[5] R. W. K. Odoni, *On a problem of Narkiewicz*, Journ. Reine Angew. Math. 288 (1976), pp. 160–167.
[6] P. Rémond, *Étude asymptotique de certaines partitions dans certains semi-groupes*, Ann. Scient. École Norm. Sup. 83 (1966), pp. 343–410.
[7] J. Rosiński and J. Śliwa, *The number of factorizations in an algebraic number field*, Bull. Acad. Pol. Sci., sér. math. phys. et astr. 24 (1976), pp. 821–826.

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On the distribution function of certain sequences (mod 1)*

by

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To the memory of P. Turán

§ 1. Introduction.

1. This paper arose from the consideration of the expression

$$R_n(\eta) := R\left(\eta\sqrt{2p(1-p)n}+pn\right)+R\left(\eta\sqrt{2p(1-p)n}-pn\right) \quad (1.1)$$

with

$$0<p<1, \quad \eta>0, \quad n\to\infty,$$

where $R(x)$ denotes generally the fractional part of x, lying in $\langle 0,1)$. The expression (1.1) occurs in the Probability Calculus. Namely, as has been shown by Uspensky [7] and Ostrowski [3], the sum

$$\sum_{\nu=0}^{n}\binom{n}{\nu}p^\nu(1-p)^{n-\nu} \quad \left(|\nu-pn|\leqslant \eta\sqrt{2np(1-p)}\right)$$

can be expressed in the form

$$\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-x^2}dx+e^{-\eta^2}\frac{1-R_n(\eta)}{\sqrt{2\pi p(1-p)n}}+O\left(\frac{1}{n}\right) \quad (n\to\infty)$$

where R_n is given by (1.1).

As a matter of fact a similar formula was first given by Laplace. However, the term $R_n(\eta)$ was missing in Laplace's deduction. The formula as it had been written down by Laplace was repeatedly used until the first quarter of this century. It was therefore of importance, that $R_n(\eta)$ does not tend with $n\to\infty$ to 0 but is *everywhere dense* in the interval between 0 and 2. This was annonced in [3] and proved in [4].

2. Since, however, very often the sequences in such connection are, not only everywhere dense, but also *uniformly distributed* that is have a constant density in every point in the corresponding interval, it appears

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to be worth while investigating whether the sequence $R_n(\eta)$ $(n\to\infty)$ is *uniformly distributed* in the interval $\langle 0,2)$.

It turns out that this is not the case and we succeeded even to obtain explicit expression for the distribution function of the $R_n(\eta)$ $(n\to\infty)$ for *irrational values* of p. This discussion presented peculiar difficulties which are overcome using a theorem about distribution of the expressions $R(\nu\lambda)$ $(\nu\to\infty)$, saying that for any interval of the length $R(n_0\lambda)$ for natural n the modulus of the "error term" is $\leqslant n_0$ ([5]; see also [2]).

Our result is contained in a corresponding result about the more general sequences,

$$R(a\sqrt{\nu}+\nu\lambda)+R(a\sqrt{\nu}-\nu\lambda)\quad(\nu\to\infty) \tag{1.2}$$

with a fixed positive a. Further, the sequence $a\sqrt{\nu}$ $(\nu\to\infty)$ can be replaced in this connection with a more general sequence

$$a_\nu=R(s_\nu)\quad(\nu\to\infty) \tag{1.3}$$

where

$$s_\nu-s_{\nu-1}\downarrow 0,\quad \nu(s_\nu-s_{\nu-1})\to\infty\quad(\nu\to\infty). \tag{1.4}$$

3. For sequences a_ν of this type it follows immediately from a result by L. Fejér, that they are uniformly distributed in $\langle 0,1)$ ([1]; see also [6]).

But, in order to carry through our discussion we have to restrict the sequence s_ν further imposing on it a condition which will be formulated in (2.10) in Section 7. In any case we show that the sequence

$$R(a\nu^\alpha)\quad(\nu\to\infty,\ 0<\alpha<1,\ a>0)$$

can be used as a_ν in our results.

We will therefore consider generally the sequence

$$\tau_\nu:=R(a_\nu+\nu\lambda)+R(a_\nu-\nu\lambda)\quad(\nu\to\infty) \tag{1.5}$$

and prove that, for any irrational λ, the density of this sequence in the interval $0<x\leqslant 1$ has the value x and in the interval $1\leqslant x<2$ the value $2-x$.

4. As to the corresponding expressions in the case of a *rational fraction* λ, we show that then the τ_ν are uniformly distributed if and only if 2λ is an integer.

We prove that the distribution function always exists for rational λ too, and derive explicitly the value of this function in a neighborhood of 1, where it is rather simple. The expression of the distribution function in the whole interval $\langle 0,2)$ can be also derived, but the expression obtained is very complicated and we omit it, as it appears to present little interest.

The sequence (1.5) is of course rather a special one. However, our discussion offers one of the very few examples of nonuniform distribution where the distribution function could be obtained explicitly.

§ 2. Discussion of the a_ν.

5. Consider a sequence of positive d_ν monotonically falling to 0 and such that

$$d_\nu\downarrow 0,\quad \nu d_\nu\to\infty\quad(\nu\to\infty). \tag{2.1}$$

Since, from a ν on, $d_\nu>1/\nu$, it follows that

$$s_n:=\sum_{\nu=1}^{n}d_\nu\to\infty\quad(n\to\infty). \tag{2.2}$$

Putting

$$a_\nu:=R(s_\nu),\quad d_\nu=s_\nu-s_{\nu-1}, \tag{2.3}$$

it follows obviously from (2.2) that the numbers a_ν infinitely often run monotonically through the interval $\langle 0,1)$. Consider the values of ν for which

$$a_{\nu-1}+d_\nu\geqslant 1. \tag{2.3a}$$

Denote these values of ν ordered monotonically by $n_\varkappa$, taking $n_1:=1$:

$$1=n_1<n_2<n_3<\ldots \tag{2.3b}$$

Then, for each $\varkappa$ we have the inequality

$$0\leqslant a_{n_\varkappa}<a_{n_\varkappa+1}<\ldots<a_{n_{\varkappa+1}-1}<1\quad(\varkappa=1,2,\ldots). \tag{2.4}$$

We denote the sequence of indices in (2.4) by $P_\varkappa$ and call it the *$\varkappa$-th run.* As, from a $\varkappa\geqslant\varkappa_0$ on, $d_{n_\varkappa}<1$ and at each run $[s_\nu]$ is increased by 1, we see that for the $\varkappa$th run

$$s_\nu=\varkappa+c+a_\nu,\quad [s_\nu]=\varkappa+c\quad(\nu\in P_\varkappa,\ \varkappa\geqslant\varkappa_0), \tag{2.4a}$$

for a constant c. Put further

$$n_{\varkappa+1}-n_\varkappa=:N_\varkappa,\quad \frac{1}{d_{n_\varkappa}}=:D_\varkappa,\quad n_\varkappa=\sum_{\varkappa=1}^{k-1}N_\varkappa+1. \tag{2.5}$$

6. For a fixed x, $0<x\leqslant 1$, denote by $N_\varkappa(x)$ the number of the a_ν from the $\varkappa$th run, which are $<x$,

$$N_\varkappa(x):=N\,(a_\nu<x,\ \nu\in P_\varkappa). \tag{2.6}$$

(Obviously $N_\varkappa(1)=N_\varkappa$.) Then

$$a_{n_\varkappa+N_\varkappa(x)-1}<x\leqslant a_{n_\varkappa+N_\varkappa(x)}$$

and therefore

$$\sum_{\sigma=n_\varkappa+1}^{n_\varkappa+N_\varkappa-1}d_\sigma<x-a_{n_\varkappa}\leqslant\sum_{\sigma=n_\varkappa+1}^{n_\varkappa+N_\varkappa}d_\sigma.$$

From (2.1) and (2.5) it follows now

$$\big(N_\varkappa(x)-1\big)\,d_{n_{\varkappa+1}} < x-a_{n_\varkappa} \leqslant N_\varkappa(x)\,d_{n_\varkappa}$$

and by (2.5)

(2.7) $$D_\varkappa(x-a_{n_\varkappa}) \leqslant N_\varkappa(x) < D_{\varkappa+1}(x-a_{n_\varkappa})+1.$$

Taking $x=1$, we obtain

(2.8) $$D_{\varkappa+1}(1-a_{n_\varkappa})+1 > N_\varkappa \geqslant D_\varkappa(1-a_{n_\varkappa}).$$

Dividing (2.8) by $D_{\varkappa+1}$ it follows

$$1-a_{n_\varkappa}+d_{n_{\varkappa+1}} > \frac{N_\varkappa}{D_{\varkappa+1}} = \frac{N_\varkappa/n_\varkappa}{1+N_\varkappa/n_\varkappa}\,(n_{\varkappa+1}d_{n_{\varkappa+1}}).$$

Here, the left side expression tends to 1 while by (2.1) $n_{\varkappa+1}d_{n_{\varkappa+1}}$ tends to ∞. It follows $\dfrac{N_\varkappa/n_\varkappa}{1+N_\varkappa/n_\varkappa} \to 0$ and therefore

(2.9) $$N_\varkappa/n_\varkappa \to 0.$$

7. We make now the supplementary hypothesis about (2.1):

(2.10) $$D_{\varkappa+1}/D_\varkappa \to 1.$$

Then, dividing all three terms of (2.8) by $D_\varkappa$ it follows

(2.11) $$N_\varkappa/D_\varkappa \to 1, \qquad N_\varkappa \to \infty.$$

Divide now all three terms of (2.7) by $N_\varkappa$:

$$(x-a_{n_\varkappa})\frac{D_\varkappa}{N_\varkappa} \leqslant \frac{N_\varkappa(x)}{N_\varkappa} \leqslant (x-a_{n_\varkappa})\frac{D_\varkappa}{N_\varkappa}\,\frac{D_{\varkappa+1}}{D_\varkappa}+\frac{1}{N_\varkappa}.$$

Using (2.10) and (2.11) it follows for $\varkappa\to\infty$

(2.12) $$\frac{N_\varkappa(x)}{N_\varkappa} \to x.$$

8. Denote, for a natural n, by $A(n,x)$ the number of all a_ν with $\nu<n$ and $a_\nu<x$:

(2.13) $$A(n,x) := N(a_\nu<x,\ \nu<n).$$

To any n corresponds a k such that $n_k\leqslant n<n_{k+1}$, and here $k\to\infty$ with $n\to\infty$. Thence we have

$$A(n,x) = \sum_{\varkappa=1}^{k-1} N_\varkappa(x)+\theta N_k = A(n_k,x)+\theta N_k,$$

where θ, as well as θ with different indices, denotes from now on a number of modulus $\leqslant 1$, not necessarily the same for different θ in a formula.

Similarly,

$$n = \sum_{\varkappa=1}^{k-1} N_\varkappa+\theta' N_k = n_k+1+\theta' N_k.$$

By Cauchy's convergence theorem it follows from (2.12):

(2.14) $$\frac{\sum_{\varkappa=1}^{k-1} N_\varkappa(x)}{\sum_{\varkappa=1}^{k-1} N_\varkappa} = \frac{A(n_\varkappa,x)}{n_\varkappa-1} \to x.$$

We have now

$$\frac{A(n,x)}{n} = \frac{A(n_k,x)+\theta N_k}{n_k-1+\theta' N_k}.$$

Dividing on the right the numerator and denumerator by $n_\varkappa$ we see that

(2.15) $$\frac{A(n,x)}{n} \to x \qquad (n\to\infty).$$

This follows also, as has been said in Section 3, from a result by L. Féjer.

§ 3. An example.

9. Consider, for $a>0$, $b\geqslant 0$ and an α with $0<\alpha<1$, the sequence

(3.1) $$s_\nu = (a\nu+b)^\alpha \qquad (\nu = 1, 2, \ldots).$$

Here the d_ν are defined by

$$d_\nu = (a\nu+b)^\alpha-(a\nu-a+b)^\alpha.$$

They are >0 and monotonically decreasing.

Writing

$$\frac{d_\nu}{(a\nu)^\alpha} = \left(1+\frac{b}{a\nu}\right)^\alpha-\left(1+\frac{b-a}{a\nu}\right)^\alpha$$

and developing we obtain $a/\nu+O(\nu^{-2})$ and

(3.2) $$d_\nu = \frac{\alpha a^\alpha}{\nu^{1-\alpha}}+O\left(\frac{1}{\nu^{2-\alpha}}\right), \qquad d_\nu\downarrow 0, \qquad \nu d_\nu\to\infty.$$

The conditions (2.1) are satisfied.

10. From (2.4a) it follows now for $\varkappa\to\infty$: $s_{n_\varkappa}\sim\varkappa$, $an_\varkappa+b\sim\varkappa^{1/\alpha}$,

(3.3) $$n_\varkappa \sim \frac{1}{a}\varkappa^{1/\alpha}.$$

From (3.3) we obtain

$$(3.4)\qquad D_\varkappa = 1/d_{n_\varkappa} \sim \frac{n_\varkappa^{1-a}}{a a^a} \sim \frac{1}{aa}\,\varkappa^{(1-a)/a}.$$

We see that our supplementary condition (2.10) is also satisfied and

$$(3.5)\qquad N_\varkappa \sim \frac{1}{aa}\,\varkappa^{(1-a)/a}.$$

Taking in particular $a := \frac{1}{2}$, $b := 0$ we obtain

$$(3.6)\qquad s_\nu = \sqrt{a\nu}, \quad n_\varkappa \sim 2\varkappa^2, \quad N_\varkappa \sim D_\varkappa \sim \frac{2\varkappa}{a} \quad (\varkappa \to \infty).$$

§ 4. Introduction of $A'(x)$, $A''(x)$, $B'(x)$, $B''(x)$.

11. If we use the notation

$$(4.1)\qquad R(\nu\lambda) =: \lambda_\nu$$

the expression (1.5) of τ_ν can be written as

$$(4.2)\qquad \tau_\nu := R(\alpha_\nu + \lambda_\nu) + R(\alpha_\nu - \lambda_\nu).$$

Our aim is to obtain an expression for the number of $\tau_\nu < 2x$ if $\nu < n$:

$$(4.3)\qquad T(x, n) := N(\nu\colon \tau_\nu < 2x,\ \nu < n) \qquad (0 < x \leqslant 1).$$

Observe that $R(\alpha_\nu + \lambda_\nu)$ has one of the values $\alpha_\nu + \lambda_\nu$ or $\alpha_\nu + \lambda_\nu - 1$ which lies in $\langle 0, 1)$ that is $\alpha_\nu + \lambda_\nu$ if $\lambda_\nu < 1 - \alpha_\nu$ and $\alpha_\nu + \lambda_\nu - 1$ if $\lambda_\nu \geqslant 1 - \alpha_\nu$. Similarly $R(\alpha_\nu - \lambda_\nu)$ is $\alpha_\nu - \lambda_\nu$ if $\lambda_\nu \leqslant \alpha_\nu$ and $\alpha_\nu - \lambda_\nu + 1$ if $\lambda_\nu > \alpha_\nu$.

τ_ν can therefore have one of the three values $2\alpha_\nu$, $2\alpha_\nu + 1$, $2\alpha_\nu - 1$ and we have obviously four cases:

(A′) $\alpha_\nu < \lambda_\nu < 1 - \alpha_\nu$, $\tau_\nu = 2\alpha_\nu + 1$,

(B′) $1 - \alpha_\nu \leqslant \lambda_\nu \leqslant \alpha_\nu$, $\tau_\nu = 2\alpha_\nu - 1$,

(A″) $\lambda_\nu > \alpha_\nu$, $\lambda_\nu \geqslant 1 - \alpha_\nu$, $\tau_\nu = 2\alpha_\nu$,

(B″) $\lambda_\nu \leqslant \alpha_\nu$, $\lambda_\nu < 1 - \alpha_\nu$, $\tau_\nu = 2\alpha_\nu$.

12. We will first discuss the distribution numbers corresponding to a *fixed run* $P_\varkappa$ assuming that our ν runs through the whole set $P_\varkappa$. We call then the corresponding distribution numbers for $\alpha_\nu < x$ in the corresponding cases $A'_\varkappa, \ldots, B''_\varkappa$ defining them as [1]

$$(4.4)\qquad A'_\varkappa(x) := N(\nu\colon \alpha_\nu < x,\ \nu \in P_\varkappa,\ \alpha_\nu < \lambda_\nu < 1 - \alpha_\nu),$$

$$(4.5)\qquad A''_\varkappa(x) := N(\nu\colon \alpha_\nu < x,\ \nu \in P_\varkappa,\ \lambda_\nu \geqslant 1 - \alpha_\nu,\ \lambda_\nu > \alpha_\nu),$$

[1] Observe that $B'_\varkappa(x) = 0$ for $x \leqslant \frac{1}{2}$, while $A'_\varkappa(x) = A'_\varkappa(\frac{1}{2})$ is constant for $\frac{1}{2} < x < 1$.

$$(4.6)\qquad B'_\varkappa(x) := N(\nu\colon \alpha_\nu < x,\ \nu \in P_\varkappa,\ 1 - \alpha_\nu \leqslant \lambda_\nu \leqslant \alpha_\nu),$$

$$(4.7)\qquad B''_\varkappa(x) := N(\nu\colon \alpha_\nu < x,\ \nu \in P_\varkappa,\ \lambda_\nu \leqslant \alpha_\nu,\ \lambda_\nu < 1 - \alpha_\nu).$$

Denote by $T_\varkappa(x)$ the number of ν from $P_\varkappa$ for which $\tau_\nu < 2x$,

$$(4.8)\qquad T_\varkappa(x) := N(\nu\colon \tau_\nu < 2x,\ \nu \in P_\varkappa).$$

Then I say that

$$(4.9)\qquad T_\varkappa(x) = A'_\varkappa(x - \tfrac{1}{2}) + B'_\varkappa(x + \tfrac{1}{2}) + A''(x) + B''(x).$$

Indeed, in the cases (A′) and (B′) if $\tau_\nu = 2\alpha_\nu \pm 1 < 2x$ it follows that $\alpha_\nu < x \mp \frac{1}{2}$, while in the cases (A″) and (B″), $2\alpha_\nu < 2x$ and therefore $\alpha_\nu < x$.

On the other hand, denoting by $N_\varkappa(x)$ the number of all $\alpha_\nu < x$ from the $\varkappa$th run, as in (2.6),

$$(4.10)\qquad N_\varkappa(x) := N(\nu\colon \alpha_\nu < x,\ \nu \in P_\varkappa),$$

we obtain easily, according as $x \leqslant \frac{1}{2}$ or $x > \frac{1}{2}$:

$$(4.11)\qquad N_\varkappa(x) = A'_\varkappa(x) + A''_\varkappa(x) + B''_\varkappa(x) \qquad (x \leqslant \tfrac{1}{2}),$$

$$(4.12)\qquad N_\varkappa(x) = A'_\varkappa(\tfrac{1}{2}) + B'_\varkappa(x) + A''_\varkappa(x) + B''_\varkappa(x) \qquad (x > \tfrac{1}{2}).$$

Our next aim is now to obtain suitable expressions for $A'_\varkappa(x)$ and $B'_\varkappa(x)$, keeping $\varkappa$ fixed.

§ 5. Use of the intervals J_σ.

13. We assume from now an until § 8 that λ is *irrational*. Assume a natural number s and consider the points of $\langle 0, 1)$:

$$(5.1)\qquad R(\sigma\lambda) = \lambda_\sigma \qquad (\sigma = 0, 1, 2, \ldots, s).$$

If we order these points monotonically,

$$(5.2)\qquad p_0 = 0 < p_1 < \ldots < p_s < 1,$$

they decompose the interval $\langle 0, 1)$ into $s+1$ intervals

$$(5.3)\qquad J_\sigma := \langle p_{\sigma-1}, p_\sigma) \quad (\sigma = 1, \ldots, s), \qquad J_{s+1} = \langle p_s, 1).$$

Then the length of J_σ is

$$(5.4)\qquad |J_\sigma| := \Delta_\sigma = p_\sigma - p_{\sigma-1} \quad (1 \leqslant \sigma \leqslant s), \qquad \Delta_{s+1} := 1 - \sigma_s.$$

Put

$$(5.5)\qquad \operatorname*{Max}_\sigma \Delta_\sigma =: l.$$

Denote now by $N_\varkappa^{(\sigma)}$ the number of the α_ν from the $\varkappa$th run, which lie in J_σ:

$$(5.6)\qquad N_\varkappa^{(\sigma)} := N(\nu\colon \alpha_\nu \in J_\sigma,\ \nu \in P_\varkappa).$$

Then, for two consecutive of these α_ν from J_σ the difference $\alpha_{\nu+1}-\alpha_\nu$ lies between $d_{n_\varkappa}$ and $d_{n_{\varkappa+1}}$. Since the number of subintervals of J_σ obtained by introducing the corresponding α_ν lies between $N_\varkappa^{(\sigma)}+1$ and $N_\varkappa^{(\sigma)}-1$ we obtain the inequality

(5.7) $$(1+N_\varkappa^{(\sigma)})d_{n_\varkappa} \geqslant \Delta_\sigma \geqslant (N_\varkappa^{(\sigma)}-1)d_{n_{\varkappa+1}} \quad (\sigma = 1, \ldots, s+1).$$

Using (2.5) it follows

(5.8) $$D_\varkappa \Delta_\sigma - 1 \leqslant N^{(\sigma)} \leqslant D_{\varkappa+1}\Delta_\sigma + 1 \quad (\sigma = 1, \ldots, s+1).$$

By (2.10) we can write

$$D_{\varkappa+1}/D_\varkappa = 1-\varepsilon_\varkappa, \quad \varepsilon_\varkappa \geqslant 0, \quad \varepsilon_\varkappa \to 0.$$

There exists therefore for each $\varkappa \geqslant 1$:

(5.9) $$\eta_\varkappa := \operatorname*{Max}_{\nu \geqslant \varkappa-1} \varepsilon_\nu \to 0 \quad (\varkappa \to \infty).$$

We assume $\varkappa$ so large that

(5.9a) $$\eta_\varkappa < 1/20.$$

We can then write the expression on the right of (5.8) as $D_\varkappa \Delta_\sigma(1-\varepsilon_\varkappa)+1$ and obtain finally for $N_\varkappa^{(\sigma)}$ the formula:

(5.10) $$N_\varkappa^{(\sigma)} = D_\varkappa \Delta_\sigma (1+\theta\eta_\varkappa)+\theta.$$

14. We will now have to use the following lemma ([5]):

LEMMA 1. *Consider subinterval, J, of the interval $\langle 0, 1)$, closed from the left, whose length is $R(n_0\lambda)$ for an integer $n_0 \geqslant 0$. Denote for a natural integer n by $N(J, n)$ the number of $\lambda_\nu = R(\nu\lambda)$ with $\nu < n$, lying in J.*

Then [2]

(5.11) $$N(J, n) = R(n_0\lambda)n + \theta n_0, \quad |\theta| < 1.$$

From this lemma it follows:

LEMMA 2. *Under the assumptions of Lemma 1 assume N a natural number and denote by $N(J, n, N)$ the number of the λ_ν lying in J with $n \leqslant \nu < n+N$. Then, applying (5.11) to the interval (modulo 1) $J+n\lambda$:*

(5.12) $$N(J, n, N) - |J|N = \theta n_0, \quad |\theta| < 1.$$

15. If we apply (5.12) to the interval $\langle p_\tau, 1-p_\tau)$, $p_\tau = R(\sigma'\lambda) < \frac{1}{2}$, the length of this interval is

$$1-2p_\tau = R(-2\sigma'\lambda),$$

our $n_0 = 2\sigma'$ will be $\leqslant 2s$, and we obtain for the number of corresponding cases for α_ν from J_σ:

(5.13) $$N(\nu:\ p_\tau \leqslant \lambda_\nu < 1-p_\tau,\ \alpha_\nu \in J_\sigma) = N^{(\sigma)}(1-2p_\tau)+2\theta s \quad (p_\tau < \tfrac{1}{2}).$$

[2] J can be an interval modulo 1, consisting of two parts adjoining 1 and 0.

Correspondingly, if $p_\tau = R(\sigma''\lambda) > \frac{1}{2}$, the length of the interval $\langle 1-p_\tau, p_\tau)$ is

$$2p_\tau - 1 = R(2\sigma''\lambda),$$

our $|n_0| = 2\sigma''$ is again $\leqslant 2s$ and we obtain for the number of corresponding cases with α_ν from J_σ:

(5.14) $$N(\nu:\ 1-p_\tau \leqslant \lambda_\nu < p_\tau,\ \alpha_\nu \in J_\sigma) = N^{(\sigma)}(2p_\tau-1)+2\theta s \quad (p_\tau > \tfrac{1}{2}).$$

§ 6. Expressions for $A'_\varkappa(x)$ and $B'_\varkappa(x)$.

16. We now define the numbers corresponding to $A'_\varkappa$ and $B'_\varkappa$, if we replace the condition $\alpha_\nu < x$ by the condition $\alpha_\nu \in J_\sigma$:

(6.1) $$\bar{A}_\sigma = N(\nu:\ \alpha_\nu < \lambda_\nu < 1-\alpha_\nu,\ \alpha_\nu \in J_\sigma,\ \nu \in P_\varkappa) \quad (p_\sigma < \tfrac{1}{2}),$$

(6.2) $$\bar{B}_\sigma = N(\nu:\ 1-\alpha_\nu \leqslant \lambda_\nu \leqslant \alpha_\nu,\ \alpha_\nu \in J_\sigma,\ \nu \in P_\varkappa) \quad (p_{\sigma-1} > \tfrac{1}{2}).$$

The number $\bar{A}_\sigma$ is obviously increased if we replace α_ν in the condition for the γ_ν intervals by $p_{\sigma-1}$, and decreased, if α_ν is replaced by p_σ. We obtain for the ν from p_σ

$$N(\nu: p_\sigma < \lambda_\nu < 1-p_\sigma,\ \alpha_\nu \in J_\sigma) \leqslant \bar{A}_\sigma \leqslant N(\nu: p_{\sigma-1} < \lambda_\nu < 1-p_{\sigma-1}, \alpha_\nu \in J_\sigma).$$

Applying now (5.13) with $\tau = \sigma$ and $\tau = \sigma-1$, we obtain

$$N^{(\sigma)}(1-2p_\sigma)-2s \leqslant \bar{A}_\sigma \leqslant N^{(\sigma)}(1-2p_{\sigma-1})+2s$$

or using (5.10)

$$[D_\varkappa \Delta_\sigma(1-\eta_\varkappa)-1](1-2p_\sigma)-2s \leqslant \bar{A}_\sigma \leqslant [D_\varkappa \Delta_\sigma(1+\eta_\varkappa)+1](1-2p_{\sigma-1})+2s$$

or

(6.3) $$D_\varkappa \Delta_\sigma(1-2p_\sigma)-\eta_\varkappa D_\varkappa \Delta_\sigma - 1 - 2s \leqslant \bar{A}_\sigma \leqslant D_\varkappa \Delta_\sigma(1-2p_{\sigma-1}) + \\ + \eta_\varkappa D_\varkappa \Delta_\sigma + 1 + 2s.$$

But, as $1-2x$ is monotonically decreasing, we have

$$(1-2p_\sigma)\Delta_\sigma < \int_{p_{\sigma-1}}^{p_\sigma}(1-2x)\,dx < (1-2p_{\sigma-1})\Delta_\sigma,$$

$$(1-2p_\sigma)\Delta_\sigma = \int_{p_{\sigma-1}}^{p_\sigma}(1-2x)\,dx + 2\theta\Delta_\sigma^2,$$

$$(1-2p_{\sigma-1})\Delta_\sigma = \int_{p_{\sigma-1}}^{p_\sigma}(1-2x)\,dx + 2\theta\Delta_\sigma^2.$$

Introducing this into (6.3) we finally obtain, as $\Delta_\sigma \leqslant l$ by (5.5)

(6.4) $$\bar{A}_\sigma = D_\varkappa \int_{p_{\sigma-1}}^{p_\sigma}(1-2x)\,dx + \theta(2l+\eta_\varkappa)\Delta_\sigma D_\varkappa + \theta(s+1).$$

17. In the case of $\bar{B}_\sigma$ we proceed in exactly the same way, the only difference beeing that $1-2p_\sigma$, $1-2p_{\sigma-1}$ are to be replaced with $2p_{\sigma-1}-1$, $2p_\sigma-1$. In this way we obtain immediately

$$D_\varkappa \Delta_\sigma(2p_{\sigma-1}-1)-\eta_\varkappa D_\varkappa \Delta_\sigma-\theta(s+1) \leqslant \bar{B}_\sigma \leqslant D_\varkappa \Delta_\sigma(2p_\sigma-1)+ +\eta_\varkappa D_\varkappa \Delta_\sigma+\theta(s+1).$$

Exactly as above we see that both $\Delta_\sigma(2p_{\sigma-1}-1)$ and $\Delta_\sigma(2p_\sigma-1)$ can be written as

$$\int_{p_{\sigma-1}}^{p_\sigma}(2x-1)\,dx+2\theta\Delta_\sigma^2,$$

and obtain finally

$$\bar{B}_\sigma = D_\varkappa \int_{p_{\sigma-1}}^{p_\sigma}(2x-1)\,dx+\theta(2l+\eta_\varkappa)\Delta_\sigma D_\varkappa+\theta(s+1). \tag{6.5}$$

18. It is now easy to obtain the asymptotic relations for $A'_\varkappa(x)$ and $B'_\varkappa(x)$.

Assume $x \leqslant \frac{1}{2}$ and

$$p_t \leqslant x < p_{t+1}, \quad p_u \leqslant \tfrac{1}{2} < p_{u+1}. \tag{6.6}$$

Then $p_t \leqslant p_u \leqslant \frac{1}{2}$ and for all intervals J_σ ($\sigma = 1, \ldots, t$), we can use the formula (6.4), while the number of indices ν between p_t and x corresponding to the case A' is, by (5.6), $\leqslant N_\varkappa^{(t+1)}$. We obtain therefore

$$A'_\varkappa(x) = \sum_{\sigma=1}^{t}\bar{A}_\sigma+\theta N_\varkappa^{(t+1)},$$

or using (6.4)

$$A'_\varkappa(x) = D_\varkappa\int_0^{p_t}(1-2x)\,dx+\theta(2l+\eta_\varkappa)D_\varkappa p_t+\theta(s+1)^2.$$

Dividing both sides by $N_\varkappa$ we obtain further, as $p_t \leqslant \frac{1}{2}$,

$$\frac{A'_\varkappa(x)}{N_\varkappa} = \frac{D_\varkappa}{N_\varkappa}\Big[\int_0^{p_t}(1-2x)\,dx+\theta(2l+\eta_\varkappa)\Big]+\frac{\theta(s+1)^2}{N_\varkappa}.$$

For $\varkappa \to \infty$ and a constant s it follows:

$$-2l \leqslant \overline{\underline{\lim}}\Big(\frac{A'_\varkappa(x)}{N_\varkappa}-\int_0^{p_t}(1-2x)\,dx\Big) \leqslant 2l \quad (x \leqslant \tfrac{1}{2}).$$

On the other hand

$$\int_{p_t}^{x}(1-2x)\,dx \leqslant x-p_t < l,$$

and we finally obtain

$$-2l \leqslant \overline{\underline{\lim}}\Big(\frac{A_\varkappa(x)}{N_\varkappa}-\int_0^{x}(1-2x)\,dx\Big) \leqslant 2l \quad (x \leqslant \tfrac{1}{2}). \tag{6.7}$$

19. To discuss $B'_\varkappa(x)$, assume $x > \frac{1}{2}$ and again (6.6). Then obviously $p_{t+1} \geqslant p_{u+1}$, $t \geqslant u$ and the intervals J_σ from $(\frac{1}{2}, x)$, for which (6.5) holds, correspond to σ with $u+2 \leqslant \sigma \leqslant t$, while the parts of $B'_\varkappa$ which correspond to a part of J_u and to a part of J_{t+1} must be estimated by $N^{(u+1)}$ and $N^{(t+1)}$. Then we have by (6.5) and (5.10)

$$B'_\varkappa(x) = \sum_{\sigma=u+2}^{t}\bar{B}_\sigma+\theta(N^{(u+1)}+N^{(t+1)})$$
$$= D_\varkappa\Big[\int_{p_{u+1}}^{p_{t+1}}(2x-1)\,dx+\theta(2l+\eta_\varkappa)\Big]+\theta(s+1)^2+\theta(3lD_\varkappa+2)$$

where the integral must be deleted if it is < 0.

Further,

$$\Big|\int_x^{p_{t+1}}(2x-1)\,dx\Big| \wedge \Big|\int_{1/2}^{u+1}(2x-1)\,dx\Big| \leqslant l$$

and we can therefore rewrite our formula as

$$B'_\varkappa(x) = D_\varkappa\Big(\int_{1/2}^{x}(2x-1)\,dx+\theta(6l+\eta_\varkappa)\Big)+2\theta\big((s+1)^2+2\big) \quad (x > \tfrac{1}{2}). \tag{6.8}$$

Observe that this formula also holds if $u = t$.

Dividing (6.8) on both side by $N_\varkappa$, we obtain

$$\frac{B'_\varkappa(x)}{N_\varkappa} = \frac{D_\varkappa}{N_\varkappa}\Big(\int_{1/2}^{x}(2x-1)\,dx+\theta(6l+\eta_\varkappa)\Big)+2\theta\frac{(s+1)^2+2}{N_\varkappa}.$$

With $\varkappa \to \infty$ it follows now for a fixed s

$$-6l \leqslant \overline{\underline{\lim}}\Big(\frac{B'_\varkappa(x)}{N_\varkappa}-\int_{1/2}^{x}(2x-1)\,dx\Big) \leqslant 6l \tag{6.9}$$

and now (6.7) and (6.8) become with $s \to \infty$, $l \to 0$:

$$\frac{A'_\varkappa(x)}{N_\varkappa} \to \int_0^x(1-2x)\,dx = x-x^2 \quad (x \leqslant \tfrac{1}{2}), \tag{6.10}$$

$$\frac{B'_\varkappa(x)}{N_\varkappa} \to \int_{1/2}^x(2x-1)\,dx = (x-\tfrac{1}{2})^2 \quad (x > \tfrac{1}{2}). \tag{6.11}$$

In particular, with $x = \frac{1}{2}$ and $x = 1$ we have

(6.12) $$\frac{A'_\varkappa(\frac{1}{2})}{N_\varkappa} \to \int_0^{1/2} (1-2x)\,dx = \tfrac{1}{4},$$

(6.13) $$\frac{B'_\varkappa(1)}{N_\varkappa} := \frac{B'_\varkappa(1-0)}{N_\varkappa} \to \tfrac{1}{4}.$$

§ 7. Density values for irrational λ.

20. Rewriting (4.11) and (4.12) as

(7.1) $$A''(x)+B''(x) = N_\varkappa(x) - A'_\varkappa(x) \quad (x \leqslant \tfrac{1}{2}),$$

(7.2) $$A''(x)+B''(x) = N_\varkappa(x) - B'_\varkappa(x) - A'_\varkappa(\tfrac{1}{2}) \quad (\tfrac{1}{2} < x < 1)$$

and using (2.12), (6.10)–(6.12) we obtain

(7.3) $$\frac{A''(x)+B''(x)}{N_\varkappa} \to x - \int_0^x (1-2x)\,dx = x^2 \quad (x \leqslant \tfrac{1}{2}),$$

(7.4) $$\frac{A''(x)+B''(x)}{N_\varkappa} \to x - \int_{1/2}^x (2x-1)\,dx - \tfrac{1}{4} = 2x - x^2 - \tfrac{1}{2}$$

$$(\tfrac{1}{2} < x < 1).$$

Resuming now the formula (4.9) we see that it becomes for $x \leqslant \frac{1}{2}$:

(7.5) $$T_\varkappa(x) = B'_\varkappa(x+\tfrac{1}{2}) + A''_\varkappa(x) + B''_\varkappa(x).$$

Dividing this by $N_\varkappa$ and using (6.11) and (7.3) it follows

(7.6) $$T_\varkappa(x)/N_\varkappa \to 2x^2 \quad (\varkappa \to \infty,\ x \leqslant \tfrac{1}{2}).$$

As to the case $x > \frac{1}{2}$, (4.9) becomes in this case by (6.10), (6.13) and (7.4)

(7.7) $$T_\varkappa(x) = A'_\varkappa(x-\tfrac{1}{2}) + B'_\varkappa(1) + A''_\varkappa(x) + B''_\varkappa(x) \quad (x > \tfrac{1}{2}),$$

and dividing this by $N_\varkappa$ we obtain with $\varkappa \to \infty$:

(7.8) $$T_\varkappa(x)/N_\varkappa \to 1 - 2(1-x)^2 \quad (\varkappa \to \infty,\ x > \tfrac{1}{2}).$$

If we introduce now

(7.9) $$\Delta(x) := \begin{cases} x^2/2 & (x \leqslant 1), \\ 1 - 2(1-x/2)^2 & (1 < x < 2) \end{cases}$$

we can write the formulas (7.7) and (7.8) as

(7.10) $$T_\varkappa(x)/N_\varkappa \to \Delta(2x) \quad (\varkappa \to \infty,\ 0 < x < 1).$$

21. Returning now to the definitions (4.3) and (4.8) of $T(x, n)$ and $T_\varkappa(x)$, we can write, taking $n = n_k$ and using (2.5):

$$T(x, n_k) = \sum_{\varkappa=1}^{k-1} T_\varkappa(x), \qquad n_\varkappa = \sum_{\varkappa=1}^{k-1} N_\varkappa + 1.$$

But then it follows from (7.10), by Cauchy's convergence theorem, that

(7.11) $$T(x, n_k)/n_k \to \Delta(2x) \quad (k \to \infty).$$

For a general natural n, we can write, with a convenient k,

$$n_k \leqslant n < n_{k+1} = n_k + N_k.$$

Then obviously $T(x, n) = T(x, n_k) + \theta N_k$, $n = n_k + \theta_1 N_k$,

$$\frac{T(x)}{n} = \frac{T(x, n_k)/n_k + \theta N_k/n_k}{1 + \theta_1 N_k/n_k}.$$

It follows now, using (2.9),

(7.12) $$\frac{T(x, n)}{n} \to \Delta(2x) \quad (k \to \infty).$$

In virtue of the definition (4.3) of $T(x, n)$ we have now proved that the *distribution function of the* τ_ν *in the interval* $(0, 2)$ *is given by* $\Delta(x)$ *in* (7.9).

Differentiating, we obtain for *the* corresponding *density* the value x in $(0, 1\rangle$ and $2-x$ in $(1, 2)$.

We remind the reader that this result has been obtained assuming λ to be *irrational*.

§ 8. An auxiliary problem.

22. Our problem, in the case of a rational λ, can be reduced to the following special problem which we solve in this § 8.

Assume a_ν as in § 2. Assume a fixed r, $0 \leqslant r < 1$, and put

(8.1) $$\bar{\tau}_\nu := R(a_\nu + r) + R(a_\nu - r).$$

What value has the distribution function of the $\bar{\tau}_\nu$ *in* $\langle 0, 2)$?

Put, assuming first $r > 0$,

(8.2) $$r' := \mathrm{Min}(r, 1-r), \qquad r'' := \mathrm{Max}(r, 1-r).$$

Since $R(a_\nu - r) = R(a_\nu + 1 - r)$, we can write

(8.3) $$\bar{\tau}_\nu = R(a_\nu + r') + R(a_\nu + r'').$$

If we assume that the positive ν satisfies

(8.4) $$\bar{\tau}_\nu < 2x, \qquad \nu < n,$$

we have to consider three cases, according as a_ν lies in the intervals (A') $\langle r', r'')$ or (B') $a_\nu < r'$ or (C') $a_\nu \geqslant r''$. Accordingly we define

(8.5) $$A'(x) := N(\nu\colon a_\nu < x, \nu < n, \; r' \leqslant a_\nu < r''),$$

(8.6) $$B'(x) := N(\nu\colon a_\nu < x, \nu < n, \; a_\nu < r'),$$

(8.7) $$C'(x) := N(\nu\colon a_\nu < x, \nu < n, \; a_\nu \geqslant r'').$$

Obviously, if $r' = \frac{1}{2} = r''$, $A'(x) = 0$.

23. As to $A'(x)$ we have, since the distribution of a_ν is uniform, for a_ν the interval $r' \leqslant a_\nu < \mathrm{Min}(x, r'')$ and therefore for $A'(x)/n$ the following limits with $n \to \infty : 0$ if $x \leqslant r'$; $x - r'$ if $r' < x < r''$, and $r'' - r'$ if $x \geqslant r''$.

Introducing the general notation

$$(D)_+ := \begin{cases} D & \text{if} \quad D \geqslant 0, \\ 0 & \text{if} \quad D < 0 \end{cases}$$

we immediately verify that

(8.8) $$A'(x)/n \to (x - r')_+ - (x - r'')_+ .$$

In the case of $B'(x)$, a_ν runs through the interval $\langle 0, \mathrm{Min}(x, r'))$ and it follows

(8.9) $$B'(x)/n \to \mathrm{Min}(x, r').$$

Finally in the case of $C'(x)$, a_ν runs through the interval $\langle r'', x)$ if $x > r''$ and then $C'(x)/n \to x - r''$, while for $x < r''$, $C'(x)$ vanishes; we can write

(8.10) $$C'(x)/n \to (x - r'')_+ .$$

We note in particular that with $n \to \infty$:

(8.11) $$B'(x - \tfrac{1}{2})/n \to \mathrm{Min}(x - \tfrac{1}{2}, r') \qquad (x \geqslant \tfrac{1}{2}),$$

(8.12) $$C'(x + \tfrac{1}{2})/n \to (x + \tfrac{1}{2} - r'')_+ = (x - \tfrac{1}{2} + r')_+ \qquad (x \leqslant \tfrac{1}{2}),$$

(8.13) $$C'(1)/n \to r'.$$

24. We can now easily find the asymptotic value of

(8.14) $$T^*(x, n) := N(\nu\colon \nu < n, \; \bar{\tau}_\nu < 2x).$$

The distribution function of the $\bar{\tau}_\nu$ is given by

$$\lim_{n\to\infty} T^*(x/2, n)/n.$$

Observe that if $\bar{\tau}_\nu < 2x$ then in the case (A'), a_ν lies in $\langle r', r'')$,

$$a_\nu + r' < 1, \quad a_\nu + r'' \geqslant 1, \quad \bar{\tau}_\nu = 2a_\nu, \quad a_\nu < x;$$

in the case (B'):

$$a_\nu < r', \quad a_\nu + r' < 2r' \leqslant 1, \quad a_\nu + r'' < 1, \quad \bar{\tau}_\nu = 2a_\nu + 1, \quad a_\nu < x - \tfrac{1}{2};$$

finally in the case (C'):

$$a_\nu \geqslant r'', \quad a_\nu + r' \geqslant 1, \quad a_\nu + r'' \geqslant 2r'' \geqslant 1, \quad \bar{\tau}_\nu = 2a_\nu - 1, \quad a_\nu < x + \tfrac{1}{2}.$$

Therefore obviously

(8.15) $$T^*(x, n) = A'(x) + B'(x - \tfrac{1}{2}) + C'(x + \tfrac{1}{2}).$$

However, $B'(x - \frac{1}{2})$ vanishes if $x < \frac{1}{2}$, while $C'(x + \frac{1}{2})$ has to be replaced with $C'(1)$ if $x \geqslant \frac{1}{2}$. Therefore we can write in particular:

(8.16) $$T^*(x, n) = A'(x) + C'(x + \tfrac{1}{2}) \qquad (x \leqslant \tfrac{1}{2}),$$

(8.17) $$T^*(x, n) = A'(x) + B'(x - \tfrac{1}{2}) + C'(1) \qquad (x \geqslant \tfrac{1}{2}).$$

Thence, using (8.8), (8.12), (8.11) and (8.13) we obtain finally

(8.18) $$T^*(x, n)/n \to (x - r')_+ + (x - \tfrac{1}{2} + r') \qquad (x \leqslant \tfrac{1}{2}),$$

(8.19) $$T^*(x, n)/n \to x - (x - r'')_+ + \mathrm{Min}(r', x - \tfrac{1}{2}) \qquad (x \geqslant \tfrac{1}{2}).$$

The formulas (8.18) and (8.19) remain true for $r = r' = 0$, $r'' = 1$, as then $\tau_\nu = 2R(a_\nu)$, and the a_ν are uniformly distributed.

The corresponding distribution functions are now:

(8.20) $$\lim_{n\to\infty} T^*\left(\frac{x}{2}, n\right)/n = \left(\frac{x}{2} - r'\right)_+ + \left(\frac{x-1}{2} + r'\right)_+ \qquad (x \leqslant 1),$$

(8.21) $$\lim_{n\to\infty} T^*\left(\frac{x}{2}, n\right)/n = \frac{x}{2} - \left(\frac{x}{2} - r''\right)_+ + \mathrm{Min}\left(\frac{x-1}{2}, r'\right)$$
$$(1 < x < 2).$$

§ 9. Distribution of the τ_ν for rational λ.

25. Obviously, for $\lambda = 0$, as the a_ν are uniformly distributed in $\langle 0, 1)$, the values of $\tau_\nu = 2R(a_\nu)$ are uniformly distributed in $\langle 0, 2)$.

In the case $\lambda = \frac{1}{2}$ we have, for *even* ν, $\tau_\nu = 2R(a_\nu)$ uniformly distributed in $\langle 0, 2)$, while, for *odd* values of ν,

$$\tau_\nu = R(a_\nu + \tfrac{1}{2}) + R(a_\nu - \tfrac{1}{2}) = 2R(a_\nu + \tfrac{1}{2})$$

are again uniformly distributed in $\langle 0, 2)$. We have therefore for the complete sequence of the τ_ν again uniform distribution in $\langle 0, 2)$.

We see that the τ_ν are uniformly distributed in $\langle 0, 2)$ if 2λ is an integer.

We are now going to show that *for any rational λ for which 2λ is not an integer the distribution of the τ_ν in $\langle 0, 2)$ is not uniform.*

If we write, using (4.3),

(9.1) $$T(x/2, n) := N(\nu\colon \tau_\nu < x, \nu < n) \qquad (0 < x < 2)$$

then the *distribution function of the* τ_ν,

$$g(x) = \lim_{n\to\infty} T(x/2, n)/n, \tag{9.2}$$

should in the case of uniform distribution, exist and have the value $x/2$. It is therefore, in order to prove our assertion, sufficient to show that, in a certain neighborhood of $x=1$, $g(x)$ exists but has a value different from $x/2$, save for $x=1$.

26. Assume now

$$\lambda = q/s, \quad 0<q<s, \quad (q,s)=1, \tag{9.3}$$

for an integer $s>2$ and an integer q satisfying (9.3). Choose an a, among one of the integers $0,1,2,\ldots,s-1$ and consider the values of ν in the arithmetical progression

$$\nu = \mu s + a \quad (\mu = 1,2,\ldots). \tag{9.4}$$

For such values of ν, we have

$$aq \equiv b_a \pmod s, \quad r_a := b_a/s \tag{9.5}$$

where b_a is one of the numbers $0,1,\ldots,s-1$, and r_a assumes exactly one each of the values σ/s, $\sigma=0,1,\ldots,s-1$. Then our τ_ν becomes

$$R(\alpha_{\mu s+a}+r_a)+R(\alpha_{\mu s+a}-r_a)$$

or, putting $\beta_\mu^{(a)} := \alpha_{\mu s+a}$,

$$\tau_\nu = \tau_\mu^{(a)} := R(\beta_\mu^{(a)}+r_a)+R(\beta_\mu^{(a)}-r_a).$$

In this way the whole sequence of the α_ν is decomposed into s partial sequences which correspond to the values of ν running through s arithmetical progressions with the difference s. The sequences of ν corresponding to these progressions satisfy the conditions of § 8 and have therefore distribution functions in $\langle 0,2)$.

It follows that the complete sequence of α_ν has a distribution function in $\langle 0,2)$, equal to the arithmetical mean of the distribution functions belonging to our partial sequences.

27. If we define

$$T^*_{r_a}(x,m) := N(\mu: \mu<m, \beta_\mu^{(a)}<x),$$

we obtain, applying (8.20) and (8.21), (we have then to replace α_ν with $\beta_\mu^{(a)}$, r with r_a and n with m):

$$T^*_{r_a}(x/2, m)/m \to \Gamma_{r_a}(x) \quad (m\to\infty), \tag{9.6}$$

putting

$$\Gamma_r(x) := \left(\frac{x}{2}-r'\right)_+ + \left(\frac{x-1}{2}+r'\right)_+ \quad (x\leqslant 1), \tag{9.7}$$

$$\Gamma_r(x) := \frac{x}{2} - \left(\frac{x}{2}-r''\right)_+ + \operatorname{Min}\left(\frac{x-1}{2}, r'\right) \quad (1<x<2). \tag{9.8}$$

For $\nu<n$ it follows from (9.4) that

$$\mu < \frac{n}{s}-\frac{a}{s} \sim \frac{n}{s} \quad (n\to\infty).$$

Replacing in (9.6) m with $n/s-a/s$ we obtain now

$$sT^*_{r_a}\left(\frac{x}{2}, \frac{n-a}{s}\right)/n \to \Gamma_{r_a}(x) \quad (x\leqslant 1). \tag{9.9}$$

28. If we consider all positive $\nu<n$ for which $\tau_\nu<x$, these values are distributed among the arithmetical progressions (9.4) corresponding to all admissible values of a. We obtain therefore

$$T\left(\frac{x}{2}, n\right) = \sum_{a=0}^{s-1} T^*_{r_a}\left(\frac{x}{2}, \frac{n-a}{s}\right) \tag{9.10}$$

and by (9.9)

$$T\left(\frac{x}{2}, n\right)/n \to \frac{1}{s}\sum_{a=0}^{s-1}\Gamma_{r_a}(x).$$

The r_a runs here through the set of fractions

$$\{\sigma/s \ (\sigma=0,1,\ldots,s-1)\}. \tag{9.11}$$

We have finally for $g(x)$ in (9.2)

$$g(x) = \frac{1}{s}\sum_r \Gamma_r(x) \tag{9.12}$$

where r runs through (9.11).

29. In order to evaluate further the right hand expression in (9.12) observe that for any function of r, defined on the set (9.11), the following formula holds in which r' is defined by (8.2):

$$\sum_r f(r') = f(0) + 2\sum_{0<r<1/2} f(r) + \delta f(\tfrac{1}{2}) \tag{9.13}$$

where δ is defined by

$$\delta = \begin{cases} 0 & s \text{ odd,} \\ 1 & s \text{ even.} \end{cases} \tag{9.14}$$

Indeed, for all σ from the open interval $(0, s/2)$ we have by (8.2), $r' = r$, while, if $s/2 < \sigma < 1$, $r' = 1 - r$ runs again through the set of the values of r between 0 and $\frac{1}{2}$.

Applying (9.13) we obtain in the sums (9.7) and (9.8)

$$\sum_r \left(\frac{x}{2} - r'\right)_+ = \frac{x}{2} + 2 \sum_{0<\sigma<s/2} \left(\frac{x}{2} - \frac{\sigma}{s}\right)_+ \quad (x \leqslant 1), \tag{9.15}$$

$$\sum_r \left(\frac{x-1}{2} + r'\right)_+ = 2 \sum_{0<\sigma<s/2} \left(\frac{\sigma}{s} - \frac{1-x}{2}\right)_+ + \delta \frac{x}{2} \quad (x \leqslant 1), \tag{9.16}$$

$$\sum_r \left(\frac{x}{2} - r''\right)_+ = \sum_r \left(\frac{x}{2} + r' - 1\right)_+ \tag{9.17}$$

$$= 2 \sum_{0<\sigma<s/2} \left(\frac{x}{2} + \frac{\sigma}{s} - 1\right)_+ + \delta \frac{x-1}{2} \quad (1 < x < 2).$$

30. We are going now to compute the values of (9.15)-(9.17) in the nterval

$$1 - 1/s \leqslant x \leqslant 1 + 1/s. \tag{9.18}$$

We can write $x = 1 + \theta/s$, $-1 \leqslant \theta \leqslant 1$. On the other hand, putting

$$t := \frac{s-1-\delta}{2}, \tag{9.19}$$

t is the greatest integer $< s/2$, and the upper limit of σ in the right hand sum in (9.15) is obviously t.

For our x and σ we have

$$\frac{x}{2} - \frac{\sigma}{s} \geqslant \frac{1}{2} - \frac{\theta}{2s} - \frac{s-1-\delta}{2s} = \frac{1+\delta}{2s} - \frac{\theta}{2s} \geqslant 0.$$

The right hand sum in (9.15) becomes therefore

$$\sum_{1\leqslant\sigma\leqslant t} \left(\frac{x}{2} - \frac{\sigma}{s}\right) = \frac{s-1-\delta}{4}\left(x - \frac{s+1-\delta}{2s}\right).$$

We obtain in (9.15)

$$\sum_r \left(\frac{x}{2} - r'\right)_+ = (s-\delta)\frac{x}{2} - \frac{(s-\delta)^2 - 1}{4s} \tag{9.20}$$

$$= (s-\delta)\frac{x}{2} - \frac{s-2\delta}{4} + \frac{1-\delta}{4s}.$$

31. As to the right hand sum in (9.16), it becomes

$$\sum_{1\leqslant\sigma\leqslant t} \left(\frac{\sigma}{s} - \frac{1-x}{2}\right)_+$$

and is, since $\dfrac{\sigma}{s} - \dfrac{1-x}{2} \geqslant \dfrac{1}{s} - \dfrac{\theta}{2s} > 0$,

$$= \sum_{1\leqslant\sigma\leqslant t} \left(\frac{\sigma}{s} - \frac{1-x}{2}\right) = \frac{s-1-\delta}{4}\left(\frac{1}{s} + \frac{s-1-\delta}{2s} + x - 1\right)$$

$$= \frac{s-1-\delta}{4} x - \frac{s-2}{8} + \frac{\delta-1}{8s}.$$

It follows therefore from (9.16)

$$\sum_r \left(r' - \frac{1-x}{s}\right)_+ = \frac{s-1}{2} x + \frac{\delta-1}{4s} - \frac{s-2}{4}.$$

Using (9.12) and (9.7) we obtain

$$s g(x) = \left(s - \frac{1+\delta}{2}\right) x - t \quad (1 - 1/s \leqslant x \leqslant 1). \tag{9.21}$$

32. As to the last sum in (9.17) we have obviously, replacing x by $1 + \theta/s$,

$$\frac{x}{2} - 1 + \frac{\sigma}{s} = \frac{\sigma}{s} - \frac{1}{2} + \frac{\theta}{2s} \leqslant \frac{\theta - 1}{2s} \leqslant 0.$$

We see that our sum vanishes and we obtain from (9.17)

$$\sum_r \left(\frac{x}{2} - r''\right)_+ = \delta \frac{x-1}{2}. \tag{9.22}$$

Further, applying (9.13), we obtain

$$\sum_r \operatorname{Min}\left(\frac{x-1}{2}, r'\right) = \delta \frac{x-1}{2} + 2 \sum_{1\leqslant\sigma\leqslant t} \operatorname{Min}\left(\frac{x-1}{2}, \frac{\sigma}{s}\right). \tag{9.23}$$

But here we have $(x-1)/2 = \theta/2s \leqslant 1/2s < 1/s \leqslant \sigma/s$. Thence, replacing in (9.23) each summand by $(x-1)/2$ it follows

$$\sum_{r} \operatorname{Min}\left(\frac{x-1}{2}, r'\right) = (\delta+2t)\frac{x-1}{2} = \frac{s-1}{2}(x-1). \tag{9.24}$$

33. Using now (9.8) and summing over r we obtain from (9.22) and (9.24) the same expression as in (9.21). It follows finally

$$sg(x) = \left(s-\frac{1+\delta}{2}\right)x-t \qquad (1-1/s \leqslant x \leqslant 1+1/s). \tag{9.25}$$

Since obviously

$$sg(x)-sx/2 = t(x-1),$$

we see that $g(x)$ in $\langle 1-1/s, 1+1/s\rangle$ is always different from $x/2$ save for $x=1$. The assertion of Section 25 is proved.

References

[1] L. Fejér, *Gesammelte Arbeiten*, Bd. II, Birkhäuser, Basel 1970, p. 843.
[2] E. Hecke, *Über analytische Funktionen und die Verteilung der Zahlen mod. eins*, Hamb. Math. Abh. 1 (1922), pp. 54–76; see also: E. Hecke, *Mathematische Werke*, Vandenhoeck und Ruprecht, Göttingen 1959, p. 329.
[3] A. M. Ostrowski, *Sur le formule de Moivre-Laplace*, C. R. Acad. Sci. Paris 223 (1946), pp. 1090–1092.
[4] — *On the remainder term of the de Moivre-Laplace formula*, to appear in Aequationes Math.
[5] — *Math. Miszellen IX, Notiz zur Theorie der Diophantischen Approximationen*, J.-B.d.D.M.V. 36 (1927), pp. 178–180.
[6] G. Pólya und G. Szegö, *Aufgaben und Lehrsätze*, Bd. I, Springer, Berlin 1925, pp. 72, 237.
[7] J. V. Uspensky, *Introduction to mathematical probability*, McGraw-Hill, New York 1937, p. 129.

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On two definitions of the integral of a p-adic function

by

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In memory of Paul Turán

In his basic paper on functions of a p-adic variable Dieudonné [1], introduced a special kind of integral (primitive) of a continuous function. A completely different definition of such an integral was more recently given by M. van der Put (see A. C. M. van Rooij and W. H. Schikhof [2]). The aim of this note is to show that these two definitions lead to the same result. This is rather surprising because there is a large set of non-constant p-adic functions of derivative 0.

Since it simplifies the discussion, we shall study the two kinds of integrals for the class of functions $f: J \to Q_p$ where p is any positive rational prime, Q_p is the field of p-adic numbers, and $J = \{0, 1, 2, \ldots\}$ is the set of all non-negative rational integers. The set J is not closed, and its p-adic closure is the set $I = \{x \in Q_p;\ |x|_p \leqslant 1\}$ of all p-adic integers which is compact.

1. Let $f: J \to Q_p$ be an arbitrary function on J. The two integrals of f are defined by the following constructions.

Write $x \in J$ in the canonic form as

$$x = x_0 + x_1 p + x_2 p^2 + \ldots$$

where $x_0, x_1, x_2, \ldots$ are digits $0, 1, \ldots, p-1$. At most finitely many of these digits are distinct from 0; so, if $x \neq 0$, let $x_s \neq 0$ be the non-vanishing digit of largest suffix s. Firstly put

$$q(0) = 0, \qquad q(x) = x_s p^s \quad \text{for} \quad x \neq 0.$$

Secondly write

$$x^{(n)} = x_0 + x_1 p + \ldots + x_{n-1} p^{n-1} \qquad (n = 1, 2, 3, \ldots)$$

so that

$$x^{(n+1)} = x^{(n)} \quad \text{for} \quad n > s.$$