \( \tau = s \) and \( \sigma = s - 1 \), we obtain

\[
N^{(s)}(1 - 2p_{s-1}) - 2s \leq \bar{A}_s \leq N^{(s)}(1 - 2p_{s+1}) + 2s
\]

or using (5.10)

\[
[D_n A_s (1 - \eta_s) - 1](1 - 2p_{s-1}) - 2s \leq \bar{A}_s \leq [D_n A_s (1 + \eta_s) + 1](1 - 2p_{s+1}) + 2s
\]

or

\[
D_n A_s (1 - 2p_s) - \eta_s D_n A_s - 2s \leq \bar{A}_s \leq D_n A_s (1 - 2p_{s-1}) + \eta_s D_n A_s + 1 + 2s.
\]

But, as \( 1 - 2s \) is monotonically decreasing, we have

\[
(1 - 2p_s) D_n < \int_{p_{s-1}}^{p_s} (1 - 2s) ds < (1 - 2p_{s-1}) D_n,
\]

\[
(1 - 2p_s) A_s = \int_{p_{s-1}}^{p_s} (1 - 2s) ds + 26D_n^2,
\]

\[
(1 - 2p_{s-1}) A_s = \int_{p_{s-1}}^{p_{s-1}} (1 - 2s) ds + 26D_n^2.
\]

Introducing this into (6.3) we finally obtain, as \( A_s \leq l \) by (5.5)

\[
(6.4) \quad \bar{A}_s = D_n \int_{p_{s-1}}^{p_s} (1 - 2s) ds + \theta(2l + \eta_s) A_s D_n + \theta(s + 1).
\]
17. In the case of $\tilde{B}_i(x)$ we proceed in exactly the same way, the only difference being that $1 - 2p_n, 1 - 2p_{n-1}$ are to be replaced with $2p_{n-1} - 1, 2p_n - 1$. In this way we obtain immediately

$$D_n D_{n+1} (2p_{n-1} - 1) - \eta_n D_n A_n - \theta(x + 1) \leq \tilde{B}_i \leq D_n D_{n+1} (2p_n - 1) + \eta_n D_n A_n + \theta(x + 1).$$

Exactly as above we see that both $A_n (2p_{n-1} - 1)$ and $A_n (2p_n - 1)$ can be written as

$$\int_{p_{n-1}}^{p_n} (2x - 1) \, dx + 2 \theta A_n,$$

and obtain finally

$$\tilde{B}_i = D_n \int_{p_{n-1}}^{p_n} (2x - 1) \, dx + \theta(2l + \eta_n) A_n D_n + \theta(x + 1). \tag{6.5}$$

18. It is now easy to obtain the asymptotic relations for $A_n(x)$ and $B_n(x)$.

Assume $x \leq \frac{1}{2}$ and

$$p_t \leq x < p_{t+1}, \quad p_u \leq \frac{1}{2} < p_{u+1}.$$  

Then $p_t \leq x \leq \frac{1}{2}$ and for all intervals $J_n (s = 1, \ldots, t)$, we can use the formula (6.4), while the number of indices $v$ corresponding to the case $A_n'$ is, by (5.6), $N^{(v+1)}$. We obtain therefore

$$A_n' (x) = \sum_{v=1}^{t} A_n + \theta N^{(v+1)},$$

or using (6.4)

$$A_n' (x) = D_n \left[ \int_{0}^{p_t} (1 - 2x) \, dx + \theta(2l + \eta_n) \right] D_n + \theta(x + 1)^3.$$

Dividing both sides by $N_n$ we obtain further, as $p_t \leq \frac{1}{2}$,

$$\frac{A_n'(x)}{N_n} = D_n N_n \left[ \int_{0}^{p_t} (1 - 2x) \, dx + \theta(2l + \eta_n) \right] + \frac{\theta(x + 1)^3}{N_n}.$$  

For $x \to \infty$ and a constant $t$ it follows:

$$-2l \leq \lim_{N_n} \left( \frac{A_n'(x)}{N_n} - \int_{0}^{p_t} (1 - 2x) \, dx \right) \leq 2l \quad (x \leq \frac{1}{2}).$$

On the other hand

$$\int_{p_t}^{x} (1 - 2x) \, dx \leq x - p_t < l,$$

and we finally obtain

$$-2l \leq \lim_{N_n} \left( \frac{A_n'(x)}{N_n} - \int_{0}^{x} (1 - 2x) \, dx \right) \leq 2l \quad (x \leq \frac{1}{2}). \tag{6.7}$$

19. To discuss $B_n'(x)$, assume $x > \frac{1}{2}$ and again (6.6). Then obviously $p_{u+1} \geq p_{t+1}, \quad t \geq u$ and the intervals $J_n (s = 1, \ldots, t)$, for which (6.5) holds, correspond to $n + 2 \leq n \leq t$ while the parts of $B_n'$ which correspond to a part of $J_n$ and to a part of $J_{n+1}$ must be estimated by $N^{(u+1)}$ and $N^{(t+1)}$. Then we have by (6.5) and (5.10)

$$B_n'(x) = \sum_{v=1}^{t} B_v + \theta \left( N^{(u+1)} + N^{(t+1)} \right)$$

$$= D_n \left[ \int_{0}^{p_t} (2x - 1) \, dx + \theta(2l + \eta_n) \right] + \theta(s+1)^3 + \theta(3lD_n + 2)$$

where the integral must be deleted if it is $< 0$.

Further,

$$\left| \int_{0}^{p_t} (2x - 1) \, dx \right| = \left| \int_{\frac{1}{2}}^{u+1} (2x - 1) \, dx \right| \leq 1,$$

and we can therefore rewrite our formula as

$$B_n'(x) = D_n \left[ \int_{0}^{p_t} (2x - 1) \, dx + \theta(2l + \eta_n) \right] + 2\theta(s+1)^3 + 2 \tag{6.8} \quad (x > \frac{1}{2}).$$

Observe that this formula also holds if $u = t$.

Dividing (6.8) on both sides by $N_n$, we obtain

$$\frac{B_n'(x)}{N_n} = D_n \left[ \int_{0}^{p_t} (2x - 1) \, dx + \theta(2l + \eta_n) \right] + 2\theta(s+1)^3 + 2 \frac{N^{(t+1)}}{N_n}.$$  

With $s \to \infty$ it follows now for a fixed $s$

$$-6l \leq \lim_{N_n} \left( \frac{B_n'(x)}{N_n} - \int_{0}^{p_t} (2x - 1) \, dx \right) \leq 6l \tag{6.9} \quad (x \leq \frac{1}{2}).$$

and now (6.7) and (6.8) become with $s \to \infty, l \to 0$:

$$\frac{A_n'(x)}{N_n} \to \int_{0}^{p_t} (1 - 2x) \, dx = x - x^2 \quad (x \leq \frac{1}{2}), \tag{6.10}$$

$$\frac{B_n'(x)}{N_n} \to \int_{0}^{p_t} (2x - 1) \, dx = (x - \frac{1}{2})^2 \quad (x > \frac{1}{2}). \tag{6.11}$$
In particular, with \( x = \frac{1}{2} \) and \( x = 1 \) we have

\[
\frac{A'_n(1)}{N_n} \rightarrow \int_{\frac{1}{2}}^{1} (1 - 2x) \, dx = \frac{1}{4},
\]

\[
\frac{B'_n(1)}{N_n} = \frac{B'_n(1) - B'_n(0)}{N_n} \rightarrow \frac{1}{4}.
\]

\[\text{§ 7. Density values for irrational } \lambda.\]

\[\text{20. Rewriting (4.11) and (4.12) as}\]

\[
A''(x) + B''(x) = N_n(x) - A'_n(x) \quad (x \leq \frac{1}{2}),
\]

\[
A''(x) + B''(x) = N_n(x) - B'_n(x) - A'_n(\frac{1}{2}) \quad (\frac{1}{2} < x < 1)
\]

and using (2.12), (6.10)–(6.12) we obtain

\[
\frac{A''(x) + B''(x)}{N_n} \rightarrow x - \int_{\frac{1}{2}}^{x} (1 - 2u) \, du = x^2 \quad (x \leq \frac{1}{2}),
\]

\[
\frac{A''(x) + B''(x)}{N_n} \rightarrow x - \int_{\frac{1}{2}}^{x} (2u - 1) \, du = \frac{1}{2} - 2x - x^2 - \frac{1}{2} \quad (\frac{1}{2} < x < 1).
\]

Resuming now the formula (4.9) we see that it becomes for \( x \leq \frac{1}{2} \):

\[
T_n(x) = B'_n(x + \frac{1}{2}) + A'_n(x) + B'_n(x).
\]

Dividing this by \( N_n \) and using (6.11) and (7.3) it follows

\[
T_n(x) / N_n \rightarrow 2x^2 \quad (x \rightarrow \infty, \ x \leq \frac{1}{2}).
\]

As to the case \( x > \frac{1}{2} \), (4.9) becomes in this case by (6.10), (6.13) and (7.4)

\[
T_n(x) = A'_n(x - \frac{1}{2}) + B'_n(1) + A'_n(x) + B'_n(x) \quad (x > \frac{1}{2}),
\]

and dividing this by \( N_n \) we obtain with \( x \rightarrow \infty \):

\[
T_n(x) / N_n \rightarrow 1 - 2(1 - x)^2 \quad (x \rightarrow \infty, \ x > \frac{1}{2}).
\]

If we introduce now

\[
A(x) := \begin{cases} 
\frac{x^2}{2} & (x \leq 1), \\
1 - 2(1 - x^2)^2 & (1 < x < 2)
\end{cases}
\]

we can write the formulas (7.7) and (7.8) as

\[
T_n(x) / N_n \rightarrow A(2x) \quad (x \rightarrow \infty, \ 0 < x < 1).
\]

\[\text{21. Returning now to the definitions (4.3) and (4.8) of } T(x, n) \text{ and } T_n(x), \text{ we can write, taking } n = n_k \text{ and using (2.3)}:
\]

\[
T(x, n_k) = \sum_{s=1}^{n_k} T_s(x), \quad n_k = \sum_{s=1}^{n_k} N_s + 1.
\]

But then it follows from (7.10), by Cauchy’s convergence theorem, that

\[
T(x, n_k) / n_k \rightarrow A(2x) \quad (k \rightarrow \infty).
\]

For a general natural \( n_k \) we can write, with a convenient \( k \),

\[
n_k \leq n < n_{k+1} = n_k + N_k.
\]

Then obviously

\[
T(x, n) = T(x, n_k) + \theta N_k, \quad n = n_k + \theta N_k,
\]

\[
\frac{T(x)}{n} = \frac{T(x, n_k) / n_k + \theta N_k / n_k}{1 + \theta N_k / n_k}.
\]

It follows now, using (2.9),

\[
\frac{T(x, n)}{n} \rightarrow A(2x) \quad (k \rightarrow \infty).
\]

In virtue of the definition (4.3) of \( T(x, n) \) we have now proved that the distribution function of the \( x \) in the interval \((0, 2)\) is given by \( A(x) \) in (7.9).

Differentiating, we obtain for the corresponding density the value \( x \) in \((0, 1)\) and \( 2 - x \) in \((1, 2)\).

We remind the reader that this result has been obtained assuming \( \lambda \) to be irrational.

\[\text{§ 8. An auxiliary problem.}\]

\[\text{22. Our problem, in the case of a rational } \lambda, \text{ can be reduced to the following special problem which we solve in this § 8.}\]

Assume \( a_0 \) as in § 2. Assume a fixed \( r, \ 0 \leq r < 1, \) and put

\[
r := R(a_0 + r) + R(a_0 - r).
\]

What value has the distribution function of the \( \tau \) in \((0, 2)\)?

Put, assuming first \( r > 0,
\]

\[
r' := \min(r, 1 - r), \quad r'' := \max(r, 1 - r).
\]

Since \( R(a_0 - r) = R(a_0 + 1 - r) \), we can write

\[
\tau = R(a_0 + r') + R(a_0 + r'').
\]

If we assume that the positive \( \tau \) satisfies

\[
\tau < 2x, \quad x < n,
\]

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we have to consider three cases, according as \( a_r \) lies in the intervals \((A') \langle r', r'' \rangle\) or \((B') a_r < r'\) or \((C') a_r \geq r''\). Accordingly we define
\[
A'(x) := N(x: a_r < x, \tau < x, x < \tau < r'),
\]
\[
B'(x) := N(x: a_r < x, \tau < x, a_r < \tau),
\]
\[
C'(x) := N(x: a_r < x, \tau < x, a_r \geq \tau).
\]
Obviously, if \( r' = \frac{1}{2} \), \( r'' = \frac{1}{2} \), \( A'(x) = 0 \).

23. As to \( A'(x) \) we have, since the distribution of \( a_r \) is uniform, for \( a_r \) the interval \( r' < a_r < \min(x, r'') \) and therefore for \( A'(x)/n \) the following limits with \( n \to \infty \): if \( x < r'; x < x < \min(x, r'') \), and \( r'' - r' \) if \( x \geq r'' \).

Introducing the general notation
\[
(D)_+ := \begin{cases} D & \text{if } D > 0, \\ 0 & \text{if } D < 0. \end{cases}
\]
we immediately verify that
\[
A'(x)/n \to (x - r')_+ - (x - r'')_+.
\]

In the case of \( B'(x) \), \( a_r \) runs through the interval \( \langle 0, \min(x, r') \rangle \) and it follows
\[
B'(x)/n \to \min(x, r').
\]

Finally in the case of \( C'(x) \), \( a_r \) runs through the interval \( \langle r'', x \rangle \) if \( x > r'' \) and then \( C'(x)/n \to x - r'' \), while for \( \langle r'', x \rangle \), \( C'(x) \) vanishes; we can write
\[
C'(x)/n \to (x - r'')_+.
\]

We note in particular that with \( n \to \infty \):
\[
B'(x - \frac{1}{2})/n \to \min(x - \frac{1}{2}, r') \quad (x \geq \frac{1}{2}),
\]
\[
C'(x + \frac{1}{2})/n \to (x + \frac{1}{2} - r')_+ = (x - \frac{1}{2} + r')_+ \quad (x \leq \frac{1}{2}),
\]
\[
C'(1)/n \to r'.
\]

24. We can now easily find the asymptotic value of
\[
T^*(x, n) := N(x: \tau < x, \tau < 2\sigma).
\]
The distribution function of the \( \tau \), is given by
\[
\lim_{n \to \infty} T^*(x/2, n)/n = \min(\frac{x - r'}{2}, \frac{x - (1 - r')}{2}) \quad (x \leq 1);
\]
\[
\lim_{n \to \infty} T^*(x/2, n)/n = \frac{x - \tau}{2} - \min(\frac{x - r'}{2}, \tau) \quad (1 < x < 2).
\]

§ 9. Distribution of the \( \tau \), for rational \( \lambda \).

25. Obviously, for \( \lambda = 0 \), as the \( a_r \) are uniformly distributed in \( \langle 0, 1 \rangle \), the values of \( \tau = 2R(a_r) \) are uniformly distributed in \( \langle 0, 2 \rangle \).

In the case \( \lambda = \frac{1}{2} \) we have, for even \( n \), \( \tau = 2R(a_r) \) uniformly distributed in \( \langle 0, 2 \rangle \), while, for odd values of \( n \),
\[
\tau = R(a_r + \frac{1}{2}) + R(a_r - \frac{1}{2}) = 2R(a_r + \frac{1}{2})
\]
are again uniformly distributed in \( \langle 0, 2 \rangle \). We have therefore for the complete sequence of the \( \tau \), again uniform distribution in \( \langle 0, 2 \rangle \).

We see that the \( \tau \) are uniformly distributed in \( \langle 0, 2 \rangle \) if \( 2\lambda \) is an integer.

We are now going to show that for any rational \( \lambda \) for which \( 2\lambda \) is not an integer the distribution of the \( \tau \) in \( \langle 0, 2 \rangle \) is not uniform.

If we write, using (4.3),
\[
T(x/2, n) := N(x: \tau < x, \tau < 2\sigma) \quad (0 < x < 2)
\]
then the distribution function of the $\tau$,

$$g(\omega) = \lim_{n \to \infty} T(\omega/2, n)/n,$$

should in the case of uniform distribution, exist and have the value $\omega/2$.
It is therefore, in order to prove our assertion, sufficient to show that,
in a certain neighborhood of $\omega = 1$, $g(\omega)$ exists but has a value different
from $\omega/2$, save for $\omega = 1$.

26. Assume now

$$\lambda = 2\sigma, \quad 0 < \sigma < \omega, \quad (\sigma, \rho) = 1,$$

for an integer $\sigma > 2$ and an integer $\rho$ satisfying (9.3). Choose an $\alpha$
among one of the integers $0, 1, 2, \ldots, s - 1$ and consider the values of $\nu$
in the arithmetical progression

$$\nu = \mu s + \alpha \quad (\mu = 1, 2, \ldots) .$$

For such values of $\nu$, we have

$$\sigma \nu = \frac{b_\alpha}{s}, \quad r_\alpha = \frac{b_\alpha}{s},$$

where $b_\alpha$ is one of the numbers $0, 1, \ldots, s - 1$, and $r_\alpha$ assumes exactly
one each of the values $\sigma/s, \sigma = 0, 1, \ldots, s - 1$. Then our $\tau$ becomes

$$R(\alpha_{s \omega + \nu} + r_\alpha) + R(\alpha_{s \omega + \nu} - r_\alpha),$$

or, putting $\beta_\alpha \sigma = \alpha_{s \omega + \nu}$

$$\tau_\alpha = \beta_\alpha \sigma = R(\beta_\sigma \sigma + r_\alpha) + R(\beta_\sigma \sigma - r_\alpha).$$

In this way the whole sequence of the $\alpha$ is decomposed into $s$ partial
sequences which correspond to the values of $\nu$ running through $s$
arithmetical progressions with the difference $s$. The sequences of $\nu$ corresponding
to these progressions satisfy the conditions of (9.3) and have therefore
distribution functions in $\langle 0, s \rangle$.

It follows that the complete sequence of $\alpha$ has a distribution function
in $\langle 0, s \rangle$, equal to the arithmetical mean of the distribution functions
belonging to our partial sequences.

27. If we define

$$T_{\alpha} \sigma(\omega, m) = \sum_{\mu < m} \beta_\mu \omega < \omega,$$

we obtain, applying (8.20) and (8.21), (we have then to replace $\sigma$, with $\beta_\mu \sigma$,
$r$ with $r_\alpha$ and $n$ with $m$):

$$T_{\alpha} \sigma(\omega/2, m)/m \to \Gamma_\rho \omega(\omega) \quad (m \to \infty),$$

putting

$$\Gamma_\rho \omega(\omega) := \frac{\omega}{2} - r + \frac{\omega - 1}{2} + r \quad (\omega \leq 1),$$

$$\Gamma_\rho \omega(\omega) := \frac{\omega}{2} - \left( \frac{\omega}{2} - r \right) + \min \left( \frac{\omega - 1}{2}, r \right) \quad (1 \leq \omega < 2).$$

For $\omega < n$ it follows from (9.4) that

$$\mu \frac{n - a}{s} \sim \frac{n}{s} \quad (n \to \infty).$$

Replacing in (9.6) $m$ with $n/s - a/s$ we obtain now

$$sT_{\alpha} \sigma(\omega/2, n)/m \to \Gamma_\rho \omega(\omega) \quad (\omega \leq 1).$$

28. If we consider all positive $\omega < n$ for which $\tau < \omega$, these values
are distributed among the arithmetical progressions (9.4) corresponding
to all admissible values of $\alpha$. We obtain therefore

$$T \left( \frac{\omega}{2}, n \right) \to \sum_{\alpha = 0}^{\omega - 1} T_{\alpha} \sigma \left( \frac{\omega}{2}, \frac{n - a}{s} \right),$$

and by (9.9)

$$T \left( \frac{\omega}{2}, n \right)/m \to \frac{1}{s} \sum_{\sigma = 0}^{n - 1} \Gamma_\sigma \omega(\omega).$$

The $\alpha$ runs here through the set of fractions

$$\{ \sigma/s \mid \sigma = 0, 1, \ldots, s - 1 \}.$$

We have finally for $g(\omega)$ in (9.2)

$$g(\omega) = \frac{1}{s} \sum \Gamma_\sigma \omega(\omega)$$

where $\sigma$ runs through (9.11).

29. In order to evaluate further the right hand expression in (9.12)
observe that for any function of $\sigma$, defined on the set (9.11), the following
formula holds in which $r'$ is defined by (8.2):

$$\sum_{r'} f(r') = f(0) + \sum_{0 < r < s/2} f(r) + \delta f(0)$$
where \( \delta \) is defined by
\[
\delta = \begin{cases} 
0 & \text{s odd}, \\
1 & \text{s even}. 
\end{cases}
\] (9.14)

Indeed, for all \( s \) from the open interval \((0, s/2)\) we have by (8.2), \( r' = r \), while, if \( s/2 < \sigma < 1 \), \( r' = 1 - r \) runs again through the set of the values of \( r \) between 0 and \( \frac{1}{2} \).

Applying (9.13) we obtain in the sums (9.7) and (9.8)
\[
\sum_r \left( \frac{x}{2} - r' \right)_+ = \frac{x}{2} + 2 \sum_{0 < \sigma < s/2} \left( \frac{x}{2} - \frac{\sigma}{s} \right)_+ \quad (x \leq 1),
\]
(9.15)
\[
\sum_r \left( \frac{x-1}{2} + r' \right)_+ = 2 \sum_{0 < \sigma < s/2} \left( \frac{\sigma}{s} - \frac{1-x}{2} \right)_+ + \delta \frac{x}{2} \quad (x \leq 1),
\]
(9.16)
\[
\sum_r \left( \frac{x}{2} - r'' \right)_+ = \sum_r \left( \frac{x}{2} + r' - 1 \right)_+ \\
= 2 \sum_{0 < \sigma < s/2} \left( \frac{\sigma}{s} + \frac{x}{2} - 1 \right)_+ + \delta \frac{x-1}{2} \quad (1 < x < 2).
\]
(9.17)

30. We are going now to compute the values of (9.15)-(9.17) in the interval
\[
1 - 1/s \leq x < 1 + 1/s.
\]
We can write \( x = 1 + \theta/s, \quad -1 \leq \theta < 1 \). On the other hand, putting
\[
t = \frac{s-1-\delta}{2},
\]
(9.18)
\[
t \quad \text{is the greatest integer } < s/2, \text{ and the upper limit of } \sigma \text{ in the right hand sum in (9.15) is obviously } t.
\]
For our \( x \) and \( \sigma \) we have
\[
\frac{x}{s} - \frac{\sigma}{s} \geq \frac{1}{2} - \frac{\theta}{2s} = \frac{s-1-\delta}{2s} - \frac{\theta}{2s} \geq 0.
\]
The right hand sum in (9.15) becomes therefore
\[
\sum_{0 < \sigma < s/2} \left( \frac{x}{2} - \frac{\sigma}{s} \right)_+ = \frac{s-1-\delta}{2} \left( x - \frac{s+1-\delta}{2s} \right).
\]
We obtain in (9.15)
\[
\sum_r \left( \frac{x}{2} - r' \right)_+ = \frac{x}{2} - \frac{(s-\delta)(x-1)}{4s}.
\]
(9.20)
\[
= \frac{(s-\delta)(x-1)}{2s} + \frac{1-\delta}{4s}.
\]

31. As to the right hand sum in (9.16), it becomes
\[
\sum_0 \left( \frac{s-1-x}{2} \right)_+ \]
and is, since \( \frac{s-1-x}{2} \geq \frac{1}{s} - \frac{\theta}{2s} > 0 \),
\[
= \sum_0 \left( \frac{s-1-x}{2} \right)_+ = \frac{s-1-\delta}{2} \left( \frac{1}{s} + \frac{s-1-\delta}{2s} + x-1 \right) \]
\[
= \frac{s-1-\delta}{2} \frac{x}{s} + \frac{1-\delta}{4s}.
\]
(9.19)
It follows therefore from (9.16)
\[
\sum_r \left( r' + \frac{1-x}{s} \right)_+ = \frac{s-1}{2} + \frac{1-\delta}{4s} - \frac{x-2}{4}.
\]

Using (9.12) and (9.7) we obtain
\[
sg(x) = \left( s - \frac{1+\delta}{2} \right) x - t \quad (1-1/s \leq x \leq 1).
\]
(9.21)

32. As to the last sum in (9.17) we have obviously, replacing \( x \) by 
\[ \frac{x}{s} \]
\[
= \frac{s-1}{2} + \frac{1+\delta}{2s} \leq \frac{\theta-1}{2s} \leq 0.
\]
We see that our sum vanishes and we obtain from (9.17)
\[
\sum_r \left( \frac{x}{2} - r'' \right)_+ = \frac{x-1}{2}.
\]
(9.22)
Further, applying (9.13), we obtain
\[
\sum_r \left( \frac{x-1}{2}, r' \right)_+ = \delta \frac{x-1}{2} + 2 \sum_{0 < \sigma < s/2} \left( \frac{x-1}{2} - \frac{\sigma}{s} \right).
\]
(9.23)
But here we have \( (x-1)/2 = \theta/2s \leq 1/2s \leq 1/s \leq \sigma/s \). Thence, replacing
in (9.23) each summand by \((x-1)/2\) it follows
\[(9.24) \quad \sum Min \left( \frac{x-1}{2}, r \right) = (6+24) \frac{x-1}{2} = \frac{x-1}{2} (x-1).\]

33. Using now (9.8) and summing over \(r\) we obtain from (9.22) and (9.24) the same expression as in (9.21). It follows finally
\[(9.25) \quad sg(x) = \left( \frac{3+1}{2} \right) x - t \quad (1-1/2 \leq x \leq 1+1/2).\]

Since obviously
\[sg(x) - ax/2 = t(x-1),\]
we see that \(g(x)\) in \(1-1/2 \leq x \leq 1+1/2\) is always different from \(x/2\) save for \(x = 1\). The assertion of Section 25 is proved.

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On two definitions of the integral of a \(p\)-adic function

by

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In memory of Paul Turán

In his basic paper on functions of a \(p\)-adic variable Diamodno [1], introduced a special kind of integral (primitive) of a continuous function. A completely different definition of such an integral was more recently given by M. van der Put (see A. C. M. van Rooij and W. H. Schikhof [2]). The aim of this note is to show that these two definitions lead to the same result. This is rather surprising because there is a large set of non-constant \(p\)-adic functions of derivative 0.

Since it simplifies the discussion, we shall study the two kinds of integrals for the class of functions \(f: J \to \mathbb{Q}_p\) where \(p\) is any positive rational prime, \(\mathbb{Q}_p\) is the field of \(p\)-adic numbers, and \(J = \{0, 1, 2, \ldots\}\) is the set of all non-negative rational integers. The set \(J\) is not closed, and its \(p\)-adic closure is the set \(I = \{x \in \mathbb{Q}_p; |x|_p < 1\}\) of all \(p\)-adic integers which is compact.

1. Let \(f: J \to \mathbb{Q}_p\) be an arbitrary function on \(J\). The two integrals of \(f\) are defined by the following constructions.

Write \(x \in J\) in the canonical form as
\[x = x_0 + x_1 p + x_2 p^2 + \ldots\]
where \(x_0, x_1, x_2, \ldots\) are digits 0, 1, \ldots, \(p-1\). At most finitely many of these digits are distinct from 0; so, if \(x \neq 0\), let \(x_{\neq 0}\) be the non-vanishing digit of largest suffix \(s\). Firstly put
\[g(0) = 0, \quad g(x) = x_sp^s \quad \text{for} \quad x \neq 0.\]

Secondly write
\[g(n) = x_0 + x_1 p + \ldots + x_{n-1} p^{n-1} \quad (n = 1, 2, 3, \ldots)\]
so that
\[g(n+1) = g(n) \quad \text{for} \quad n > s.\]