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Linear independence of 'logarithms' in linear varieties

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1. Introduction. In this paper we prove a result analogous to Baker's theorem on linear independence of logarithms (see [1], Th. 2.1, p. 10, and [2]) in the case of linear varieties. Our main result is the following:

THEOREM 1. *Let $M_1, \ldots, M_n$ be complex $d\times d$ invertible matrices and let X_i be the vector space over Q spanned by the eigenvalues of M_i $(i = 1, \ldots, n)$. Suppose that*

(i) $\exp M_i \in M(d; \bar{Q})$ *for* $i = 1, \ldots, n$;

(ii) $X_1 + \ldots + X_n$ *is a direct sum.*

Then, for every choice of matrices $C_0, C_1, \ldots, C_n$, not all zero and with algebraic coefficients,

$$C_0 + C_1M_1 + \ldots + C_nM_n \neq 0.$$

The proof of Theorem 1 and of other similar results is in Section 3. In Section 4 we produce several counterexamples: partly to fix some limits to further extensions of Baker's theorem and partly to prevent false conjectures.

2. Notation and auxiliary results. Let $M(d; C)$ be the ring of complex $d\times d$ matrices. An element $A \in M(d; C)$ is said to be a *block matrix* whenever

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix},$$

where the A_i's are $r_i \times r_i$ matrices, $r_1 + \ldots + r_k = d$, and each A_i has exactly one eigenvalue.

If each element of a subgroup H of $M(d; C)$ or of $\mathrm{GL}(d; C)$ is a block matrix, H will be called a *block group*; if at least one of the conjugate subgroups of H is at the same time a triangulable and a block group, H will be said *block triangulable.*

When there will be no confusion we shall use the word triangular instead of upper triangular; we warn the reader that we use the same letter for a matrix $A \in M(d; C)$ and the linear function $A: C^d \to C^d$ associated with that matrix. I hope no confusion is possible.

PROPOSITION 1. *Any (multiplicative) commutative group $H \subseteq \mathrm{GL}(d; C)$ is block triangulable.*

Proof. Let $A \in H$. If λ is an eigenvalue of A, for any $h \geqslant 1$, the subspace of C^d

$$W_\lambda^h = \ker(A - \lambda I)^h$$

is an eigenspace for all the functions in H. For, if $B \in H$ and $x \in W_\lambda^h$, then

$$(A-\lambda I)^h \cdot Bx = B \cdot (A - \lambda I)^h x = 0.$$

Hence if we let

$$W_\lambda^\infty = \bigcup_{n=1}^{\infty} \ker(A - \lambda I)^h$$

we obtain a decomposition of C^d of the form

$$C^d = W_{\lambda_1}^\infty \oplus \ldots \oplus W_{\lambda_k}^\infty,$$

where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of A.

Let $r_i = \dim W_{\lambda_i}^\infty$ $(i = 1, \ldots, k)$; let $\{f_1, \ldots, f_{r_1}\}$ be a basis for $W_{\lambda_1}^\infty$, $\{f_{r_1+1}, \ldots, f_{r_1+r_2}\}$ a basis for $W_{\lambda_2}^\infty$ and so on; then $\{f_1, \ldots, f_d\}$ is a basis for C^d. If $\{e_1, \ldots, e_d\}$ is the canonical basis and the matrix E gives the change of basis $e_i \mapsto f_i$, then every element $B \in K = EHE^{-1}$ can be written in the form

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix},$$

where the B_i's are $r_i \times r_i$ matrices. After this transformation, it could happen that there is an element $D \in K$ which is not a block matrix, for one of its blocks, say D_1, has more than one eigenvalue. Applying the above argument to D_1 and to the space $W_{\lambda_1}^\infty$, we get a decomposition

$$W_{\lambda_1}^\infty = T_1^1 \oplus \ldots \oplus T_{n_1}^1$$

into subspaces T_j^1 $(j = 1, \ldots, n_1)$ of lower dimension. Hence it is possible to transform H into a block group in a finite number of steps.

As for triangulating, we can clearly suppose that the matrices of H have exactly one block. Let $A \in H$, λ_A the eigenvalue of A, and let

$$V_A = \ker(A - \lambda_A I).$$

LEMMA 1. *In the preceding hypotheses, we have*

$$V^1 = \bigcap_{A \in H} V_A \neq \{0\}.$$

Proof. If we had $V^1 = \{0\}$, we could find a finite sequence $A_1, \ldots, A_s$ of matrices in H such that

$$\dim V_{A_1} > \dim(V_{A_1} \cap V_{A_2}) > \ldots > \dim(V_{A_1} \cap V_{A_2} \cap \ldots \cap V_{A_s}) = 0.$$

Since $\dim(V_{A_1} \cap \ldots \cap V_{A_{s-1}}) > 0$ and $V_{A_1} \cap \ldots \cap V_{A_{s-1}}$ is an eigenspace for A_s, we have

$$(A_s - \lambda_{A_s} I)^h|_{V_{A_1} \cap \ldots \cap V_{A_{s-1}}} = 0$$

for h sufficiently large, and therefore $V_{A_1} \cap \ldots \cap V_{A_s} \neq \{0\}$, which is a contradiction. The lemma is proved.

Now observe that $A|_{V^1} = \lambda_A I$ for any $A \in H$; then choosing a basis for V^1 and completing it to a basis for C^d (and changing basis, as before), we have done the first step for the triangulation, because the matrices so transformed have the form

$$A = \left(\begin{array}{ccc|c} \lambda_A & & 0 & \\ & \ddots & & \\ 0 & & \lambda_A & A' \\ \hline & 0 & & \end{array}\right).$$

Let

$$V_A^2 = \begin{cases} V_A & \text{if} \quad V_A \neq V^1, \\ \ker(V_A - \lambda_A I)^2 & \text{if} \quad V_A = V^1. \end{cases}$$

Under this definition, V_A^2 contains property V_A^1 for every A and is an eigenspace for every element in H. Let us define

$$V^2 = \bigcap_{A \in H} V_A^2.$$

We have, as in the proof of the lemma, $V^2 \supseteq V^1$ and $V^2 \neq V^1$; choose any subspace W_2 (and any base for it) such that

$$V^2 = V^1 \oplus W^2;$$

then $A|_{V^2} = \lambda_A I + f_A$, where f_A is a linear function such that $f_A: W_2 \to V^1$. In other words, with this new basis our matrices take the form

$$A = \left(\begin{array}{cc|c} \lambda_A I & f_A & \\ & \lambda_A I & A'' \\ 0 & & \end{array}\right)$$

and this completes the second step for the triangulation. Proceeding in this way, we obtain a sequence of subspaces

$$\{0\} \subsetneqq V^1 \subsetneqq V^2 \subsetneqq \ldots \subsetneqq V^l = C^d$$

the last one being C^d, for which the triangulation holds. ■

PROPOSITION 2. *Let A be a complex $d \times d$ matrix such that all the coefficients of $\exp A$ are algebraic. Then there is a matrix $B \in \mathrm{GL}(d; \overline{Q})$ such that BAB^{-1} and $\exp(BAB^{-1}) = B \cdot \exp(A) \cdot B^{-1}$ are block triangular and have algebraic coefficients everywhere outside the diagonal.*

Proof. We can suppose that $\exp A$ is a block matrix (possibly after a suitable transformation) and moreover that it has only one eigenvalue: call it λ. Since A commutes with $\exp A$ (remember the series expansion), A is a block matrix too and

$$W_1 = \ker(\exp A - \lambda I)$$

is an eigenspace for A. Change the basis so that the elements $w_1, \ldots, w_{r_1}$ of the new basis $\{w_1, \ldots, w_d\}$ form a basis for W_1; the matrix A will take the form

$$A = \left(\begin{array}{c|c} A_1 & X \\ \hline 0 & \end{array}\right)$$

where $\exp A_1 = \lambda I$; hence $A_1 = (\log \lambda) I$, as it follows from the following lemma.

LEMMA 2. *Let $X \in M(r; C)$ be such that $\exp X = \lambda I$. Then, chosen a determination for the logarithm,*

$$X = (\log \lambda) I.$$

Proof. Let $A \in \mathrm{GL}(r; C)$ be such that $Y = AXA^{-1}$ is triangular. Then $\exp Y = A \cdot \exp(X) \cdot A^{-1} = \lambda I$, hence all the terms on the diagonal of Y are equal to $\log \lambda$. More precisely,

$$Y = (\log \lambda) I + N,$$

where N is a nilpotent matrix. It follows

$$\exp Y = \lambda I \cdot \exp N = \lambda I,$$

hence

$$\exp N = I.$$

On the other hand,

$$\exp N = I + N + \frac{N^2}{2!} + \ldots + \frac{N^{r-1}}{(r-1)!}.$$

Let $N = (n_{ij})$; looking at the $(i, i+1)$-terms in the preceding equation, one observes that they appear only in N, so that $n_{i,i+1} = 0$ for every i.

Moreover this implies that the $(i, i+2)$-terms appear only in N, so that $n_{i,i+2} = 0$ for every i, and so on, until we conclude that $N = 0$. The lemma is proved.

Let us define $W_2 = \ker(\exp A - \lambda I)^2$: W_2 is an eigenspace for A. Change the basis again, so that the elements $w_1, \ldots, w_{r_1}, w_{r_1+1}, \ldots, w_{r_1+r_2}$ of the new basis $\{w_j\}$ form a basis for W_2 (the basis of W_1 is left unchanged); we have

$$\exp A = \left(\begin{array}{cc|c} \lambda I & B' & \\ \hline & \lambda I & B'' \\ 0 & & \end{array}\right), \qquad A = \left(\begin{array}{cc|c} (\log \lambda) I & A' & \\ \hline & A_2 & A'' \\ 0 & & \end{array}\right).$$

Since $\exp A_2 = \lambda I$, it follows from the lemma that $A_2 = (\log \lambda) I$. Go on this way until, for a certain k, $W_k = \ker(\exp A - \lambda I)^k$ is the whole space C^d. Then A has become triangular, all the terms on its diagonal are equal to $\log \lambda$, and moreover all the transformations we have done involved algebraic numbers, because the equations defining the W_j's have algebraic coefficients. If we put

$$A = (\log \lambda) I + K$$

we have

$$\exp A = \lambda I \cdot \exp K = \lambda I \left(I + K + \frac{K^2}{2!} + \ldots + \frac{K^{d-1}}{(d-1)!} \right).$$

The coefficients of K can be found solving the polynomial equations with algebraic coefficients above, and therefore they are algebraic. ■

3. Results and proofs. We can now prove Theorem 1.

Proof of Theorem 1. By Proposition 2, there exists for every i, $i = 1, \ldots, n$, an invertible matrix B_i with algebraic coefficients such that $B_i M_i B_i^{-1}$ and $B_i \cdot \exp(M_i) \cdot B_i^{-1}$ are block triangular. The blocks of the matrices $M_i' = B_i M_i B_i^{-1}$ will be of the type

$$aI + N,$$

where N is nilpotent algebraic. We can also write

$$M_i' = \begin{pmatrix} m_{11}^{(i)} & & 0 \\ & m_{22}^{(i)} & \\ 0 & & \ddots \; m_{dd}^{(i)} \end{pmatrix} \text{ (mod. algebraic matrices).}$$

Let $C_i' = B_i C_i B_i^{-1} = (c_{hk}')$. Then

$$(*) \qquad C_i' M_i' \equiv (c_{hk}' m_{kk}^{(i)}) \bmod M(d; \overline{Q})$$

and $C_i M_i = B_i^{-1} C_i' M_i' B_i$. Suppose that $C_0 + C_1 M_1 + \ldots + C_n M_n = 0$. The coefficients of the matrices $C_i M_i$'s $(i = 1, \ldots, n)$ are elements of

$(\bar{Q} \otimes_Q X_i) \oplus \bar{Q}$. Since the X_i's are spanned by logarithms of algebraic numbers, this implies, by Baker's theorem, that

$$C_i M_i \equiv 0 \bmod M(d; \bar{Q})$$

hence

$$C_i' M_i' \equiv 0 \bmod M(d; \bar{Q}).$$

But the last equation implies, by (*), that $C_i' = 0$ and therefore $C_i = 0$ for $i = 1, \ldots, n$. Finally take the difference to show that $C_0 = 0$. ∎

THEOREM 2. *Let $H \in \mathrm{GL}(d; C)$ be a triangulable subgroup (in particular, by Proposition 1, one can suppose that H is a commutative subgroup). Let $M_1, \ldots, M_n$ be matrices of $T_e(H)$ (the tangent space at the point zero) such that:*

(i) $\exp M_i \in \mathrm{GL}(d; \bar{Q})$ *for* $i = 1, \ldots, n$;

(ii) *the matrix $\lambda_1 M_1 + \ldots + \lambda_n M_n$ is invertible for every choice of not all zero integers $\lambda_1, \ldots, \lambda_n$.*

Then, for every choice of algebraic matrices in $T_e(H)$, $C_0, C_1, \ldots, C_n$, not all zero, we have

$$C_0 + C_1 M_1 + \ldots + C_n M_n \neq 0.$$

Proof. Suppose H has been triangulated: on the diagonal of each element of H there are logarithms of algebraic numbers. Suppose $C_0, C_1, \ldots, C_n$ are algebraic matrices of $T_e(H)$ not all zero and such that

$$C_0 + C_1 M_1 + \ldots + C_n M_n = 0. \tag{1}$$

Give to the pairs (h, k), with $h \leqslant k$, the lexicographic order:

$$(h, k) \leqslant (h', k') \quad \text{if} \quad \begin{cases} h < h' & \text{or} \\ h = h' & \text{and } k \leqslant k'. \end{cases}$$

Let (r, s) be the least pair such that there exists an index i, $0 \leqslant i \leqslant n$, for which the coefficient $c_{rs}^{(i)}$ of the matrix C_i is not zero. Considering the (r, s)-term in equation (1), we obtain

$$c_{rs}^{(0)} + \sum_{i=1}^{n} c_{rs}^{(i)} m_{ss}^{(i)} = 0.$$

It follows, by Baker's theorem, that there are integers $\lambda_1, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 m_{ss}^{(1)} + \ldots + \lambda_n m_{ss}^{(n)} = 0.$$

But this contradicts (ii), for the matrix $\lambda_1 M_1 + \ldots + \lambda_n M_n$ is triangular and has a zero on the diagonal. ∎

We want now to prove some theorems similar to those of [1], p. 11. Among them, the following is the only one holding in the full generality for a linear group.

THEOREM 3. *Let $M_1, \ldots, M_n$ be matrices of $M(d; C)$, not all nilpotent and such that $\exp M_i$ is algebraic for $i = 1, \ldots, n$. If $\lambda_1, \ldots, \lambda_n$ are algebraic numbers such that the numbers $1, \lambda_1, \ldots, \lambda_n$ are linearly independent over Q, then*

$$\exp(\lambda_1 M_1 + \ldots + \lambda_n M_n)$$

is transcendental.

Proof. Suppose that

$$\lambda_1 M_1 + \ldots + \lambda_n M_n = M_{n+1}$$

and $\exp M_{n+1}$ is algebraic. Let Y_i $(i = 1, \ldots, n+1)$ be the vector space over Q spanned by the eigenvalues of the $\lambda_i M_i$'s (define $\lambda_{n+1} = 1$). If $y_i \in Y_i$, then $y_i = \lambda_i a_i$ where a_i is the logarithm of an algebraic number. If follows from Baker's theorem that

$$y_1 + \ldots + y_{n+1} = 0$$

implies

$$y_1 = y_2 = \ldots = y_{n+1} = 0$$

that is, the sum $Y_1 + \ldots + Y_{n+1}$ is direct. Apply Theorem 1 to conclude the proof. ∎

In the rest of the section $H \subseteq \mathrm{GL}(d; C)$ will be a triangulable group (which we shall suppose triangulated); $M_1, \ldots, M_n$ will be algebraic matrices of $T_e(H)$, not all nilpotents and such that $\exp M_i \in H \cap \mathrm{GL}(d; \bar{Q})$ for $i = 1, \ldots, n$. Then the following theorems hold:

THEOREM 4. *For every choice of algebraic matrices of $T_e(H)$, $C_0, C_1, \ldots, C_n$, the matrix*

$$B = C_0 + C_1 M_1 + \ldots + C_n M_n$$

is either nilpotent or transcendental.

Proof. If B is not nilpotent, it has a non-zero coefficient on the diagonal, say b_{jj}. Then

$$b_{jj} = c_{jj}^{(0)} + c_{jj}^{(1)} m_{jj}^{(1)} + \ldots + c_{jj}^{(n)} m_{jj}^{(n)}$$

is transcendental by Theorem 2.2 of [1]. ∎

THEOREM 5. *For every choice of algebraic matrices of $T_e(H)$ $C_0, C_1, \ldots, C_n$, where C_0 is not nilpotent, the matrix*

$$X = \exp(C_0 + C_1 M_1 + \ldots + C_n M_n)$$

is transcendental.

Proof. Suppose that $X\in \mathrm{GL}(d;\overline{Q})$ and let

$$M_{n+1} = C_0+C_1M_1+\dots+C_nM_n.$$

Since C_0 is not nilpotent, it has a non-zero coefficient on the diagonal, say $c_{jj}^{(0)}$. It follows that

$$c_{jj}^{(0)} = -(c_{jj}^{(1)}m_{jj}^{(1)}+\dots+c_{jj}^{(n)}m_{jj}^{(n)})+m_{jj}^{(n+1)}$$

and this contradicts Theorem 2.2 of [1]. ■

THEOREM 6. *Suppose that $M_1,\dots,M_n$ are invertible matrices and that $A_1,\dots,A_n$ are algebraic matrices of $T_e(H)$ satisfying the condition*

(α) *for every choice of not all zero integers $h_0, h_1, \dots, h_n$, the matrix $h_0I+h_1A_1+\dots+h_nA_n$ is invertible.*

Then the matrix $\exp(A_1M_1+\dots+A_nM_n)$ *is transcendental.*

Proof. It is sufficient to show that $A_1M_1+\dots+A_nM_n\neq 0$ for every choice of algebraic matrices of $T_e(H)$ satisfying the condition:

(β) if $h_1,\dots,h_n$ are not all zero integers, the matrix $h_1A_1+\dots+h_nA_n$ is invertible.

In fact, the theorem follows when we substitute n with $n+1$ and put $A_{n+1}=-I$. Suppose $A_1M_1+\dots+A_nM_n=0$. By Theorem 2, there are integers $a_1,\dots,a_n$, not all zero, such that

$$a_1M_1+\dots+a_nM_n = D = (d_{ij})$$

where D is degenerate; that is, D has a zero on the diagonal. Suppose that $a_n\neq 0$ and $d_{jj}=0$. Substituting, we obtain

$$(a_nA_1-a_1A_n)M_1+\dots+(a_nA_{n-1}-a_{n-1}A_n)M_{n-1} = -A_nD$$

and looking at the (j,j)-term

$$(a_n\lambda_{jj}^{(1)}-a_1\lambda_{jj}^{(n)})m_{jj}^{(1)}+\dots+(a_n\lambda_{jj}^{(n-1)}-a_{n-1}\lambda_{jj}^{(n)})m_{jj}^{(n-1)} = 0.$$

By Theorem 2.4 of [1] the numbers $a_n\lambda_{jj}^{(k)}-a_k\lambda_{jj}^{(n)}$ are linearly dependent over Q and therefore the $\lambda_{jj}^{(k)}$'s are linearly dependent over Q too. But this contradicts (β) and the theorem is proved. ■

Remark. If a matrix of the type

$$C_0+C_1M_1+\dots+C_nM_n$$

is not zero, we can easily find a lower bound for its size. It suffices to apply well-known theorems (see for instance Theorem 3.1 of [1], p. 22) with a slight correction, depending on the changes of basis we performed in our proofs. More precisely, if

$$\text{size } C_i\leqslant\gamma \quad\text{and}\quad \text{size } M_j\leqslant\mu$$

e have, defining $C_i' = B_iC_iB_i^{-1}$ and $M_j' = B_jM_jB_j^{-1}$,

$$\text{size } C_i'\leqslant\gamma+\mu \quad\text{and}\quad \text{size } M_j'\leqslant\mu,$$

ie unwritten constants depending only upon d.

4. Counterexamples.

EXAMPLE 1. Theorem 2 is not true for any linear group, for the hypo-nesis (ii) is not sufficient. In fact, let

$$M_1 = \begin{pmatrix} ix & 0\\ 0 & 2ix\end{pmatrix},\qquad M_2 = \begin{pmatrix} 0 & ix\\ ix & 0\end{pmatrix}.$$

'hen

$$\exp M_1 = \begin{pmatrix}\cos x+i\sin x & 0\\ 0 & \cos 2x+i\sin 2x\end{pmatrix},$$

$$\exp M_2 = \begin{pmatrix}\cos x & \sin x\\ \sin x & \cos x\end{pmatrix}$$

nd both these matrices are algebraic when we choose $x = q\pi$, $q\in \mathbf{Z}-\{0\}$. f λ_1 and λ_2 are integers, not both zero, we have

$$\det(\lambda_1M_1+\lambda_2M_2) = (-2\lambda_1^2+\lambda_2^2)x^2\neq 0.$$

)n the other hand,

$$M_1-\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}M_2 = 0.$$

Note that $M_1M_2\neq M_2M_1$.)

EXAMPLE 2. The following statement is false:

— if, for every choice of not all zero integers $\lambda_1,\dots,\lambda_n$, we have $_1M_1+\dots+\lambda_nM_n\neq 0$, then we have also

$$a_1M_1+\dots+a_nM_n\neq 0$$

or every choice of not all zero algebraic numbers $a_1,\dots,a_n$.

Observe first that, for $a\neq b$,

$$\exp\begin{pmatrix}a & x\\ 0 & b\end{pmatrix} = \begin{pmatrix}e^a & x\dfrac{e^a-e^b}{a-b}\\ 0 & e^b\end{pmatrix}.$$

.et

$$M_1 = \begin{pmatrix}\log 3 & \log 3/2\\ 0 & \log 2\end{pmatrix},\qquad \exp M_1 = \begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix},$$

$$M_2 = \begin{pmatrix} \log 3 & \sqrt{2}\log 3/2 \\ 0 & \log 2 \end{pmatrix}, \qquad \exp M_2 = \begin{pmatrix} 3 & \sqrt{2} \\ 0 & 2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \log 3 & \sqrt{3}\log 3/2 \\ 0 & \log 2 \end{pmatrix}, \qquad \exp M_3 = \begin{pmatrix} 3 & \sqrt{3} \\ 0 & 2 \end{pmatrix}.$$

There are clearly three non-zero algebraic numbers α, β, γ such that

$$\alpha M_1 + \beta M_2 + \gamma M_3 = 0$$

(solve the equations $\alpha + \beta + \gamma = 0$ and $\alpha + \sqrt{2}\beta + \sqrt{3}\gamma = 0$) but this is impossible with rational numbers.

EXAMPLE 3. In Theorem 1 it would be desirable to substitute condition (ii) with the following:

— if λ_i is an eigenvalue of M_i ($i = 1, \ldots, n$), then $\lambda_1, \ldots, \lambda_n$ are linearly independent over the rationals.

This example shows that this is impossible. Let

$$M_1 = \begin{pmatrix} \log 3 & \log 3/2 \\ 0 & \log 2 \end{pmatrix}, \qquad \exp M_1 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \log 6 & 0 \\ -\log 2 & \log 3/2 \end{pmatrix}, \qquad \exp M_2 = \begin{pmatrix} 6 & 0 \\ -9/4 & 3/2 \end{pmatrix}.$$

The matrices M_1, M_2 satisfy our new condition (but not condition (ii) of Theorem 1) and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_1 - \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} M_2 = 0.$$

EXAMPLE 4. The following statement is false:

— if, for every choice of not all zero integral matrices $B_1, \ldots, B_n$, we have $B_1 M_1 + \ldots + B_n M_n \neq 0$, then for every choice of not all zero algebraic matrices $C_0, C_1, \ldots, C_n$

$$C_0 + C_1 M_1 + \ldots + C_n M_n \neq 0.$$

In fact, let

$$M_1 = \begin{pmatrix} \log 3 & \sqrt{2}\log 3/2 \\ 0 & \log 2 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \log 6 & 0 \\ (-\sqrt{2}/2)\log 2 & \log 3/2 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} M_1 - \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 2 \end{pmatrix} M_2 = 0,$$

but a similar equation is impossible with integral matrices as coefficients.

'or, if $B_1 M_1 + B_2 M_2 = 0$, in the sum there must be no term of the form $\sqrt{2}m\log 3$ or $\sqrt{2}n\log 2$ ($m, n \in \mathbf{Z} - \{0\}$); hence B_1 and B_2 must have the orm

$$B_1 = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \qquad B_2 = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}$$

ien

$$B_1 M_1 + B_2 M_2 = \begin{pmatrix} c\log 6 & a\log 2 \\ d\log 6 & b\log 2 \end{pmatrix}.$$

hich implies $B_1 = B_2 = 0$.

EXAMPLE 5. Theorems 4 and 5 are false in the general case of a linear roup. Let $x, y \in \overline{Q} - \{0\}$,

$$M_1 = \begin{pmatrix} \log 2 & x \\ 0 & \log 2 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \log 2 & 0 \\ y & \log 2 \end{pmatrix}.$$

'o prove that Theorem 4 is false in the general case, observe that

$$M_1 - M_2 = \begin{pmatrix} 0 & x \\ -y & 0 \end{pmatrix}, \qquad \det(M_1 - M_2) = xy \neq 0.$$

.s for Theorem 5, let $C_0 = M_1 - M_2$, $C_1 = -I$, $C_2 = I$.

EXAMPLE 6. Theorem 6 is false in the general case of a linear group. et M_1 and M_2 be as in Example 1,

$$A_1 = \alpha I, \qquad A_2 = \begin{pmatrix} 0 & -\alpha \\ -2\alpha & 0 \end{pmatrix},$$

here α is an algebraic number of degree $\geqslant 3$. We have

$$A_1 M_1 + A_2 M_2 = 0$$

at $\det(h_0 I + h_1 A_1 + h_2 A_2) = (h_1^2 + 2h_2^2)\alpha^2 + 2h_0 h_1 \alpha + h_0^2$ is different from ero if h_0, h_1, h_2 are not all zero.

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