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On the distributions of multiplicative functions

by

G. JOGESH BABU (Calcutta)

1. Introduction. A complex-valued arithmetic function h is said to be *multiplicative* if $h(1) = 1$ and $h(mn) = h(m)h(n)$, whenever m and n are positive integers, prime to each other. Necessary and sufficient conditions for a real-valued multiplicative function to have a distribution which is not degenerate at zero (see the definition below) were obtained by Bakstys [1] and Galambos [2]. The main tool used by both of them is the one developed by Zolotarev in 1962 for the investigation of products of independent random variables. The method involves effective use of the so-called characteristic transforms and the known results on mean-values of multiplicative functions. In this paper we shall give a straightforward proof, which uses no ideas more sophisticated than the well known Kolmogorov's three series theorem from probability theory, Turán–Kubilius inequality and a result of Halász [3] on the mean-values of multiplicative functions. We actually use a very particular case of Halász's result. We shall also find the spectrum of the distribution of h, if it exists.

2. Notations and definitions. Let m, n, N denote positive integers; p, q denote primes; and k, j denote non-negative integers.

DEFINITION. A complex-valued arithmetic function f is said to be *additive* if $f(mn) = f(m)+f(n)$, whenever $(m, n) = 1$. Further, if $f(p^k) = f(p)$ for all $k \geqslant 1$ and primes p, then f is said to be *strongly additive.*

For any set B of positive integers, let

$$\nu_N(B) = \frac{1}{N} \# \{m\colon 1 \leqslant m \leqslant N \text{ and } m \in B\},$$

$$\overline{\pi}(B) = \limsup_{N\to\infty} \nu_N(B) \quad \text{and} \quad \underline{\pi}(B) = \liminf_{N\to\infty} \nu_N(B).$$

DEFINITION. A set of positive integers B is said to *have density*, if $\overline{\pi}(B) = \underline{\pi}(B)$. The common value $\pi(B)$ is called the *density of* B.

DEFINITION. A real-valued arithmetic function f is said to *have a distribution*, if there exists a distribution function F such that the set $(m\colon f(m) < c)$ has density $F(c)$, for all continuity points c of F.

Finally we define, for any multiplicative function r,

$$r_n(m) = \prod_{p\leqslant n,\ p^k\|m} r(p^k)$$

and

$$r^n(m) = \prod_{p>n,\ p^k\|m} r(p^k).$$

3. Main results. Suppose h is a multiplicative function such that the series

$$\sum_{h(p)=0} \frac{1}{p} \tag{1}$$

is divergent. Then $\pi(m\colon h(m)=0)=1$. That is, h has a degenerate distribution. This follows from the fact that for any n,

$$\begin{aligned}\bar{\pi}(m\colon h(m)\neq 0) &\leqslant \bar{\pi}(m\colon h_n(m)\neq 0)\\ &\leqslant \bar{\pi}(m\colon p\nmid m \text{ for } p\leqslant n \text{ with } h(p)=0)\\ &= \prod_{h(p)=0,\ p\leqslant n}\left(1-\frac{1}{p}+\frac{1}{p^2}\right).\end{aligned}$$

The product above tends to zero as $n\to\infty$.

So from now on we shall assume that the series (1) converges. Let us note here that in this case, the set $(m\colon h(m)=0)$ has density less than unity.

THEOREM 1. *Let h be a real-valued multiplicative function such that the series*

$$\sum_{h(p)<0} 1/p \tag{2}$$

converges. If for some real number $a>1$, the three series

$$\sum{}' \frac{1}{p}, \quad \sum{}'' \frac{1}{p}(\log|h(p)|)^2, \quad \sum{}'' \frac{1}{p}\log|h(p)| \tag{3}$$

converge, then h has a distribution, where $\sum''$ denotes the sum over all primes such that $1/a<|h(p)|<a$ and $\sum'$ denotes the sum over the remaining primes.

We need the following lemma.

LEMMA 1. *Let h be a real-valued multiplicative function. Suppose the three series (3) converge; then for any $\varepsilon>0$,*

$$\lim_{n\to\infty}\bar{\pi}(m\colon |\log|h^n(m)||>\varepsilon,\ h^n(m)\neq 0)=0.$$

Proof. The lemma follows easily on applying Turán–Kubilius inequality (see Kubilius [5], Lemma 3.1, p. 31) to the additive function $f^{(n)}$ defined by

$$f^{(n)}(m) = \sum_{p>n,\ p|m,\ h(p)\neq 0} \log|h(p)|.$$

Proof of Theorem 1. Let $\{Z_p\}$ be a sequence of independent discrete random variables defined on some probability space satisfying for any real number x,

$$P(Z_p=x) = \left(1-\frac{1}{p}\right)\sum_{h(p^k)=x} p^{-k}.$$

Let

$$X_p = \begin{cases}|Z_p| & \text{if } Z_p\neq 0,\\ 1 & \text{otherwise},\end{cases}\qquad Y_p=\begin{cases}0 & \text{if } Z_p=0,\\ 1 & \text{otherwise}\end{cases}$$

and

$$U_p=\begin{cases}0 & \text{if } Z_p\geqslant 0,\\ 1 & \text{if } Z_p<0.\end{cases}$$

Clearly $Z_p = X_pY_pe^{i\pi U_p}$ and the sequences $\{X_p\}$, $\{Y_p\}$, $\{U_p\}$ are sequences of independent random variables. We have

$$\sum_p P(U_p\neq 0) = \sum_p P(Z_p<0)\leqslant \sum_p\frac{1}{p^2}+\sum_{h(p)<0}\frac{1}{p}<\infty.$$

So, with probability 1, $U_p=0$ for all sufficiently large p. Thus the series $\sum_p U_p$ converges a.e. As a result $\prod_p \exp(i\pi U_p)$ converges a.e. By the convergence of the three series (3) and by the Kolmogorov's three series theorem (see Halmos [4], p. 199) we have that the series $\sum_p \log X_p$ converges a.e. Thus $\prod_p X_p$ converges a.e. The convergence of the product $\prod_p Y_p$ follows, since $\prod_p Y_p=0$ if $Y_p=0$ for some p and 1 otherwise. So $\prod_p Z_p$ converges a.e. Let H be the distribution function of $Z=\prod_p Z_p$. We shall show that H is the distribution of h. First we note that convergence of the first series in (3) implies the convergence of series (1). We have

$$\begin{aligned}\nu_N(m\colon h^n(m)\leqslant 0) &\leqslant \nu_N(m\colon p^2|m \text{ for some } p>n)+\\ &\quad+\nu_N(m\colon q\|m \text{ for some } q>n \text{ such that } h(q)\leqslant 0)\\ &\leqslant \sum_{p>n}\frac{1}{p^2}+\sum_{q>n,\ h(q)\leqslant 0}\frac{1}{q}.\end{aligned}$$

Thus, we have by the convergence of the series (1) and (2), that

$$\bar{\pi}(m:\ h^n(m) \leqslant 0) \to 0 \tag{4}$$

as $n \to \infty$. Also note that for any interval I, the set $(m:\ h_n(m) \in I)$ has density $P(\prod_{p \leqslant n} Z_p \in I)$.

Let $c > 0$ be a continuity point of H. For any $\varepsilon > 0$, we have

$$\begin{aligned}
\bar{\pi}(m:\ h(m) < c) &\leqslant \bar{\pi}\big(m:\ h_n(m) < (c/h^n(m)),\ h^n(m) > 0\big) + \bar{\pi}(m:\ h^n(m) \leqslant 0)\\
&\leqslant \bar{\pi}(m:\ h_n(m) < ce^{\varepsilon},\ h^n(m) > 0) + \bar{\pi}(m:\ h^n(m) \leqslant 0) +\\
&\qquad + \bar{\pi}(m:\ |h^n(m)| \leqslant e^{-\varepsilon})\\
&\leqslant \pi(m:\ h_n(m) < ce^{\varepsilon}) + 2\bar{\pi}(m:\ h^n(m) \leqslant 0) +\\
&\qquad + \bar{\pi}\big(m:\ \big|\log|h^n(m)|\big| \geqslant \varepsilon,\ h^n(m) \neq 0\big)\\
&= P(\prod_{p \leqslant n} Z_p < ce^{\varepsilon}) + 2\bar{\pi}(m:\ h^n(m) \leqslant 0) +\\
&\qquad + \bar{\pi}\big(m:\ \big|\log|h^n(m)|\big| \geqslant \varepsilon,\ h^n(m) \neq 0\big)\\
&\to P(Z \leqslant ce^{\varepsilon}) = H(ce^{\varepsilon}),
\end{aligned}$$

as $n \to \infty$, by (4) and by Lemma 1. Thus we have

$$\bar{\pi}(m:\ h(m) < c) \leqslant P(Z \leqslant c). \tag{5}$$

Similarly, it follows that

$$\underline{\pi}(m:\ h(m) < c) \geqslant P(Z < c). \tag{6}$$

Since c is a continuity point of H, we have

$$\pi(m:\ h(m) < c) = P(Z < c).$$

Validity of the above equality for $c < 0$ follows by a similar argument. This completes the proof of Theorem 1.

We shall now show that under the hypothesis of Theorem 1, the spectrum of the distribution of h is the closure of the set $\{h(m):\ m \geqslant 1\}$. Recall that the spectrum of a distribution function F is the smallest closed subset of the real-line whose F-measure is 1.

THEOREM 2. *Under the hypothesis of Theorem 1, for any integer $m \geqslant 1$, $h(m)$ belongs to the spectrum of the distribution of h.*

Proof. Let Z_p and H be as in the proof of Theorem 1. Let $h(m) = 0$ for some integer $m \geqslant 1$. Then there exists a prime q and an integer k such that $h(q^k) = 0$. For any $\varepsilon > 0$, we have that

$$P\Big(\Big|\prod_p Z_p\Big| < \varepsilon\Big) \geqslant P(Z_q = 0) \geqslant q^{-k}(1-q^{-1}) > 0.$$

So zero belongs to the spectrum if $h(m) = 0$ for some $m \geqslant 1$. We shall now show that if $h(m) \neq 0$ for some integer $m \geqslant 1$, then $h(m)$ belongs to the spectrum of H. Since $\prod_p Z_p$ converges a.e., $\prod_{p>N} Z_p \to 1$ a.e. as $N \to \infty$. So for any $\varepsilon > 0$ there exists an integer $N > m$ such that

$$P\Big(\Big|\prod_{p>N} Z_p - 1\Big| < (\varepsilon/|h(m)|)\Big) > 1/2.$$

By the independence of Z_p we have

$$\begin{aligned}
P\Big(\Big|\prod_p Z_p - h(m)\Big| < \varepsilon\Big) &\geqslant P\Big(\Big|\prod_{p>N} Z_p - 1\Big| < (\varepsilon/|h(m)|),\ \prod_{p \leqslant N} Z_p = h(m)\Big)\\
&\geqslant \frac{1}{2}\prod_{p \leqslant N} P(Z_p = h^{(p)}(m)) \geqslant \frac{1}{2m}\prod_{p \leqslant N}\Big(1-\frac{1}{p}\Big) > 0
\end{aligned}$$

where

$$h^{(p)}(m) = h(p^r) \quad \text{if} \quad p^r \| m.$$

This completes the proof of Theorem 2.

Remark 1. By Theorem 2, it follows that the multiplicative function in Theorem 1 has a non-degenerate distribution unless $h(m) = 1$ for all integers m.

THEOREM 3. *Let h be a real-valued multiplicative function such that the three series (3) converge. If the series (2) diverges, then h has a non-degenerate symmetric distribution.*

We start the proof with some lemmas.

LEMMA 2. *Let f be a multiplicative function such that for all m, $|f(m)| \leqslant 1$. If*

$$\sum_p \frac{1}{p}\big[1 - \mathrm{Re}(f(p)p^{it})\big] = \infty \tag{7}$$

for all real numbers t, then f has zero mean-value. That is

$$\frac{1}{N}\sum_{m=1}^{N} f(m) \to 0$$

as $N \to \infty$.

For a proof see [3].

LEMMA 3. *Let f be a multiplicative function taking only the two values $+1$ and -1. If the series*

$$\sum_{f(p)=-1} 1/p \tag{8}$$

diverges, then f has zero mean-value. In particular,

$$\pi(m\colon f(m) = 1) = \pi(m\colon f(m) = -1) = \tfrac{1}{2}.$$

Proof. We shall show that (7) holds for all real numbers t. The divergence of the series (8) implies (7) for $t = 0$. The result follows if we show that for any set Q of primes and $t \neq 0$

$$\sum_{p\in Q} \frac{1}{p}[1+\mathrm{Re}(p^{it})] + \sum_{p\notin Q} \frac{1}{p}[1-\mathrm{Re}(p^{it})] = \infty. \tag{9}$$

Suppose for some real number $t \neq 0$, (9) does not hold. Since for any $|z| \leqslant 1$,

$$1-\mathrm{Re}(z^2) \leqslant 4(1-\mathrm{Re}\, z)$$

we have

$$\sum_{p} \frac{1}{p}[1-\mathrm{Re}(p^{2it})] < \infty. \tag{10}$$

But (10) is false, because $\log(\zeta(\sigma)/|\zeta(\sigma-iv)|) \to \infty$ as $\sigma \to 1^+$ for every real number $v \neq 0$. Here ζ denotes the Riemann zeta function. So (9) holds for all non-zero real numbers t. This completes the proof of Lemma 3.

We use only Lemma 3. Lemma 2 is presented here just to prove Lemma 3. Let r be the multiplicative function defined by

$$r(m) = |h(m)|$$

and let u be the multiplicative function defined by

$$u(p^k) = \begin{cases} 1 & \text{if} \quad h(p^k) \geqslant 0, \\ -1 & \text{if} \quad h(p^k) < 0. \end{cases}$$

Clearly $h = ru$. Let

$$B_{k,n} = \{km\colon u^n(m) = \delta \text{ and } (p, m) = 1 \text{ for all } p \leqslant n\},$$

where $\delta = +1$ or -1.

LEMMA 4. *For any positive integer k not divisible by any prime $p > n$,*

$$\pi(B_{k,n}) = \frac{1}{2k}\prod_{p\leqslant n}\left(1-\frac{1}{p}\right).$$

Proof. By Lemma 3, we have for any $n \geqslant 2$, that

$$\pi(m\colon u^n(m) = 1) = \pi(m\colon u^n(m) = -1) = \tfrac{1}{2}.$$

So as $N \to \infty$,

$$\nu_N(B_{k,n}) = \frac{1}{N}\Big(\sum_{u^n(m)=\delta,\, m\leqslant N/k} 1 - \sum_{p\leqslant n}\ \sum_{m\leqslant N/kp,\ u^n(m)=\delta} 1 + \ldots\Big)$$

$$\to \frac{1}{k}\prod_{p\leqslant n}\left(1-\frac{1}{p}\right)\pi(m\colon u^n(m) = \delta) = \frac{1}{2k}\prod_{p\leqslant n}\left(1-\frac{1}{p}\right).$$

This completes the proof of Lemma 4.

Let

$$A_n = \{k\colon (p, k) = 1 \text{ for all } p > n \text{ and } r_n(k) \in I,\ u_n(k) = \delta'\}$$

and

$$B_n = \{m\colon r_n(m) \in I,\ u_n(m) = \delta',\ u^n(m) = \delta\}$$

where I is an interval, $\delta = +1$ or -1 and $\delta' = +1$ or -1.

We have

LEMMA 5. *For any interval I, the density of B_n is*

$$\tfrac{1}{2}\pi(m\colon r_n(m) \in I,\ u_n(m) = \delta').$$

Proof. Clearly

$$B_n = \bigcup_{k\in A_n} B_{k,n}, \quad \nu_N(B_{k,n}) \leqslant \frac{1}{k} \quad \text{and} \quad \sum_{k\in A_n}\frac{1}{k} \leqslant \prod_{p\leqslant n}\left(1-\frac{1}{p}\right)^{-1}.$$

So

$$\Big|\nu_N(B_n) - \sum_{k\in A_n}\pi(B_{k,n})\Big|$$
$$\leqslant \sum\nolimits_1 \nu_N(B_{k,n}) + \sum\nolimits_1 \pi(B_{k,n}) + \sum\nolimits_2 |\nu_N(B_{k,n}) - \pi(B_{k,n})|$$
$$\leqslant \sum\nolimits_1 2/k + \sum\nolimits_2 |\nu_N(B_{k,n}) - \pi(B_{k,n})|,$$

where $\sum_1$ denotes the sum over $k \in A_n$, $k > k'$ and $\sum_2$ denotes the sum over the remaining $k \in A_n$. Since $\sum_1 1/k \to 0$ as $k' \to \infty$, it follows from the above inequality and from Lemma 4, that

$$\pi(B_n) = \sum_{k\in A_n}\pi(B_{k,n}) = \sum_{k\in A_n}\frac{1}{2k}\prod_{p\leqslant n}\left(1-\frac{1}{p}\right)$$
$$= \tfrac{1}{2}\pi(km\colon k \in A_n \text{ and } (m, p) = 1 \text{ for all } p \leqslant n)$$
$$= \tfrac{1}{2}\pi(m\colon r_n(m) \in I,\ u_n(m) = \delta').$$

This completes the proof of the lemma.

Proof of Theorem 3. By Theorem 1, r has a distribution F. By Remark 1 it follows that F is not degenerate at zero. Let $c > 0$ be a continuity point of F. We have

$$\begin{aligned} \nu_N(m\colon h(m) < c) &= \nu_N(m\colon r(m) < c,\, u(m) = 1) + \nu_N(m\colon r(m) > -c,\ u(m) = -1) \\ &= \nu_N(m\colon r(m) < c,\, u(m) = 1) + \nu_N(m\colon u(m) = -1). \end{aligned}$$

By Lemma 3, we have $\pi(m\colon u(m) = -1) = \frac{1}{2}$. We shall show that the next to the last term in the equation above tends to the limit $\frac{1}{2}F(c)$, as $N \to \infty$.

For any $\varepsilon > 0$, we have

$$\begin{aligned} (11)\quad & \nu_N(m\colon r_n(m) < ce^{-\varepsilon},\, u(m) = 1) - \nu_N(m\colon r^n(m) = 0) - \\ &\qquad - \nu_N(m\colon r^n(m) \neq 0 \text{ and } |\log r^n(m)| \geqslant \varepsilon) \\ &\leqslant \nu_N(m\colon r(m) < c,\, u(m) = 1) \\ &\leqslant \nu_N(m\colon r^n(m) = 0) + \nu_N(m\colon r_n(m) < ce^{\varepsilon},\, u(m) = 1) + \\ &\qquad + \nu_N(m\colon r^n(m) \neq 0 \text{ and } |\log r^n(m)| \geqslant \varepsilon). \end{aligned}$$

By Lemma 5, we have for any positive real number b, as $N \to \infty$

$$\begin{aligned} & \nu_N(m\colon r_n(m) < b,\, u(m) = 1) \\ &= \nu_N(m\colon r_n(m) < b,\, u_n(m) = 1,\, u^n(m) = 1) + \\ &\quad + \nu_N(m\colon r_n(m) < b,\, u_n(m) = -1,\, u^n(m) = -1) \\ &\to \tfrac{1}{2}\{\pi(m\colon r_n(m) < b,\, u_n(m) = 1) + \pi(m\colon r_n(m) < b,\, u_n(m) = -1)\} \\ &= \tfrac{1}{2}\pi(m\colon r_n(m) < b). \end{aligned}$$

Thus for any real number $b > 0$

$$(12)\qquad \pi(m\colon r_n(m) < b,\, u(m) = 1) = \tfrac{1}{2}\pi(m\colon r_n(m) < b).$$

Since, by the convergence of the series (1), $\bar{\pi}(m\colon r^n(m) = 0) \to 0$ as $n \to \infty$, and since F is continuous at c, we have by (11), (12) and Lemma 1

$$\nu_N(m\colon r(m) < c,\, u(m) = 1) \to \tfrac{1}{2}F(c),$$

as $N \to \infty$. So if $c > 0$ is a continuity point of F, then $\pi(m\colon h(m) < c)$ exists and equals $\frac{1}{2}(1 + F(c))$. Similarly it follows that, if $c < 0$ is a continuity point of F, then

$$\pi(m\colon h(m) < c) = \tfrac{1}{2}\pi(m\colon r(m) > -c) = \tfrac{1}{2}(1 - F(-c)).$$

Thus h has a non-degenerate symmetric distribution.

The following corollary is an immediate consequence of Theorems 1 and 3.

COROLLARY. *A real-valued multiplicative function has a distribution, if the three series* (3) *converge.*

Conversely, we have the following

THEOREM 4. *Suppose h is a real-valued multiplicative arithmetic function having a distribution which is not degenerate at zero. Then the three series* (3) *converge.*

Proof. Since h has a distribution which is not degenerate at zero, convergence of the series (1) follows from what we have noted at the beginning of this section. So $\pi(m\colon h(m) = 0) < 1$. Hence, for some positive real numbers $b < c$,

$$\pi(m\colon b < |h(m)| < c) > 0.$$

If f is the additive arithmetic function defined by

$$f(p^k) = \begin{cases} \log|h(p^k)| & \text{if } h(p^k) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

then we have,

$$\underline{\pi}(m\colon \log b < f(m) < \log c) > 0.$$

As in the proof of Theorem 2 of E. M. Paul [6], it follows that $\sum_p \frac{1}{p}(f'(p))^2 < \infty$. That is, for some $a > 1$, the first two series in (3) converge. As in the proof of Theorem 1, it follows that there exists a distribution function F such that

$$(13)\qquad \nu_N(m\colon |h(m)|\, e^{-A_N} < c) \to F(c)$$

as $N \to \infty$, for *each* continuity point c of F, where

$$A_N = \sum_{p \leqslant N,\ 1/a < |h(p)| < a} \frac{1}{p} \log|h(p)|.$$

Since $|h|$ has a distribution H and since $\pi(m\colon h(m) = 0) < 1$, we have from (13) that the sequence $\{A_N\}$ is bounded. Let θ be a limit point of the sequence $\{A_N\}$. We have from (13) that $H(ce^{\theta}) = F(c)$ for all c. Since H is not concentrated at zero, the sequence $\{A_N\}$ has only one limit point. Thus A_N tends to a limit as $N \to \infty$. This completes the proof of the theorem.

Remark 2. From the theorems above, it follows that a multiplicative function h has a distribution which is not degenerate at zero if and only if the series in (3) converge. Of course all this analysis is done under the assumption that the series (1) converges.

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STAT.-MATH. DIVISION
INDIAN STATISTICAL INSTITUTE
Calcutta, India

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On the remainder term of the prime number formula I. On a problem of Littlewood

by

J. Pintz (Budapest)

1. In the present paper we shall deal with the prime number formula and other well known sums depending on the prime numbers. Let us define

$$(1.1)\qquad \Delta_1(x) \overset{\text{def}}{=} \pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{p\leqslant x} 1 - \int_0^x \frac{dr}{\log r},$$

$$(1.2)\qquad \Delta_2(x) \overset{\text{def}}{=} \Pi(x) - \operatorname{li} x \overset{\text{def}}{=} \sum_{\nu\geqslant 1} \frac{1}{\nu}\pi(x^{1/\nu}) - \operatorname{li} x,$$

$$(1.3)\qquad \Delta_3(x) \overset{\text{def}}{=} \theta(x) - x \overset{\text{def}}{=} \sum_{p\leqslant x} \log p - x,$$

$$(1.4)\qquad \Delta_4(x) \overset{\text{def}}{=} \psi(x) - x \overset{\text{def}}{=} \sum_{n\leqslant x} \Lambda(n) - x.$$

All these sums depend on the nontrivial zeros $\varrho = \beta + i\gamma$ $(0 < \beta < 1)$ of $\zeta(s)$. The corresponding formula has the simplest character in the case of $\Delta_4(x)$ where the formula of Riemann–von Mangoldt states: if

$$(1.5)\qquad \tilde{\Delta}_4(x) = \psi_0(x) - x = \frac{\psi(x+0)+\psi(x-0)}{2} - x \qquad (x > 1)$$

then we have

$$(1.6)\qquad \tilde{\Delta}_4(x) = -\sum_{\varrho} \frac{x^{\varrho}}{\varrho} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) - \frac{\zeta'}{\zeta}(0)$$

where $\varrho = \beta + i\gamma$ stand for the zeros with $0 < \beta < 1$. This shows that

$$(1.7)\qquad \Delta_4(x) = -\sum_{\varrho} \frac{x^{\varrho}}{\varrho} + O(\log x).$$