ACTA ARITHMETICA
XXXVI (1980)

On a problem of E. Landau

by

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1. The starting point of these investigations was the following remark of Landau in his *Handbuch der Lehre von der Verteilung der Primzahlen* from 1909: "... Die Tatsache, daß $\sum_{\varrho} x^{\varrho}/\varrho$ gerade in der Nähe der Primzahlen und der höheren Primzahlpotenzen und sonst in der Nähe keiner Stelle >1 ungleichmässig konvergiert, deutet auf einen arithmetischen Zusammenhang zwischen den komplexen Wurzeln ϱ der Zetafunktion und den Primzahlen hin. Ich habe keine Ahnung, worin derselbe bestecht." Much later in 1930 Titchmarsh wrote in his *Cambridge Tract* much less dramatically (and more cautiously) after exposing Landau's results that "... It is clear that the numbers ϱ are closely connected with prime numbers. No more explicit relation between them that what was given by the above formulae has been discovered."

These sentences reflect the situation (valid even today) that we know a lot about primes, a lot about the ϱ's, we can use the ϱ's to deduce fine properties of the primes but after all we do not *understand* why. The aim of the present note is — shortly expressed — to find another type of connection between finitely many ϱ's and finitely many primes and draw various conclusions from these. We shall omit the discussion of analogous results for other ζ-type functions.

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* Paul Turán died on 26. September 1976. He worked on this article in the very last days of his life. Up to formula (6.8), except for the footnote to the sentence following (6.2), it is presented here in his own words. The remaining exposition was prepared by János Pintz, based on the as yet unpublished manuscript of the forthcoming book of Paul Turán [4]. An abbreviated version of this paper without proofs is contained in [8].

Perhaps the best characterization of the personal courage and scientific devotion which made this article possible can be found in what Paul Turán himself wrote about the last work of another mathematician ([9]): "Being aware of a grave illness and still being able to concetrate on questions beyond life shows passion, devotion, energy and one gets the impression of a heroic last ditch fight against death."

2. The germ of the results we are going to discuss is contained in the following two theorems contained in my paper [1]. Throughout this paper let b be a constant with

(2.1) $$0 < b \leqslant 1,$$

$c(\delta, \varepsilon, \ldots)$ explicitly calculable positive functions of the parameters $\delta, \varepsilon, \ldots$ and c explicitly calculable positive constants not necessarily the same at different occurrences. Further, for

(2.2) $$N \leqslant N_1 < N_2 \leqslant 2N$$

let $Z(\tau, N_1, N_2)$ be defined by

(2.3) $$Z(\tau, N_1, N_2) = \sum_{N_1 \leqslant n \leqslant N_2} \Lambda(n) e^{-i\tau \log n},$$

$\Lambda(n)$ being the von Mangoldt symbol. Then we have

THEOREM A. *Suppose the inequality*

(2.4) $$|Z(\tau, N_1, N_2)| < c \frac{N \log^{100} N}{\tau^b}$$

holds with a positive constant c for all τ-values satisfying

(2.5) $$|\tau - T| \leqslant \tfrac{1}{2} T^b$$

and all (N_1, N_2)-pairs with

(2.6) $$\tfrac{1}{2} T^a \leqslant N \leqslant N_1 < N_2 \leqslant 2N.$$

Then $\zeta(s) \neq 0$ in the parallelogram

(2.7) $$\sigma \geqslant 1 - b^3/a^2, \quad |t - T| \leqslant \tfrac{1}{2} T^b \quad (s = \sigma + it)$$

provided $T > c$.

3. Further, we have

THEOREM B. *Suppose that with a $0 < B \leqslant 1$, $\frac{1}{2} \leqslant \theta < 1$, and $T > 10$, $\zeta(s)$ does not vanish in the parallelogram*

(3.1) $$\sigma > \theta, \quad |t - T| \leqslant \tfrac{1}{2} T^B.$$

Then the inequality

(3.2) $$|Z(\tau, N_1, N_2)| < \frac{c(\delta)}{B} \cdot \frac{N \log^3 N}{\tau^B}$$

holds for all τ's with

(3.3) $$|\tau - T| \leqslant \frac{1-\delta}{2} T^B$$

and for all pairs (N_1, N_2) satisfying

(3.4) $$T^{B/(1-\theta)} \leqslant N \leqslant N_1 < N_2 \leqslant 2N$$

provided $T > c(\delta)$, $0 < \delta < 1$.

Especially, if

(3.5) $$B = b, \quad \theta = 1 - b^3/a^2,$$

this gives

COROLLARY I. *If $0 < b \leqslant 1$, $b^3/a^2 \geqslant \frac{1}{2}$, $T > 10$, $0 < \delta < 1$ and $\zeta(s) \neq 0$ in the parallelogram (2.7), then the inequality*

(3.6) $$|Z(\tau, N_1, N_2)| < \frac{c(\delta)}{b} \cdot \frac{N \log^3 N}{\tau^b}$$

holds for all τ with

(3.7) $$|\tau - T| \leqslant \frac{1-\delta}{2} T^b$$

and for all pairs (N_1, N_2) satisfying

(3.8) $$T^{a^2/b^2} \leqslant N \leqslant N_1 < N_2 \leqslant 2N.$$

Corollary I and Theorem A are "essentially" of inverse character; this indicates already the importance of the finite exponential sums $Z(\tau, N_1, N_2)$ in connection with zero-free finite parallelograms in the critical strip.

4. Of Theorems A and B Theorem A is incomparably deeper. Still more surprising is Theorem C from my paper [1].

THEOREM C. *Suppose that for a*

(4.1) $$0 < b \leqslant 1, \quad b^3/a^2 \geqslant \tfrac{1}{2}$$

there is a single $\tau_0 \geqslant 2$ such that the inequality

(4.2) $$|Z(\tau_0, N_1, N_2)| < c \frac{N \log^{100} N}{\tau_0^b}$$

holds for all pairs (N_1, N_2) satisfying

(4.3) $$\tau_0^a \leqslant N \leqslant N_1 < N_2 \leqslant 2N.$$

Then $\zeta(s) \neq 0$ on the segment

(4.4) $$\sigma \geqslant 1 - b^3/a^2, \quad t = \tau_0.$$

Here I cannot prove the corresponding inverse theorem.

5. Theorems A, C and B, however interesting they are, offer in the form given above no conclusion on Landau's problem. They are all right with respect to the resulting zero-free domains; they are finite parallelograms or even segments. But to the occurring primes, actually all sufficiently large primes occur in Theorem A, *in the hypothesis*. Nevertheless two remarks will be helpful. First, in Theorem B the conclusion refers to *all* sufficiently large primes owing to (2.6). Secondly, in Theorem A inequality (2.4) is trivially satisfied if

$$c\,\frac{N\log^{100}N}{\tau^{b}} > N,$$

which is true for all

(5.1) $$N > \exp\left\{\left(2T^{b}\cdot\frac{1}{c}\right)^{1/100}\right\},$$

i.e. restriction (2.6) actually refers only to all pairs (N_1, N_2) satisfying only

(5.2) $$T^{a} \leqslant N \leqslant N_1 < N_2 \leqslant 2N \leqslant 2\exp(2T)^{b/101}.$$

So Theorem A — with (2.6) replaced by (5.2) — and Corollary I are "essentially" of converse character and also have the additional advantage that their hypotheses depend only on *finitely many primes*, resp. *finite parallelograms*. So they are already Landau-type results. However, interval (5.2) seems to be much too large; so the exciting question arises what are "the shortest" intervals for the pairs (N_1, N_2) which still can replace interval (5.2) without violating the surprising conclusion of Theorem A (or even Theorem C).

6. We dealt with such questions in our papers [2] and [3]. In [2] we proved that Theorem C is still valid in the form that $\zeta(s) \neq 0$ for

(6.1) $$\sigma \geqslant 1-b^2, \qquad \tau = \tau_0$$

provided $0 < b < c$, $T > c(b)$, and (2.6) resp. (2.5) is replaced by

(6.2) $$T^2 \leqslant N \leqslant N_1 < N_2 \leqslant 2N \leqslant T^8,$$

resp.

$$T \leqslant \tau_0 \leqslant 2T.$$

In [3] improvement of principal importance is shown (1), namely, that

(1) Actually in [3] only the corresponding theorem is stated and the proof will be contained in the forthcoming book of Paul Turán [4].

if $D \geqslant 4$ and is fixed, and further $0 < b < c(D)$, then if for a suitable $\tau_0 = c(D, b)$ and a $\tau > \tau_0$ the inequality

$$\tau^{D(1-b^{1/6})} \leqslant N \leqslant N_1 < N_2 \leqslant \min(2N, \tau^{D(1+b^{1/6})})$$

holds, then $\zeta(s) \neq 0$ on the segment

$$\sigma = 1-b^2, \qquad t = \tau.$$

The principal novelty is the occurrence of the essentially arbitrary number D. Its real importance lies in the fact that from it one could deduce the following relationship for the values of $Z(\tau, N_1, N_2)$ *independently* of ζ-roots. If $D \geqslant 4$ and fixed, if $0 < b < c(D)$ and for a suitable $c(D, b)$ for $T > c(D, b)$ and for all τ with

(6.3) $$T-\sqrt{T} \leqslant \tau \leqslant T+\sqrt{T}$$

the inequality

(6.4) $$|Z(\tau, N_1, N_2)| < c\,\frac{N\log^{100}N}{\tau^{b}}$$

holds for

(6.5) $$\tfrac{1}{2}T^{D(1-b^{1/6})} \leqslant N_1 \leqslant N_2 \leqslant \min(2N, 2T^{D(1+b^{1/6})}),$$

then the inequality

(6.6) $$|Z(\tau, M_1, M_2)| \leqslant c\,\frac{M\log^{10}M}{\sqrt{\tau}}$$

holds for the "τ-range"

(6.7) $$T-\tfrac{1}{2}\sqrt{T} \leqslant \tau \leqslant T+\tfrac{1}{2}\sqrt{T}$$

and the "(M_1, M_2)-range"

(6.8) $$T^{1/2b^2} \leqslant M \leqslant M_1 < M_2 \leqslant 2M.$$

This can be expressed shortly as follows: if the sums $Z(\tau, N_1, N_2)$ satisfy inequality (6.4) for the short prime-range (6.5) then they satisfy inequality (6.6) of the same type (which is in the case of $b < \frac{1}{2}$ even sharper) for the unbounded prime-range (6.8) (which is, however, as remarked in (5.1), (5.2), essentially bounded, but the domain in which (6.6) is non-trivial (see (5.2)) is incomparably larger than the interval (6.5)).

The fact described above shows that there is a connection between the distribution of primes in the interval (6.5) and the distribution of primes in the interval

(6.9) $$T^{1/2b^2} \leqslant p \leqslant \exp(T^{1/20})$$

(where (6.6) is non-trivial); it seems as if the distribution of primes in the interval (6.5) had an influence upon the distribution of primes in the far longer interval (6.9).

Further, it is interesting to note that the fact described above is an assertion containing exclusively primes; ζ-roots occur only in the proof of the statement; still I cannot see any possibility of proving it directly if we remain in the realm of integers.

7. As remarked already, Theorem A and Corollary I are essentially of inverse character qualitatively but not in all the quantitative respects. An inverse theorem for the sums $Z(\tau, N_1, N_2)$ which is more satisfactory also in the quantitative respects can be proved, however, by supposing the truth of Lindelöf's conjecture, i.e. of the assertion that for an arbitrarily small $\varepsilon > 0$ the inequality

$$|\zeta(s)| \leqslant c(\varepsilon) t^{\varepsilon} \tag{7.1}$$

holds.

In the following let $\frac{1}{4} > \delta > 0$ be arbitrarily small and let us fix it. About a and b we suppose that

$$6\delta^2 \log \frac{3}{\delta} \leqslant b \leqslant 1, \tag{7.2}$$

$$\delta \leqslant b/a \leqslant 1/2. \tag{7.3}$$

These imply

$$1/\delta \geqslant a \geqslant \delta^2. \tag{7.4}$$

In the following theorems and corollaries we always suppose the truth of Lindelöf's conjecture (7.1).

Then we assert the following

THEOREM D. *Suppose that with a $\tau_0 > 10$ the inequality*

$$|Z(\tau\ , N_1, N_2)| < c\,\frac{N \log^{100} N}{\tau_0^b} \tag{7.5}$$

holds for

$$\tau_0^a \leqslant N \leqslant N_1 < N_2 \leqslant \min(2N, \tau_0^{a(1+2\delta)}). \tag{7.6}$$

Then $\zeta(s) \neq 0$ on the segment

$$\sigma \geqslant 1 - \frac{b}{a}(1-2\delta), \quad t = \tau_0,$$

provided $\tau_0 > c(\delta)$.

Theorem D has also a "semi-local" form, which is naturally a trivial consequence of it, namely we have

COROLLARY II. *Suppose, for a $T > 10$, the validity of (7.5) for all τ_0 with*

$$|\tau_0 - T| \leqslant \tfrac{1}{2} T^b \tag{7.7}$$

and for all pairs (N_1, N_2) with

$$\tfrac{1}{2} T^a \leqslant N \leqslant N_1 < N_2 \leqslant \min(2N, T^{a(1+3\delta)}). \tag{7.8}$$

Then $\zeta(s) \neq 0$ in the parallelogram

$$\sigma \geqslant 1 - \frac{b}{a}(1-2\delta), \quad |t-T| \leqslant \frac{1}{2} T^b$$

provided $T > c(\delta)$.

To see the strength of Corollary II we shall reformulate Theorem B with the choice

$$\theta = 1 - \frac{b}{a}(1-2\delta), \quad B = b.$$

Then, for

$$0 < \frac{b}{a} < \frac{1}{2(1-2\delta)},$$

this gives

COROLLARY III. *Suppose for a $T > 10$ that $\zeta(s) \neq 0$ in the parallelogram*

$$\sigma \geqslant 1 - \frac{b}{a}(1-2\delta), \quad |t-T| \leqslant \frac{1}{2} T^b.$$

Then the inequality

$$|Z(\tau_0, N_1, N_2)| \leqslant \frac{c(\delta)}{b} \frac{N \log^3 N}{\tau_0^b}$$

holds for all pairs (N_1, N_2) with

$$T^{a/(1-2\delta)} \leqslant N \leqslant N_1 < N_2 \leqslant 2N$$

and all τ_0 with

$$|\tau_0 - T| \leqslant \frac{(1-\delta)}{2} T^b$$

provided $T > c(\delta)$.

The comparison of Corollaries II and III shows an almost completely inverse character, even in every quantitative respect. Therefore it would be highly interesting to prove Corollary II without supposing Lindelöf's conjecture.

On applying Corollary II and III in succession, the already mentioned phenomenon, namely that the distribution of primes in a short range "have influence" upon the distribution of primes in a far longer range, can also be stated in a more transparent form, supposing the truth of Lindelöf's conjecture. This is given by

COROLLARY IV. *If δ is an arbitrarily small fixed positive number, a, b satisfy (7.2)–(7.4) and for a $T > c(\delta)$ the inequality*

$$|Z(\tau_0, N_1, N_2)| < c\frac{N\log^{100}N}{\tau_0^b}$$

holds for all

$$|\tau_0 - T| \leqslant \tfrac{1}{2}T^b$$

and for all pairs (N_1, N_2) with

$$\tfrac{1}{2}T^a \leqslant N \leqslant N_1 < N_2 \leqslant \min(2N, T^{a(1+3\delta)}),$$

then the inequality

$$Z(\tau_0, N_1, N_2) < \frac{c(\delta)}{b}\frac{N\log^3 N}{\tau_0^b}$$

holds for all τ_0 with

$$|\tau_0 - T| \leqslant \frac{1-\delta}{2}T^b$$

and for all pairs (N_1, N_2) with

$$T^{a/(1-2\delta)} \leqslant N \leqslant N_1 < N_2 \leqslant 2N.$$

8. So far δ was arbitrarily small and fixed and the fixed pair (a, b) satisfied (7.2), (7.3). Now we fix besides δ also b and let $S_0(\delta, b)$ be the set of a-values satisfying (7.3) for which (7.5), (7.7) and (7.8) hold.

Further, let $S_1(\delta, b)$ be the set of a-values satisfying (7.3).

Put

(8.1) $$\inf_{a\in S_0(\delta,b)} a = a_0; \quad \inf_{a\in S_1(\delta,b)} a = a_1.$$

Corollary II shows that if S_0 is non-empty, then immediately $\zeta(s) \neq 0$ in the parallelogram

(8.2) $$D_1\colon\ \sigma > 1 - \frac{b}{a_0}(1-2\delta), \qquad |t-T| \leqslant \tfrac{1}{2}T^b.$$

This is, in a sense, the best possible, if $a_0 > (1+\delta)a_1$. Namely, let us suppose that $\zeta(s) \neq 0$ would hold in the somewhat larger parallelogram

(8.3) $$D_2\colon\ \sigma > 1 - \frac{b}{a_0}(1+\delta), \qquad |t-T| \leqslant \frac{1}{2(1-\delta)}T^b.$$

Then we could apply Theorem B with

$$\theta = 1 - \frac{B}{a_0}(1+\delta), \qquad B = b + \frac{\log\dfrac{1}{1-\delta}}{\log T}.$$

This would imply that for $T > c(\delta)$ the inequality

$$|Z(\tau_0, N_1, N_2)| < \frac{c(\delta)}{b}\frac{N\log^3 N}{\tau_0^b}$$

holds for all τ_0 with

$$|\tau_0 - T| \leqslant \tfrac{1}{2}T^b$$

and for all pairs (N_1, N_2) with

$$T^{a_0/(1+\delta)} \leqslant N \leqslant N_1 < N_2 \leqslant 2N.$$

But this means, owing to the definition of $S_0(\delta, b)$, $S_1(\delta, b)$ and $a_0 > (1+\delta)a_1$, that

$$\frac{a_0}{1+\delta} \in S_0(\delta, b),$$

which contradicts the definition of a_0 in (8.1).

Hence we obtain (supposing Lindelöf's conjecture) the interesting

COROLLARY V. *If δ is arbitrarily small, b does not violate (7.2), both are fixed, and $a_0 > a_1(1+\delta)$ (both defined in (8.1)), then $\zeta(s)$ does not vanish in the parallelogram D_1 (see (8.2)) but has certainly a zero in the "narrow" frame*

$$D_2 \setminus D_1$$

(with D_2 in (8.3)).

9. As to Landau's problem, the following picture can be formed, based on the above-mentioned results. An essential role is played by the finite sums $Z(\tau_0, N_1, N_2)$ containing primes only from a finite interval of a surprisingly small size, depending only on τ_0, from which one can infer the existence of a finite zero-free segment ($\sigma > d$, $t = \tau_0$) where the size of the segment depends on the "strength of the mod 1 interference behaviour of the numbers $\frac{\tau_0}{2\pi}\log p$". Naturally if τ_0 runs in an interval then one can infer the existence of finite zero-free parallelograms. Moreover (supposing Lindelöf's conjecture), one of these zero-free parallelograms is the best possible in the sense that a "slightly bigger" parallelogram is no longer zero-free, which shows a very interesting phenomenon, namely that from the behaviour of primes of a finite interval one can locate a ζ-root in a well-defined relatively small finite domain.

10. Now we turn to the proof of Theorem D. We shall need the following

LEMMA 1. *For an integer $k \geqslant 2$, real γ and positive λ let*

$$h_k(\gamma, \lambda) \overset{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{(2)} \left(\frac{e^{\lambda w} - e^{-\lambda w}}{2\lambda w}\right)^k e^{\gamma w} dw. \tag{10.1}$$

Then we have in the case of $|\gamma| > k\lambda$

$$h_k(\gamma) = 0 \tag{10.2}$$

and in the case of $|\gamma| \leqslant k\lambda$

$$|h_k(\gamma, \lambda)| \leqslant 1/\lambda. \tag{10.3}$$

Further, $h_k(\gamma, \lambda)$ is monotonically increasing in γ if $-k\lambda \leqslant \gamma \leqslant 0$ and monotonically decreasing in γ if $0 \leqslant \gamma \leqslant k\lambda$.

For the proof of this see [5], Lemma I and [6], Lemma II.

In the course of proof a very important role will be played by the so-called second main theorem of the powersum theory, which we state here in a special case (and in a slightly modified form) as

LEMMA 2. *If $z_1, \ldots, z_n$ are arbitrary complex numbers, and $n \leqslant N$ and m is an arbitrary positive real number, then there exists a positive integer k^* satisfying*

$$m \leqslant k^* \leqslant m + N \tag{10.4}$$

such that

$$\left|\sum_{i=1}^{n} z_i^{k^*}\right| \geqslant \left(\frac{1}{8e\left(\frac{m}{N}+1\right)}\right)^N |z_1|^{k^*}. \tag{10.5}$$

For the proof see V.T. Sós and P. Turán [7].

Finally we note that (7.2)–(7.4) implies

$$\max\left(\frac{1}{b}, \frac{1}{a}, b, a\right) < c(\delta). \tag{10.6}$$

11. Let

$$s_0 \overset{\text{def}}{=} 1 + \frac{\delta b}{2a} + i\tau_0 \overset{\text{def}}{=} 1 + \mu + i\tau_0, \tag{11.1}$$

and let k be an integer restricted at present only by

$$\frac{b}{1-\delta/3} \log\tau_0 \leqslant k \leqslant \left(1 + \frac{2\delta}{3}\right) b \log\tau_0. \tag{11.2}$$

(The length of this interval is certainly $\geqslant 1$ if $\tau_0 > c(\delta)$.)

Further, let

$$J(s_0) = -\frac{1}{2\pi i} \int\limits_{(2)} \varphi(w)^k \frac{\xi'}{\xi}(s_0 + w) dw \tag{11.3}$$

where

$$\varphi(w) = e^{\frac{a}{b}w} \frac{e^{\frac{\delta a}{3b}w} - e^{-\frac{\delta a}{3b}w}}{2\frac{\delta a}{3b} w}. \tag{11.4}$$

Using the Dirichlet series of ξ'/ξ we obtain from Lemma 1

$$J(s_0) = \sum_{n \in I} \frac{\Lambda(n)}{n^{i\tau_0}} \frac{h_k\left(\frac{ka}{b} - \log n, \frac{\delta a}{3b}\right)}{n^{1+\mu}}, \tag{11.5}$$

where I stands for the interval

$$\left[\exp\left(k\frac{a}{b}\left(1 - \frac{\delta}{3}\right)\right), \exp\left(k\frac{a}{b}\left(1 + \frac{\delta}{3}\right)\right)\right]. \tag{11.6}$$

Let us note that owing to (11.2) and $0 < \delta < 1/4$

$$I \subset [\tau_0^a, \tau_0^{a(1+2\delta)}], \tag{11.7}$$

and thus we can make use of our assumption (7.5) and (7.6).

12. In order to use (7.5) and (7.6) we split the sum in (11.5) in at most

$$\left[\frac{2k\frac{a}{b}, \frac{\delta}{3}}{\log 2}\right] + 1 < 2a\delta \log\tau_0 < c(\delta)\log\tau_0 \tag{12.1}$$

partial sums of the form

$$\sum_{N_1 \leqslant n \leqslant N_2}, \quad N_2 < 2N_1 \tag{12.2}$$

owing to (10.6), (11.2) and (11.6).

This gives

$$|J(s_0)| < c(\delta)\log\tau_0 \max\left|\sum_{N_1 \leqslant n \leqslant N_2} \frac{\Lambda(n)}{n^{i\tau_0}} \frac{h_k\left(\frac{ka}{b} - \log n, \frac{\delta a}{3b}\right)}{n^{1+\mu}}\right| \tag{12.3}$$

where max refers to

$$\exp\left(k\frac{a}{b}\left(1-\frac{\delta}{3}\right)\right) \leqslant N_1 < N_2 \leqslant \min\left(2N_1,\ \exp\left(k\frac{a}{b}\left(1+\frac{\delta}{3}\right)\right)\right). \tag{12.4}$$

Denoting the inner sum in (12.3) by H and introducing $g(x)$ by

$$g(x) = \sum_{N_1 \leqslant n \leqslant x} \frac{\Lambda(n)}{n^{i\tau_0}}, \tag{12.5}$$

we get the representation

$$H = \int_{N_1}^{N_2} \frac{h_k\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)}{x^{1+\mu}}\, dg(x). \tag{12.6}$$

Integration by parts, use of (10.3), (10.6) and (7.5)–(7.6) give

$$|H| < c(\delta)\frac{N_2\log^{100}N_2}{\tau_0^b}\left(\frac{1}{N_1^{1+\mu}} + \int_{N_1}^{N_2}\left|\frac{d}{dx}\frac{h_k\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)}{x^{1+\mu}}\right|dx\right)$$

$$< c(\delta)\frac{N_2\log^{100}N_2}{\tau_0^b}\left(\frac{1}{N_1^{1+\mu}} + c(\delta)\int_{N_1}^{N_2}\left|\frac{d}{dx}\frac{1}{x^{1+\mu}}\right|dx + \right.$$

$$\left. + \frac{1}{N_1^{1+\mu}}\int_{N_1}^{N_2}\left|\frac{d}{dx}h\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)\right|dx\right)$$

$$< c(\delta)\frac{N_2\log^{100}N_2}{\tau_0^b}\frac{1}{N_1^{1+\mu}}\left(1+\int_{N_1}^{N_2}\left|\frac{d}{dx}h_k\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)\right|dx\right).$$

Owing to Lemma 1 the function $h_k\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)$ consists of at most two monotonic parts and by (10.3) and (7.3)

$$\left|h_k\left(\frac{ka}{b}-\log x, \frac{\delta a}{3b}\right)\right| \leqslant \frac{3b}{a\delta} \leqslant \frac{3}{2\delta}, \tag{12.8}$$

and thus using also $N_2 \leqslant \min\ (2N_1, \tau_0^{a(1+2\delta)})$ we get from (12.7) and (10.6)

$$|H| < c(\delta)\frac{\log^{100}N_2}{\tau_0^b} < c(\delta)\big(a(1+2\delta)\big)^{100}\frac{\log^{100}\tau_0}{\tau_0^b} < c(\delta)\frac{\log^{100}\tau_0}{\tau_0^b}. \tag{12.9}$$

Thus we obtain from (12.1), (12.3) and (12.9) the final estimate for $J(s_0)$ as

$$|J(s_0)| < c(\delta)\frac{\log^{101}\tau_0}{\tau_0^b}. \tag{12.11}$$

13. Next we shift the line of integration in (11.3) to $\operatorname{Re} w = -2-\mu = -u$.

Using the well-known estimate

$$\zeta(-1+it) = O\big(\log(|t|+2)\big), \tag{13.1}$$

we get for this integral the upper bound

$$I = c\int_{-\infty}^{\infty}\log(2+|\tau_0+v|)\exp\left(-k\frac{a}{b}u\right)\left|\frac{e^{\frac{\delta a}{3b}(-u+iv)} - e^{-\frac{\delta a}{3b}(-u+iv)}}{2\frac{\delta a}{3b}(-u+iv)}\right|^k dv. \tag{13.2}$$

We shall use the simple inequality

$$\left|\frac{e^z-e^{-z}}{2z}\right| = \left|\frac{1}{2}\int_{-1}^{1}e^{zr}dr\right| < e^{|\operatorname{Re} z|}. \tag{13.3}$$

We write the last factor in the integrand in (13.2) as

$$|\ \ |^k = |\ \ |^{k-2}\,|\ \ |^2$$

and, applying (13.3) to the first factor, we get

$$|I| < c\exp\left(-\left(1-\frac{\delta}{3}\right)^{k-2}\frac{a}{b}2\right)\times \tag{13.4}$$

$$\times\int_{-\infty}^{\infty}\log(2+\tau_0+|v|)\frac{\left|e^{-\frac{a}{b}\left(2+\frac{\delta}{3}\right)u}+e^{-\frac{a}{b}\left(2-\frac{\delta}{3}\right)u}\right|^2}{\left(2\frac{\delta a}{3b}\right)^2(u^2+v^2)}\,dv,$$

which for $\tau > c(\delta)$ owing to (7.2), (7.3) and (11.2) is

$$< c(\delta)\tau_0^{-1,99a}\int_{-\infty}^{\infty}\frac{\log(2+\tau_0+|v|)}{4+v^2}\,dv \tag{13.5}$$

$$< c(\delta)\tau_0^{-1,99a}\log\tau_0 < c(\delta)\tau_0^{-3b}\log\tau_0.$$

We shall make use of the simple fact that from (11.4)

$$|\varphi(w)| \leqslant \frac{1}{\frac{\delta a}{3b}|\operatorname{Im} w|} \quad \text{for} \quad \operatorname{Re} w \leqslant 0. \tag{13.6}$$

Thus for the residuum in $w = 1-s_0$ we get $\varphi^k(1-s_0)$, which is by (13.6) less in absolute value than

$$\left(\frac{1}{\frac{\delta a}{3b}\tau_0}\right)^k \leqslant \left(\frac{3}{2\delta}\cdot\frac{1}{\tau_0}\right)^k < 4^{-k} < \tau_0^{-b} \tag{13.7}$$

for $\tau_0 > 6/\delta$, owing to (7.3), (11.2) and $\delta < 1/4$.

Hence we obtain from (12.10), (13.5) and (13.7), writing

$$\varphi(\varrho - s_0) = z_\varrho, \tag{13.8}$$

the inequality

$$\Big|\sum_\varrho z_\varrho^k\Big| < c(\delta)\tau_0^{-b}\log^{101}\tau_0 \quad \text{for} \quad \tau_0 > c(\delta). \tag{13.9}$$

14. Next we consider the contribution of the zeros with

$$|\gamma - \tau_0| \geqslant \frac{18\,b}{\delta\;a}. \tag{14.1}$$

(13.6) gives for

$$|\gamma - \tau_0| \geqslant \frac{3b}{\delta a}\,\nu \tag{14.2}$$

the inequality

$$|\varphi(\varrho - s_0)| = |z_\varrho| \leqslant 1/\nu. \tag{14.3}$$

Hence

$$\sum_{\frac{3b}{\delta a}\nu \leqslant |\gamma-\tau_0| < \frac{3b}{\delta a}(\nu+1)} |z_\varrho| < c\frac{3b}{\delta a}\log\left(\tau_0 + \frac{3b}{\delta a}(\nu+1)\cdot\frac{1}{\nu^k}\right) < c(\delta)\;\frac{\log(\tau_0+\nu)}{\nu^k} \tag{14.4}$$

and thus

$$\sum_{|\gamma-\tau_0|\geqslant 18b/\delta a} |z_\varrho|^k < c(\delta)\sum_{\nu=6}^{\infty}\frac{\log(\tau_0+\nu)}{\nu^k} < c(\delta)\;\frac{\log\tau_0}{6^{k-1}}, \tag{14.5}$$

which is

$$< c(\delta)\tau_0^{-b}\log\tau_0 \tag{14.6}$$

owing to (11.2) and $\delta < 1/4$.

15. Next we have to eliminate the contribution of the zeros with

$$|\gamma - \tau_0| \leqslant \frac{18\,b}{\delta\;a}, \qquad \beta \leqslant \frac{1}{2} + \frac{\delta\,b}{2\,a}. \tag{15.1}$$

For this purpose it is enough to remark that for $\operatorname{Re} w \leqslant -U < 0$ we have, owing to (11.4) and (13.3),

$$|\varphi(w)| \leqslant \exp\left(-\frac{a}{b}\left(1-\frac{\delta}{3}\right)U\right), \tag{15.2}$$

and so from (11.1) we get for the zeros in (15.1)

$$|z_\varrho|^k = |\varphi(\varrho - s_0)|^k \leqslant \exp\left(-\frac{a}{b}\left(1-\frac{\delta}{3}\right)\frac{1}{2}\,k\right), \tag{15.3}$$

which in turn, owing to (11.2) and (7.3), is

$$\leqslant \tau_0^{-a/2} \leqslant \tau_0^{-b}. \tag{15.4}$$

Since the number of terms in (15.1) is

$$< c(\delta)\log\tau_0, \tag{15.5}$$

we obtain for the contribution of all zeros with (15.1) the upper bound

$$c(\delta)\log\tau_0\cdot\tau_0^{-b}. \tag{15.6}$$

Thus from (13.9), (14.5) and (15.6) we get the basic inequality

$$|Z| \stackrel{\text{def}}{=} \Big|\sum{}' z_\varrho^k\Big| < c(\delta)\tau_0^{-b}\log^{101}\tau_0, \tag{15.7}$$

where the summation is extended to the ϱ's satisfying

$$\beta > \frac{1}{2} + \frac{\delta\,b}{2\,a}, \qquad |\gamma - \tau_0| < \frac{18b}{\delta a}. \tag{15.8}$$

16. In order to get a lower bound for the power sum Z in (15.7) we shall use Lemma 2. We choose

$$m = \frac{b}{1-\delta/3}\;\log\tau_0 \tag{16.1}$$

and we shall need an upper bound N for the number of zeros satisfying (15.8). Here — and only here — we shall use the Lindelöf conjecture or rather its well known consequence, namely that, denoting the number of zeros in the parallelogram

$$\sigma \geqslant \tfrac{1}{2} + \eta, \qquad T \leqslant t \leqslant T+1 \tag{16.2}$$

(η positive, arbitrarily small) by $H_\eta(T)$, we have

$$\lim_{T\to\infty}\frac{H_\eta(T)}{\log T} = 0. \tag{16.3}$$

This implies at once that for $\tau_0 > c(\delta)$ we may choose

$$N = \delta^3\log\tau_0. \tag{16.4}$$

Since $\delta < 1/4$ we get by (7.2)

$$\delta^3 < \left(1 + \frac{2\delta}{3} - \frac{1}{1-\delta/3}\right)b, \tag{16.5}$$

and so the integer k^* in (10.4) will surely satisfy (11.2) in view of (16.1), (16.4) and (16.5).

Suppose that Theorem D is false and there is a $\varrho_1 = \beta_1 + i\tau_0$ zero with

$$\beta_1 \geqslant 1 - \frac{b}{a}(1-2\delta). \tag{16.6}$$

Since for real

$$\left|\frac{e^x - e^{-x}}{2x}\right| \geqslant 1, \tag{16.7}$$

we have by (11.1), (11.4) and (16.6)

$$|z_{\varrho_1}|^{k^*} = |\varphi(\varrho_1 - s_0)|^{k^*} \geqslant \exp\left(k^* \frac{a}{b}(\beta_1 - 1 - \mu)\right) \tag{16.8}$$

$$\geqslant \exp\left(-k^*(1-\tfrac{3}{2}\delta)\right) \geqslant \exp\left(-(1-\tfrac{5}{6}\delta)\log\tau_0\right) = \tau_0^{-b+b\delta\frac{5}{6}}.$$

On the other hand, by $0 < \delta < 1/4$ and (7.2) we have from (16.1) and (16.4)

$$8e\left(\frac{m}{N}+1\right) = 8e\left(\frac{b}{\left(1-\frac{\delta}{3}\right)\delta^3}+1\right) < 8e\left(\frac{1}{\frac{11}{12}\delta^3}+1\right) < \left(\frac{3}{\delta}\right)^3, \tag{16.9}$$

and so

$$\left(\frac{1}{8e\left(\frac{m}{N}+1\right)}\right)^N > \left(\frac{1}{\left(\frac{3}{\delta}\right)^3}\right)^{\delta^3\log\tau_0} = \tau_0^{-3\delta^3\log\frac{3}{\delta}} \geqslant \tau^{-\frac{b\delta}{2}}. \tag{16.10}$$

Thus we infer from Lemma 2 the existence of an integer k^* satisfying (10.4) and thus, by (16.5), also (11.2), such that for the power sum Z in (15.7) the inequality

$$|Z| > \left(\frac{1}{8e\left(\frac{m}{N}+1\right)}\right)^N |z_1|^{k^*} \tag{16.11}$$

holds, from which by (16.8) and (16.10) we get

$$|Z| > \tau_0^{-\frac{b\delta}{2}} \cdot \tau_0^{-b+\frac{5}{6}b\delta} = \tau_0^{-b+\frac{b\delta}{3}}. \tag{16.12}$$

This and (15.7) together give

$$\tau_0^{-b+\frac{b\delta}{3}} < c(\delta)\tau_0^{-b}\log^{101}\tau_0, \tag{16.13}$$

i.e.

$$\tau_0^{\frac{b\delta}{3}} < c(\delta)\log^{101}\tau_0. \tag{16.14}$$

Now, by (7.2), this implies

$$\tau_0^{2\delta^3\log\frac{3}{\delta}} < c(\delta)\log^{101}\tau_0, \tag{16.15}$$

which is surely false for $\tau_0 > c(\delta)$. This contradiction proves Theorem D.

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Received on 2. 1. 1978 (1017)