we use Stark's inequality

$$L(1, \chi) \geq \frac{C}{\log(n)} \cdot \frac{2\alpha}{\alpha} \cdot \left(1 - \frac{\alpha}{\alpha}ight)^{1/n} \cdot \frac{1}{c_k(\alpha)}$$

for every $\alpha$, with $1 - (8\log(\sqrt{f}))^{-1} \leq \alpha \leq 2$, where $d, c_k$ denote the discriminant and the Dedekind zeta-function for $k$, and $f$ is defined by $|\text{disc}(k(\sqrt{f}))| = 2k$; $C$ is an effectively computable constant, $g(n) = n!$ and $g(n) = n$, if one assumes the Generalized Riemann Hypothesis (see [7], [3], (4.2) ff., cf. also [5], (9) there it should read $2d$ instead of $2d$).

Assuming the Generalized Riemann Hypothesis, one has

$$d_k(\alpha) \geq 188 + o(1) \quad \text{for} \quad n \to \infty$$

(3)

If $G$ has only one class, this implies $M(G) \leq 1/2$. Comparing this with (1), (2), (3) and with [3], (4.3) to (4.9) ff., one gets (i) and (iii) of the theorem for $k$ with $n > 2$. The case $k = Q$ is well known. For quadratic fields $k$ one has only finitely many ideal numbers $(b, k)$, as follows using the Brauer–Siegel Theorem instead of (2). Similarly one gets (ii) of the theorem (for a similar theorem and proof, see [1], Satz 20).

References


On linear forms of a certain class of $G$-functions and $p$-adic $G$-functions

by

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1. Introduction. In the present paper we employ the ideas of Baker ([2], [3], Chapter 10) and the Siegel–Shidlovskii theory ([3], [10], [15], [16]) to examine the linear forms of certain $G$-functions. We have two main aims, firstly to generalize the results of Galochkin [8], and thus obtain for $G$-functions an analogue of Makarov's [11] result concerning $E$-functions, and secondly to find $p$-adic analogues to the results obtained. Our studies have been motivated by a recent paper of Flicker [6], where he obtains $p$-adic analogues of the results of Galochkin [7] and Nurmagomedov [13]. Here we shall obtain similar $p$-adic analogues in connection with the papers [2], [4], [5], [8], [11], [17], [18]. In particular we shall give lower bounds in terms of all the coefficients for the $p$-adic valuations of linear forms in the values of certain $G$-functions.

I should like to express my thanks to the referee for valuable suggestions.

2. Main results. Let $I$ denote the field of rational numbers or an imaginary quadratic field. We consider $r$ systems of $G$-functions

$$f_{i1}(x), \ldots, f_{i2}(x), \quad \xi_i \geq 1, \quad i = 1, \ldots, r$$

(in [8] $\xi_i = 1$ ($i = 1, \ldots, r$)), and assume that these functions satisfy the corresponding systems of differential equations

$$y''_i = Q_{i0}(x) + \sum_{i=1}^{\xi_i} Q_{ij}(x)y_{ij}, \quad i = 1, \ldots, r, \quad j = 1, \ldots, \xi_i$$

where all $Q_{ij}(x) \in I(x)$. We thus immediately obtain, for $i = 0, 1, \ldots$,

$$y'_0 = Q_{00}(x) + \sum_{i=1}^{\xi_i} Q_{i0}(x)y_{ij}, \quad i = 1, \ldots, r, \quad j = 1, \ldots, \xi_i$$

where all $Q_{ij}(x) \in I(x)$.

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We assume that the functions (1) belong to the class $G(I, A, B, D)$ $(A, B, D$ positive constants, $B > 1, D > 1)$, by which we mean that these functions are of the form

$$f_q(z) = \sum_{\mathbf{r}} a_{\mathbf{r}} z^\mathbf{r}, \quad \mathbf{r} = 1, \ldots, r, \quad z = 1, \ldots, s, \quad a_{\mathbf{r}} \neq 0$$

where all $a_{\mathbf{r}} \in I$, and the following conditions are satisfied:

(i) There exists a constant $\gamma_1$ such that

$$|a_{\mathbf{r}}| \leq \gamma_1 A^{\mathbf{r}} \quad \gamma_1 = 1, \ldots, r, \quad j = 1, \ldots, s, \quad a_{\mathbf{r}} \neq 0, 1, \ldots,$$

(ii) There exist a constant $\gamma_2$ and a sequence $\{b_n\}$ of natural numbers, such that $b_n \leq \gamma_2 B^n (n = 0, 1, \ldots)$ and all the numbers $a_{\mathbf{r}} b_n (i = 1, \ldots, r, \quad j = 1, \ldots, s, \quad r = 0, \ldots, n)$ are integers in $I$;

(iii) There exist a constant $\gamma_3$, a sequence $\{d_n\}$ of natural numbers, and a polynomial $T(z) = I[z]$ with integer coefficients, not all zero, such that $d_n \leq \gamma_3 D^n (n = 0, 1, \ldots)$ and all the functions

$$\frac{d_n}{l!} [T(z)]^l Q_{G/1}(z), \quad l = 1, \ldots, r, \quad j = 1, \ldots, s, \quad v = 0, \ldots, s, \quad i = 0, \ldots, n,$$

are polynomials in $I[z]$ with integer coefficients.

The following notations are used,

$$S = s_1 + \cdots + s_r, \quad \delta = \max \{\deg T(z) + r, \deg T(z) + s(z), \}, \quad b = \max \{\deg T(z) + s(z), \},$$

where $\deg T(z)$ denotes the maximum of the absolute values of the coefficients of the polynomial $T(z)$. We shall obtain lower bounds for the absolute values of the linear forms

$$L(x) = a_{\mathbf{r}} + \sum_{i=1}^{s} \sum_{j=1}^{r} x_{ij} f_q(x),$$

where all $x_{ij}$ are integers in $I$, not all zero. Let us denote

$$h_i = \max \{1, |x_{ij}|\}, \quad i = 1, \ldots, r, \quad h_0 = \max \{|x_{0j}|, h_i\}.$$

Our main results are given in the following theorems and corollaries.

**Theorem 1.** Let the $G$-functions (1) belong to the class $G(I, A, B, D)$, $A > 1$, and assume that these functions and the function $f_0(z) = 1$ are linearly independent over $C(z)$. Let $L(x)$ be any linear form given by (4). Then there exists positive constants $\gamma_1, \gamma_2, \gamma_3$, depending only on $S$ and the functions (1), such that, if $a, \beta, \omega, \tau$ are positive numbers satisfying

$$\tau > (1 + S) (\delta + 2) a^{-1} - \beta > \gamma_2$$

then

$$|L(q^{-1})| > q^{-1} H^{-1} \left(1 + (d + 2) a^{-1} - \beta \right) \omega^{-1}$$

for all natural numbers $q$ satisfying

$$q > \max \{B + S + (d + 2), \}, \quad T(q^{-1}) \neq 0,$$

where $\lambda, H, E, F$ and $T$ are given by

$$\lambda = \gamma_2 + S(a + 1) \left(1 + (S + 1) x, \right), \quad T(q^{-1}) = \gamma_2 + S(a + 1) \left(1 + (S + 1) x, \right), \quad E = \gamma_2 + S(a + 1) \left(1 + (S + 1) x, \right), \quad F = \gamma_2 + S(a + 1) \left(1 + (S + 1) x, \right).$$

This theorem admits the following corollary.

**Corollary 1.** Let the functions (1) and a linear form $L(x)$ be as in Theorem 1. Let $\gamma_1, \gamma_2$ be given. Then there exist positive constants $\lambda, \gamma_1, \gamma_2$ depending only on $S$ and the functions (1), such that

$$|L(q^{-1})| > q^{-1} H^{-1} a^{-1}$$

for all natural numbers $q$ satisfying $q \geq C, \quad T(q^{-1}) \neq 0$.

We note that if $\omega = \tau$ are positive numbers satisfying

$$\omega > S(a + 2), \quad \tau > (\delta + 2) (a + 2) \omega^{-1},$$

then we find positive numbers $a = a(\omega, \tau, \gamma_4)$ and $\beta = \beta(\omega, \tau, \gamma_4)$ such that (5) is valid. Thus the special case $s_i = 1 (i = 1, \ldots, r)$ of Theorem 1 gives an analogue of Galochkin's [8] result.

Now let $Q$ be any fixed prime number, $Q_p$ the $p$-adic completion of $Q$, $Q_p$ a $p$-adically complete algebraically closed extension of $Q_p$, and let $|\cdot|_p$ denote the $p$-adic valuation on $Q_p$ satisfying $|p|_p = p^{-1}$. If the functions (1) belong to the class $G(Q, A, B, D)$, then the power series of these functions converge $p$-adically for all $|z|_p < B^{-1}$, since

$$1 > |b_i a_{ij}|_p = |b_i|_p |a_{ij}|_p \geq |a_{ij}|_p b_i^{-1},$$
which gives, for all \( i = 1, \ldots, r, j = 1, \ldots, s_i, \)

\[
\left| a_{ij} \right|_p \leq B \gamma_{ij}^{r\delta}, \quad v = 0, 1, \ldots
\]

The \( p \)-adic \( G \)-functions (1) are thus defined at least for all \( |z|_p < B^{-1} \).

In the following theorem we give a lower bound for \( |L(z)|_p \) at some of these points \( z \).

**Theorem 2.** Let the \( G \)-functions (1) belong to the class \( G(Q, A, B, D), AB \geq 1 \), and assume that these functions and the function \( f_0(z) = 1 \) are linearly independent over \( C_{\mathbb{Q}}(z) \). Assume that \( L(z) \) is any linear form (4) with \( a_0 \in \mathbb{Z} \). Then there exist positive constants \( I_1, I_2, I_3, \) depending only on \( S \) and the functions (1), such that if \( a, b, \omega > 2, r \) and \( \delta \) are positive numbers satisfying

\[
(11) \quad (r - (1 + (S + 1))\alpha (d + 2 + b(\omega - 2)) \omega^{-1} - \beta) a > I_1,
\]

then

\[
(12) \quad \left| L \left( \frac{q}{q_1} \right) \right|_p > q^{-\lambda_1 - \lambda_2^{-1}-\omega^{-1}(\omega - 1) - \omega^{-1} - \omega^{-1} - \omega^{-1}}
\]

for all natural numbers \( q, q_1 \) satisfying

\[
(13) \quad (q, q_1) = 1, \quad q > \left( Mq_1 \right)^{(1 + (S + 1))\omega^{-1}},
\]

where \( \lambda_1, \lambda_2, K, \) and \( M \) are given by

\[
H_1 = h_0 \prod_{i=1}^{r} h_i^5, \quad \lambda_1 = I_1 + (S + 1)(\omega + 1)(1 + (S + 1)\alpha),
\]

\[
K = I_1^{-1} - 2beDS(d + 1)\omega^{-1}(\omega - 1) - \omega^{-1} - \omega^{-1} - \omega^{-1},
\]

\[
M = KB^{(1 - \omega^{-1})}.
\]

This theorem implies the following corollary.

**Corollary 2.** Let the functions (1) and a linear form \( L(z) \) be as in Theorem 2. Let \( \delta, \omega, 0 < \delta < 1 \), be given. Then there exist positive constants \( \delta, \lambda_1, \lambda_2, K, \) depending only on \( \delta, \omega \) and the functions (1), such that

\[
(15) \quad \left| L \left( \frac{q}{q_1} \right) \right|_p > q^{-\lambda_1 - \lambda_2^{-1}-\omega^{-1}}
\]

for all natural numbers \( q, q_1 \) satisfying \( (q, q_1) = 1, q > (C_1 q_1)^{\delta}, \left| q \right|_p < q^{-1+\delta}, \)

\[
T(q/q_1) \neq 0.
\]

We note that if we wish to find a lower bound depending only on \( \max \{h_i\} \), then the proof of Theorem 2 can be used with weaker conditions than (13). We are thus able to improve on a certain special case of Flicker's Theorem [5]. This will be given in Section 8.

**3. Lemmas I.** We may now proceed to the lemmas needed in the proof of Theorem 1. We assume in all lemmas that the functions (1) satisfy the conditions given in Theorem 1. Let \( [x] \) denote the integer part of \( x \), and let \( \omega > S(d + 2) \) be a positive number. By \( \gamma, \gamma_1, \ldots \) we denote positive constants depending only on \( S \) and the functions (1).

**Lemma 1.** Let \( a_i (i = 1, \ldots, r) \) be natural numbers, and let \( n_0 = n = \max \left\{ n_i \right\} \geq 3, N = n_1 + \ldots + n_r \). There exist \( S+1 \) polynomials

\[
C_i(z) = \sum_{\mu_1 + \ldots + \mu_i = n_i} a_i \mu_i z^{\mu_i}, \quad i = 0, j = 0; i = 1, \ldots, r, j = 1, \ldots, n_1,
\]

not all identically zero, with the following properties:

(1') All \( C_i(z) \) are integers in \( I, \) and satisfy

\[
|c_{ij} | \leq \gamma_{ij}^{(a_1 - 1)\log \sigma}, \quad E_0 = (AB)^{(2\omega - 1)} e^{-1};
\]

(2') We have

\[
R_0(z) = C_0(z) + \sum_{i=1}^{r} \sum_{j=1}^{n_i} C_i(z) f_i(z) = \sum_{i=0}^{\infty} g_i z^i,
\]

where

\[
g_0 = 0, \quad v = 0, 1, \ldots, v + N - \left\lfloor \frac{N}{\omega} \right\rfloor - 2,
\]

and, for all \( |z| < 2A^{-1}, \)

\[
|R_0(z)| \leq \gamma_{ij}^{(a_1 - 1)\log \sigma} F_0|z|^{n_0 + N - \left\lfloor \frac{N}{\omega} \right\rfloor - 1}, \quad F_0 = E_0 e^{-1}. \]

**Proof.** We need only refer to the proof of [3], Lemma 2, which uses Siegel's lemma.

We then construct new linear forms from the form \( R_0(z) \) as follows,

\[
R_k(z) = \frac{A_k}{k!} (T(z))^k R_0(z), \quad k = 1, 2, \ldots
\]

The use of (3) gives an equality

\[
R_k(z) = \frac{A_k}{k!} (T(z))^k \left\{ C_0^0(z) + \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{\mu_1 + \ldots + \mu_i = n_i} A_i \left( \frac{A_{ij}}{k!} (T(z))^k C_i^{a_i - 1}(z) Q_{ij}(z) + \sum_{\mu_1 + \ldots + \mu_i = n_i} A_i \left( \frac{A_{ij}}{k!} (T(z))^k C_i^{a_i - 1}(z) Q_{ij}(z) f_i(z) \right), \quad k = 1, 2, \ldots
\]
from which it follows that

\begin{equation}
R_k(z) = C_{0k}(z) + \sum_{i=1}^{r} \sum_{j=1}^{r} C_{ij}(z) f_{ij}(z), \quad k = 0, 1, \ldots,
\end{equation}

where

\begin{align*}
C_{0k}(z) &= C_{00}(z), \quad C_{k0}(z) = C_{ij}(z), \\
C_{0k}(z) &= \frac{d_k}{k!} (T(z))^k C_{0k}(z) + \\
&\quad + \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=0}^{k} \frac{C_{ij}^{(k-l)}(z)}{(k-l)!} (T(z))^{k-l} \frac{d_l}{l!} B_{ijl}(z),
\end{align*}

\begin{align*}
C_{Uk}(z) &= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=0}^{k} \frac{C_{ij}^{(k-l)}(z)}{(k-l)!} (T(z))^{k-l} \frac{d_l}{l!} B_{ijl}(z),
\end{align*}

for which we denote $B_{ijl}(z) = (T(z))^l Q_{ijl}(z) = \prod_{v=0}^{l} (T(z))^v q_{ijl}(z)$. For these $i, j, r, l$ we have, by (2) and (3),

\begin{equation}
Q_{ijl+1}(z) = Q_{ijl}(z) + \sum_{k=1}^{s} Q_{ijk}(z) Q_{ijk}(z),
\end{equation}

and thus we obtain the recursive formulae

\begin{equation}
B_{ijl+1}(z) = T(z) B_{ijl}(z) - T'(z) B_{tijl}(z) + \sum_{k=1}^{s} B_{ijk}(z) R_{ijk}(z),
\end{equation}

by means of which these functions $B_{ijl}(z)$ can be easily considered. We see that, for all $l = 0, 1, \ldots, B_{ijl}(z)$ is a polynomial of degree $\leq s_l$. Thus we have the estimates

\begin{equation}
\deg C_{0k}(z) \leq n+k, \quad \ord C_{0k}(z) \geq n-m_k-k,
\end{equation}

for the degrees and orders at $z = 0$ of the polynomials $C_{0k}(z)$ ($i = 0, j = 0; i = 1, \ldots, r, j = 1, \ldots, s_j; k = 0, 1, \ldots$).

For the sake of simplicity, let the functions $f_{00}(z) = 1$, in some order, be $G_0(z), G_1(z), \ldots, G_n(z)$, and for $k = 0, 1, \ldots, S$ let $m_k = n_k$, when $G_k(z)$ is one of the functions $f_{ij}(z)$ ($j = 1, \ldots, s_j; i = 0$ if $i = 0$). We then have, by (16) and (19),

\begin{equation}
R_k(z) = \sum_{i=0}^{n} G_{ik}(z) G_k(z), \quad k = 0, 1, \ldots,
\end{equation}

where $G_{ik}(z)$ are polynomials of degree $\leq n+k$, and the order of $G_{ik}(z)$ at $z = 0$ is at least $n-m_i-k$.

The following lemma is valid (the constants $\gamma_0, \gamma_S$ are given in the proof of the lemma).

**Lemma 2.** If $m_k \geq \lfloor N/\omega \rfloor + \gamma_0$ ($i = 0, \ldots, S$), then the determinant $D(z)$ of the matrix $G = (G_{ik}(z))_{k=0, \ldots, n}$ has the form

\begin{equation}
D(z) = s^{(S+1)-\lfloor N/\omega \rfloor - \gamma_S} D_1(z),
\end{equation}

where $D_1(z) \neq 0$ is a polynomial of degree $\leq \lfloor N/\omega \rfloor + \gamma_S$.

**Proof.** First we prove that $D(z) \neq 0$. For the general theory of this step we refer to [15] and [10], Chapter 4. We follow Galychkin's [8] deduction and write here those parts of the proof which will also be of use in the proof of Lemma 5.

Assume that $D(z) = 0$, and let $l+1 (< S+1)$ be the rank of $G$. By Lemma 1 we have $l \geq 0$. The matrix $G$ has a minor $\neq 0$ of order $l+1$. Assume that the functions $G_{ik}(z)$ ($i = 0, \ldots, S$) are numbered in such a way that this minor is

\begin{equation}
D_k(z) = \det(G_{ik}(z))_{k=0, \ldots, l} \neq 0.
\end{equation}

Further we may assume that $m_k = \max(m_k, \ldots, m_k)$. Using the general theory as in [8], p. 147, or [18], pp. 11-14, we obtain an inequality

\begin{equation}
\ord D_k(z) \geq \ord G_{i0}(z) \ldots G_{i(l-1)}(z) R_{il}(z) - \gamma_{11},
\end{equation}

The use of the properties of $G_{ik}(z)$ and $R_{ij}(z)$ gives us

\begin{equation}
\ord D_k(z) \geq \sum_{i=0}^{l-1} (m_i - m_i - 1) + n - \left\lfloor \frac{N}{\omega} \right\rfloor - l - \gamma_{11},
\end{equation}

\begin{equation}
\deg D_k(z) < (l+1)n + (S+1)^2 d.
\end{equation}

Since $\ord D_k(z) = \deg D_k(z)$, it follows that

\begin{equation}
m + \left\lfloor \frac{N}{\omega} \right\rfloor + \gamma_S > \sum_{i=0}^{S} m_i,
\end{equation}

where we denote $\gamma_S = (S+1)^2 d + 2$. Now $m_i = n$ for some $i = 1, \ldots, S$, and thus the above inequality is impossible if $m_i > \lfloor N/\omega \rfloor + \gamma_0$ ($i = 0, \ldots, S$). This gives $D(z) \neq 0$.

By choosing $\gamma_0 = (S+1)^2 d$ we can now establish the truth of Lemma 2 (as in [18], p. 15, for instance).
Let, for \( k = 0, 1, \ldots \),
\[
\begin{align*}
\tau_k(x) &= q^{k+1}r_k(x) = c_{0k}(x) + \sum_{i=1}^{\nu} \sum_{j=1}^{s_i} c_{ij}(x)f_j(x),
\end{align*}
\]
where \( q \) is a natural number and \( c_{ik}(x) = q^{i+k}C_{ik}(x) \). By using the above results we obtain the following lemma, which gives certain important properties of linear forms (22).

Lemma 3. Let \( n > [N/\omega] + \gamma \) (i = 1, \ldots, r). Let \( q \) be a natural number satisfying \( q > 2A \), \( T(q^{-1}) \neq 0 \), and denote \( \theta = q^{-1} \). There exist \( S+1 \) numbers \( k_0, k_1, \ldots, k_S \), such that \( k_0 + k_1 + \cdots + k_S \leq [N/\omega] + \gamma \), and the linear forms
\[
\tau_{k_0}(\theta), \tau_{k_1}(\theta), \ldots, \tau_{k_S}(\theta)
\]
are linearly independent. Further all the numbers \( c_{ik}(\theta) \) (i = 0, j = 0; i = 1, \ldots, r; j = 1, \ldots, \nu; k = 0, 1, \ldots, [N/\omega] + \gamma \) are integers in \( I \), and satisfy
\[
\begin{align*}
\max_i |c_{ik}(\theta)| &\leq (1 - \theta)^{-2} r \left( \frac{\log N}{e} \right)^{\nu(k)} \left( \frac{a_n}{e(k)} \right)^{\nu(k)},
\end{align*}
\]
where \( \nu(k) = \max(1, \min(n, k)) \). If \( N > 4\gamma \) and \( q > 4A \), then
\[
|\tau_k(\theta)| \leq (1 - \theta)^{-2} r \left( \frac{\log N}{e} \right)^{\nu(k)} \left( \frac{a_n}{e(k)} \right)^{\nu(k)}
\]
\[
\times q^{-N [N/\omega] + \gamma \nu(k) + \nu(k) + \nu(k)^2} \left( \frac{a_n}{e(k)} \right)^{\nu(k)} \left( 1 + \frac{N}{\nu} \right)^{\nu(k)},
\]
(23)
which gives
\[
|B_{ijkl}(\theta)| \leq (1 - \theta)^{-2} r \left( \frac{\log N}{e} \right)^{\nu(k)} \left( \frac{a_n}{e(k)} \right)^{\nu(k)},
\]
(25)
\[
|B_{ijkl}(\theta)| \leq (1 - \theta)^{-2} r \left( \frac{\log N}{e} \right)^{\nu(k)} \left( \frac{a_n}{e(k)} \right)^{\nu(k)},
\]
(26)

Since \( B_{ijkl}(\theta) \equiv 0 \), (25) is also valid for \( l = 0 \).

By Lemma 1 we get
\[
\max_{0 < \theta < A} \left| \frac{C_{ijkl}(\theta)}{(\theta - b)^l} \right| \leq \max_{0 < \theta < A} \left[ \left( \frac{a_n}{e(k)} \right)^{\nu(k)} \right]^{\nu(k)},
\]
and thus, by (iii), (17), (22), (25) and \( \max_{0 < \theta < A} \left| \frac{a_n}{e(k)} \right| \leq 5 \left( \frac{a_n}{e(k)} \right)^{\nu(k)} \), the inequality (23) immediately follows.

To prove (24) we put \( \Gamma_1 = k_0 + n - [N/\omega] - k_0 \), which follows from (2) of Lemma 1. Now \( \omega > 8(d + 2) \), \( N > 4\gamma \), and \( q > 4A \), and thus we obtain
\[
(1 - \Gamma_1) \theta < 2q^{-1} - (2A)^{-1}, \quad \Gamma_1 = 1, \ldots, [N/\omega] + \gamma.
\]
Therefore the circles \( |z - \theta| < \Gamma_1 \theta \) lie in the circle \( |z| < (2A)^{-1} \), and thus the use of Cauchy's integral formula
\[
\int_{|s| = (2A)^{-1}} \frac{B_{ijkl}(\theta)}{(z - \theta)^{\nu(k)}} \, ds
\]
(24)
(25)
(26)
(27)
(28)

We use the notation \( f(x) \equiv g(x) \) if
\[
|f(x)| \leq |g(x)|, \quad g(x) = \sum_{i=0}^{b} a_i x^i, \quad b \geq 0,
\]
and \( |a_i| \leq b \) (i = 0, 1, \ldots). Since \( B_{ijkl}(x) = T(x)Q_{ijkl}(x) \), we have, for all \( |x| < (2A)^{-1} \), \( B_{ijkl}(x) < b(1 - x)^{-1} \) (i = 1, \ldots, r; j = 1, \ldots, \nu; \nu = 0, \ldots, \nu). Using (15) we obtain, for all these \( i, j, \nu \),
\[
B_{ijkl}(x) < b(1 - x)^{-1} \int_0^{1 - b(1 - x)^{-1}} (\nu + S) \, ds, \quad l = 1, 2, \ldots,
\]
(15)
\[ n_i > \frac{\log k_i + \log H}{\log q} - 1 \geq \left(1 + \frac{\hat{S}r}{\omega} \right) \frac{\log H}{\log q} + y_1 - 1 \geq \frac{N}{\omega} + y_1 - 1 \geq \frac{N}{\omega} + y, \]

\[ N > \frac{S N}{\omega} + S(y_1 - 1) > 4S y_12 \geq 4y_{12}. \]

Thus we can use Lemma 3, by which we find \( B \) linear forms, say \( r_i(\theta) \) \((i = 1, \ldots, S)\), \( \theta = q^4 \), that together with the form \( L(\theta) \) are linearly independent. Then the determinant \( \Delta \) of these linear forms is different from zero and, by Lemma 3, an integer in \( I \), whence \( |\Delta| \geq 1 \).

The determinant \( \Delta \) can be expressed in the form

\[
\begin{vmatrix}
L(\theta) & x_{11} & \cdots & x_{1i} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
x_{S1} & \cdots & x_{Si} & x_{Ss} \\

r_1(\theta) & c_{11}(\theta) & \cdots & c_{1i}(\theta) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
r_S(\theta) & c_{S1}(\theta) & \cdots & c_{Si}(\theta) \\

\end{vmatrix}
\]

and thus, again using Lemma 3, we obtain the estimates

\[
1 \leq |\Delta| \leq |L(\theta)| |y|^{2 \omega - (N/\omega - 1)2} \frac{1}{\log^N ((1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \times
\]

\[
\times \left( \prod_{i=1}^{S} \left( \frac{c_{ni}}{\nu(\nu_i)} \right)^{v_i} \right) q^{N+(d+1)(N+\eta_1)+} + |y|^{2 \omega - (N/\omega - 1)2} \frac{1}{\log^N ((1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \max_{\theta_i \in [0,1]} |\theta_i| q^{-\eta_1} \times \]

\[
\times \left( \prod_{i=1}^{S} \left( \frac{c_{ni}}{\nu(\nu_i)} \right)^{v_i} \right) \max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \right) \right) \]

Since (see [7], p. 411; we have \( n > N/\omega + y_1 + 1 \geq N/\omega + y_{12} \))

\[
\prod_{i=1}^{S} \left( \frac{c_{ni}}{\nu(\nu_i)} \right)^{v_i} \leq \left( \frac{eS n}{N/\omega + y_{12}} \right)^{N/\omega + y_{12}} \leq \left( \frac{eS n}{N} \right)^{N/\omega + y_{12}},
\]

it follows that

\[
1 \leq |\Delta| \leq |L(\theta)| E^N q^{N+(d+1)(N+\eta_1)+} + E^N q^{(d+2)(N+\eta_1)+} \max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \right) \]

Our next purpose is to prove that \( 2 \Delta \geq 1 \). By (26) we have

\[
\max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \max_{\theta_i \in [0,1]} \left( 1 + \frac{N}{\omega} \frac{(1 - \theta)^{-2} 3B^D)^{2} E_\theta^N} N ! \right) \right) \]

and thus our purpose is achieved provided that

\[
N \log F + (d+2)(N/\omega + y_{12}+2) \log q < \log H.
\]

Since \( H > q^4 \), we obtain, by (5) and (25),

\[
(d+2)(y_{12}+2) \log q \leq \log q \left( \frac{1}{\omega} - \frac{(1 + \hat{S}r)(d+2)}{\omega} - \beta \right) \log H.
\]

With this inequality we use the results obtained from (7) and (28), namely

\[
(1 + \hat{S}r) \log F < \beta \log q, \quad N \log q \leq (1 + \hat{S}r) \log H.
\]

This implies

\[
\log H > \beta \log H + \frac{(1 + \hat{S}r)(d+2)}{\omega} \log H + (d+2)(N/\omega + y_{12}+2) \log q.
\]

Thus (29) is valid, which implies, by (28) and (30), that

\[
1 < 2 \Delta = |L(\theta)| E^N q^{N+(d+1)(N+\eta_1)+} + \max_{\theta_i \in [0,1]} |\theta_i| q^{-\eta_1} \times \left( \frac{eS n}{N/\omega + y_{12}} \right)^{N/\omega + y_{12}},
\]

which gives (6) in the case \( H > q^4 \).

If \( H < q^4 \), then in considering the linear form \( L_1(\theta) = [2q]L(\theta) \), we obtain, as above,

\[
1 \leq |L_1(\theta)| q^{2 \omega} (2^{d+2} \log^2 H)^{(1 + \hat{S}r)(d+1)(\omega + \beta)} \log^2 H.
\]

Since \( E \leq F \), the use of (5), (8) and (30) now easily yields (6), and thus Theorem 1 is proved.

5. Proof of Corollary 1. Let \( \varepsilon, 0 < \varepsilon < 1 \), be given. For proving Corollary 1 we denote

\[
\alpha = \frac{100S(y_1+1)}{7\varepsilon}, \quad \beta = \frac{\varepsilon}{100S}, \quad \omega = \frac{100S(d+2)}{\varepsilon}, \quad \tau = \frac{8}{10S}.
\]

We then have

\[
\left( \frac{1 + \hat{S}r - (d+2)}{\omega} - \beta \right) \alpha > \frac{7\varepsilon a}{100S} > \gamma_1.
\]

Now define the constants \( \lambda, E, F \) by (8), where \( \alpha, \beta, \omega, \tau \) have the values given above, and let

\[ \text{On linear forms of a certain class of } G\text{-functions} \]
(31) \[ C = \max \{p^{\alpha_0 + \beta_0} - 4, A + 1\}. \]

In this way we immediately obtain, for all \( q \geq C \),
\[ S^r + (1 + S^r) \left( \frac{d + 1 + \log E}{\omega} + \frac{\log q}{\omega} \right) < \varepsilon, \]
and thus Corollary 1 follows from Theorem 1.

6. Lemmas II. We shall now consider lemmas required for the proof of Theorem 2. The functions \( f(\xi) \) are assumed to satisfy the conditions of this theorem. Let \( \omega > (S + 1)/(d + 2) \) be a positive number, and denote by \( t_i, t_i^+ \) positive constants depending only on \( S \) and the functions \( f(\xi) \).

Although we shall construct the approximation polynomials in a different manner from that used in Lemma 1, earlier considerations can be used in many parts of this section. We replace Lemma 1 by the following lemma.

**Lemma 4.** Let \( n_i (i = 0, \ldots, r) \) be natural numbers, and assume that \( n_0 \geq \max \{n_i\} \geq 3 \). Let \( N' = n_0 + n_1 + \cdots + n_r \). There then exist \( S + 1 \) polynomials
\[ P_{ij}(z) = \sum_{n=0}^{n_{ij} - 1} a_{n_{ij}} x^n; \quad i = 0, j = 0; \quad i = 1, \ldots, r; \quad j = 1, \ldots, s_i, \]
not all identically zero, with the following properties:

1. All \( P_{ij} \in \mathbb{Z} \) and satisfy
\[ |P_{ij}| \leq 1 + |x| \left( 1 + 2^r \right); \quad |K_i| = (AB)^{\omega - 1} 2^{r - 1}; \]

2. We have
\[ U_s(z) = P_{o0}(z) + \sum_{j=1}^{r_i} \sum_{i=0}^{s_j} P_{ij}(z) f_{ij}(z) = \sum_{i=0}^{S} t_i z^i, \]
where
\[ t_i = 0; \quad i = 0, 1, \ldots, r; \quad \nu = 0, 1, \ldots, N' - [N'/\omega] - 2, \]
and, for all \( |z| < B^{-1} \),
\[ |U_s(z)| \leq 1 + |x| \left( 1 + 2^r \right) 2^{r - 1}. \]

**Proof.** Part \( 1^{(a)} \) can be proved as in Lemma 1, see [8], Lemma 2. Since \( (\text{let } P_{ij} = 0 \text{ for all } \mu \geq n_i) \)
\[ t_{ij} = P_{o0} + \sum_{i=0}^{r_i} \sum_{j=0}^{s_j} P_{ij} a_{n_{ij}, r_{ij}, \mu}, \quad \nu = 0, 1, \ldots, \]
and \( |a_{n_{ij}, r_{ij}, \mu}| \leq y_2 B^{r_{ij} - 1} \) (p. 276), we obtain (let \( y_2 \geq 1 \))
\[ |U_s(z)| \leq \max_{\nu \geq N' - [N'/\omega] - 1} \left( (t_i s_i z_j^i) \right) \leq y_2 (B |z|)^{N' - [N'/\omega] - 1}, \]
which proves the last part of Lemma 4.

Again we construct linear forms
\[ U_s(z) = \frac{d}{k!} (T(z))^k U_0(z), \quad k = 1, 2, \ldots, \]
and thus obtain
\[ (32) \]
and
\[ U_s(z) = P_{o0}(z) + \sum_{j=1}^{r_i} \sum_{i=0}^{s_j} P_{ij}(z) f_{ij}(z), \quad k = 0, 1, \ldots, \]
where \( P_{ij}(z) \) are polynomials having exactly the same representation \( (17) \) as the polynomials \( C_{ij}(z) \) (we only replace \( C \) by \( P \)). Since \( \deg P_{ij}(z) < n_i \), the degrees of these polynomials satisfy
\[ \deg P_{ij}(z) \leq n_i + k d, \quad i = 0, j = 0; \quad i = 1, \ldots, r; \quad j = 1, \ldots, s_i; \quad k = 0, 1, \ldots, \]
As in Section 3, denote now
\[ U_s(z) = \sum_{i=0}^{S} G_i(z) G_i(z), \quad k = 0, 1, \ldots, \]
where the functions \( G_i(z) \) are, in some order, the functions \( f_0(z) \) and \( f_0(z) \) are the corresponding polynomials \( P_{ij}(z) \). Let \( n_i \geq n_i \), when \( G_i(z) \) is one of the functions \( f_0(z) \) (\( j = 1, \ldots, s_i \); \( j = 0 \) if \( i = 0 \)). We obtain the following lemma (the constants \( t_i, t_i^\prime \) are given in the proof of the lemma).

**Lemma 5.** If \( m_i > [N'/\omega] + I_i \) (\( i = 0, \ldots, S \)), then the determinant \( D(z) \) of the matrix \( G = (G_{ij}(z))_{k=0, \ldots, S} \) has the form
\[ D(z) = z^{N' - [N'/\omega] - I_i} D_i(z), \]
where \( D_i(z) \neq 0 \) is a polynomial of degree \( < [N'/\omega] + I_i \).

**Proof.** If \( D(z) = 0 \), then, as in the proof of Lemma 2, we come to the inequality \( (31) \), which now gives
\[ \deg D_i(z) < \sum_{i=0}^{S} m_i + (S + 1)^{d}, \]
since
\[ \ord D_i(z) > N' - \left[ \frac{N'}{\omega} \right] - (S + 1) - I_i. \]
we obtain
\[ \sum_{i=1}^{n} m_i \leq \left[ \frac{N}{\omega} \right] + \Gamma_\nu, \]
where \( \Gamma_\nu = (S+1)^4(d+1) + \Gamma_\nu \). We thus have a contradiction if \( m_i \geq [N/\omega] + \Gamma_\nu \) for some \( i \in \{0, \ldots, S\} \), which means that \( D(x) \neq 0 \). The truth of Lemma 5 is now easily verified. We can choose \( \Gamma_\nu = S+1 \).

Instead of Lemma 3 we use the following lemma, whose proof utilizes Lemmas 4 and 5.

**Lemma 6.** Let \( n_i > [N/\omega] + \Gamma_\nu \) for \( i = 0, \ldots, r \). Let \( q_i \), \( q_0 \) be any natural numbers satisfying \( q_i < q_0 \); \( (q, q_0) = 1 \); \( \| q_0 \| < B^{-1} \); \( \| q_i \| < B^{-1} \); \( (q, q_0) \neq 0 \); and denote \( \theta = q/q_0 \). Then there exist \( S+1 \) numbers \( s_0, s_1, \ldots, s_S \) such that
\[ k_0 + k_1 + \ldots + k_S \leq \left[ \frac{N}{\omega} \right] + \Gamma_\nu, \]
and the linear forms
\[ U_{s_k}(\theta), U_{s_{k+1}}(\theta), \ldots, U_{s_{k+S}}(\theta) \]
are linearly independent. Further all the numbers
\[ p_{s_k}(\theta) = q_i^{s_i+kd} p \]
are rational integers, and for those values of \( i, j \), \( k \) we have the estimates
\[ \left(34 \right) \quad \max_{j} \left| p_{s_k}(\theta) \right| \leq \left[ b \left( \frac{S}{d+1} \right) \right]^{\left[ \frac{N}{\omega} \right] + \Gamma_\nu}, \]
and the linear forms
\[ U_{s_k}(\theta), U_{s_{k+1}}(\theta), \ldots, U_{s_{k+S}}(\theta) \]
are linearly independent. Further all the numbers
\[ p_{s_k}(\theta) = q_i^{s_i+kd} p \]
are rational integers, and for those values of \( i, j \), \( k \) we have the estimates
\[ \left(35 \right) \quad \left| U_{s_k}(\theta) \right| \leq \left| \sum_{k=0}^{S} B_k \right|^{N/\omega} \cdot \left[ b \left( \frac{S}{d+1} \right) \right]^{\left[ \frac{N}{\omega} \right] + \Gamma_\nu}. \]

**Proof.** The first part follows from Lemma 5 (see the proof of [7], Lemma 6) and it is also clear that the numbers \( p_{s_k}(\theta) \) (for \( i = 0, j = 0; i = 1, j = 1, \ldots, s_1; k_0, \ldots, k_S \) are integers, because the coefficients of \( p_{s_k}(\theta) \) are integers and (33) is valid. As in the proof of Lemma 3, we obtain (now \( \omega > 1 \), for all \( i, j, k, n, s_i, r, r \), \( n = n_0 = \max \left\{ s_i \right\} \)),
\[ B_{s_k}(\theta) \leq \left[ b \left( 1 + \varepsilon + \ldots + 3^{S-j} \right) \right] \left( \frac{S}{d+1} \right)^{1+\varepsilon}, \]
where
\[ B_{s_k}(\theta) \leq \left[ b \left( 1 + \varepsilon + \ldots + 3^{S-j} \right) \right] \left( \frac{S}{d+1} \right)^{1+\varepsilon}, \]
and this thus the first assumption of Lemma 3 is satisfied. The constant \( \Gamma_\nu \) in (18) can be assumed to be greater than \( 1 \), and therefore it also follows, by (21) and (22), that \( \| q_0 \| < b^{-1} \).

By Lemma 6 we find linearly independent linear forms \( L(\theta) \) and, say \( U_{s_k}(\theta), \ldots, U_{s_{k+S}}(\theta) \), where \( \theta = q/q_0 \) and \( k_0 + \ldots + k_S < [N/\omega] + \Gamma_\nu \). If \( A \) is the determinant of these linear forms, then the number
\[ N^' = q^{[N'/\omega]} (b^{[N'/\omega]}), \]
and the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
and thus the use of Lemma 4 gives
\[ \left| U_{s_k}(\theta) \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \leq \left| \sum_{i=0}^{S} t_i \left( \frac{\theta}{q_0} \right)^i \right| \]
By using (34) we obtain the upper bound
\[ |A'| \leq (S + 1) |L_{S^*}^{\beta\log N} K_1^{N^*} q^{N^*} \max \{ |x_0| q^{-\rho_1} q^{-\rho_2} \} \times \]
\[ \times \prod_{i=1}^{s} \left\{ \left( 2b (d + 1)^2 D_b q^{\delta_{i2}} \left( \frac{em}{p_k} \right) \right)^{N(p_k)} \right\} \].

From (37) it follows that \( \max_{i,j} \{ |x_0| q^{-\rho_1} q^{-\rho_2} \} \leq q H_1^{-\tau} \), and thus (see p. 282) we can find \( \delta \) such that
\[ |A'| \leq K N^* q^{-\rho_1} q^{-\rho_2} d^{N^* (a + \Gamma_1) - 1} H_1^{-\tau}. \]

By this inequality we immediately obtain \(|q|_{l_p} = 1\)
\[ |A| \geq |A'| \geq K N^* q^{-\rho_1} q^{-\rho_2} d^{N^* (a + \Gamma_1) - 1} H_1^{-\tau}. \]

The determinant \( \Delta \) can be expressed in a similar form to (27), from which
\[ \Delta = XL(\theta) + \sum_{i=1}^{s} X_i U_{l_p}(\theta), \]
where \( \max_{i} \{ |X_i| \} \leq 1 \), whence
\[ |A| \leq \max_{i} \{ |X_i| \} \leq 1. \]

We next prove that \( |U_{l_p}(\theta)| < |A| \) for all \( i = 1, \ldots, S \). By (35) and (38) this is satisfied provided that
\[ H_1 > M N^* q^{-\rho_1} q^{-\rho_2} d^{N^* (a + \Gamma_1) - 1} 2^N q^{-2} q^{-\rho_1 - \rho_2}, \]
and, since \( |q|_{l_p} < q^{-\rho_1 - \rho_2} \), this inequality follows from the inequality
\[ H_1 > M N^* q^{-\rho_1} q^{-\rho_2} d^{N^* (a + \rho_2) + (a + \rho_2) + \Gamma_1}. \]

By (11)
\[ \tau \log H_1 > \beta \log H_1 + (d + 2) \log q + \Gamma_1 \log q, \]
whence, by (13) and (37)
\[ \tau \log H_1 > \beta \log q + (d + 2) \log q + \log H_1 + \Gamma_1 \log q \]
\[ \geq \frac{\beta \log q}{1 + (S + 1) \tau} N^* + \frac{(d + 2) \log q}{\omega} + \frac{N^*}{\omega} \log q + \Gamma_1 \log q \]
\[ \geq N^* \log q + (d + 2 \log q + \Gamma_1 \log q. \]

Thus (40) is valid, and then (38) and (39) give
\[ |L(\theta)| > (K q_1)^{-N^*} q^{-\rho_1} q^{-\rho_2} H_1^{-\tau}. \]

By (37) we have
\[ N^* \log (K q_1) + N^* (1 + \frac{d}{\omega} + \frac{\log (K q_1)}{\log q}) \log H_1. \]

By using (36) and (41) we now obtain (12) in the case \( H_1 > q^s \). The proof can be completed in a manner analogous to the proof of Theorem 1. Thus Theorem 2 is true.

8. Proof of Corollary 2 and certain additional results. Let \( \varepsilon > 0 \) be given. To prove Corollary 2 we use Theorem 2, choosing
\[ \alpha = \frac{100(S + 1)(T + 1)}{6 \varepsilon}, \quad \beta = \frac{\varepsilon}{100(S + 1)}, \quad \omega = \frac{100(S + 1)(d + 2)}{\varepsilon}, \]
\[ \tau = \frac{\varepsilon}{100(S + 1)}, \quad \delta = \frac{6 \varepsilon}{100(S + 1)}. \]

Further let \( h, K, M \) be given by (14), where \( a, \beta, \omega, \tau \) have the values given above, and denote
\[ C = M, \quad c = (1 + (S + 1) \tau)^{\beta^{-1}}. \]

We now see immediately that (11) is valid, and we also have
\[ (S + 1) \tau + (S + 1) \tau \left( \frac{d}{\omega} + \frac{\log (K q_1)}{\log q} \right) < \varepsilon, \]
if \( q, q_1 \) are natural numbers satisfying \( q > (C q_1)^s \). Thus Corollary 2 follows from Theorem 2.

Our next purpose is to improve slightly on one special case of Flicker's [6] Theorem, namely the case when the algebraic number field \( K \) equals \( Q \). To do this we follow the proof of Theorem 2, but choose
\[ H_1 = h^{n-1}, \quad n = \left[ \frac{\log (h H_1^{1/2})}{\log q} \right]. \]

where \( h = h_a = \max \{ |x_0| \} \). It is important that \( n = n' = \min \{ n_i \} \)
which means that \( q_i \) does not appear in (38) or (40). In this way we arrive at the following theorem (where we can assume, without loss of generality, that \( r = 1 \)).

Theorem 3. Let the assumptions of Theorem 2 be valid. Then there exist positive constants \( \Gamma_1, \Gamma_2, \Gamma_3, \) depending only on \( S \) and the functions (1), such that if \( a, \beta, \omega, \tau, \delta > 0, \) and \( \delta > \) positive numbers satisfying (11), then
\[ \left| L \left( \frac{q_1}{q} \right) \right| > q^{-1} h^{(S + 1)(\varepsilon - (1 + \omega)(1 + d \omega + \log H_1 \log q)^{-2})}. \]
for all natural numbers \(q, q_1\) satisfying
\[
(q, q_1) = 1, \quad q > \max \{M^{1/(r+1)\beta - 1}, q_1\}, \quad |q|_p < q^{-1+\delta}, \quad T \left( \frac{q}{q_1} \right) \neq 0,
\]
where \(\lambda, K, M\) are given by (14), and \(h = \max \{\{w_i\}\}.

Now let \(f_j(x) (j = 1, \ldots, m)\) be \(p\)-adic \(\mathcal{G}\)-functions satisfying
\[
y_j = Q_{j\beta}(x) + \sum_{k=1}^{s} Q_{j\phi}(x)y_k, \quad j = 1, \ldots, m,
\]
where all the functions \(Q_{j\phi}(x)\in \mathcal{Q}(x)\), and assume that these functions \(f_j(x) (j = 1, \ldots, m)\) belong to the class \(G(\mathcal{Q}, A, B, D)\), where the positive numbers \(A, B, D\) are not smaller than 1. Suppose that our functions do not satisfy any equation of the form
\[
P(x, f_1(x), \ldots, f_m(x)) = 0,
\]
where \(P(x, a_1, \ldots, a_m) \neq 0\) is a polynomial in \(\mathcal{Q}[x, a_1, \ldots, a_m]\) of degree \(\leq a_0\) with respect to \(a_1, \ldots, a_m\). By applying Lemma 7 of [7] and our Theorem 3 to the functions
\[
f_i(x) = f_{i\alpha}(x), \quad 1 \leq i_1 + \cdots + i_m \leq s, \quad s \leq a_0,
\]
where \(i_j (j = 1, \ldots, m)\) are non-negative integers, we obtain the following corollary giving the special case \(K = \mathcal{Q}\) of Flicker's [8] Theorem.

**Corollary 3.** Let the functions \(f_j(x) (j = 1, \ldots, m)\) be as above, and let \(s, 0 < s < 1, \beta\) be given. Let \(P(x, a_1, \ldots, a_m) \neq 0\) be any polynomial in \(\mathcal{Q}[x, a_1, \ldots, a_m]\) of total degree \(s \leq a_0\), and of height \(\leq H\). There exist positive constants \(c_1, c_2, c_3, c_4\) depending only on \(s, m, s, a_0\) and the functions \(f_j(x) (j = 1, \ldots, m)\), such that
\[
\left| P \left( \frac{f_1(x)}{q}, \ldots, \frac{f_m(x)}{q} \right) \right|_p > q^{-c_4 H^{(1+s)/(s+1)}}
\]
for all natural numbers \(q, q_1\) satisfying \((q, q_1) = 1, \quad q > \max \{a_0, q_1\}, \quad |q|_p < q^{-c_4}, \quad q/a_0\) different from the poles of the functions \(Q_{j\phi}(x) (j = 1, \ldots, m, k = 0, \ldots, m)\).

**9. Examples.** From Lemmas 7–9 of [7] it follows that we can apply our results to the functions
\[
f_i(x) = \log^t(1 + a_i x), \quad t = 1, \ldots, r, \quad j = 1, \ldots, s,
\]
where \(a_1, \ldots, a_s\) are distinct non-zero numbers in \(I\). Corollaries 1 and 2 give the following results concerning logarithms and \(p\)-adic logarithms, respectively.

**Corollary 4.** Let \(\varepsilon, 0 < \varepsilon < 1, \beta\) be given. Assume that \(a_1, \ldots, a_s\) are distinct non-zero numbers in \(I\), and let
\[
P_i(x) = \sum_{j=1}^{s} a_j x^j, \quad i = 1, \ldots, r,
\]
be polynomials with integer coefficients in \(I\), not all identically zero. Further let \(x_s\) be any integer in \(I\). There then exist positive constants \(C_i, \lambda_i, \delta_i\) depending only on \(s, C_i, \lambda_i, \delta_i\), such that for all natural numbers \(q > C_i\), \(|q|_p < q^{-1+\delta_i}\) we have
\[
\left| x_s + P_i \left( \log \left( 1 + a_i q \right) \right) + \cdots + P_r \left( \log \left( 1 + a_s q \right) \right) \right|_p > q^{-\lambda_i H^{(1+s)/(s+1)}},
\]
where \(H = \prod_{i=1}^{r} h_i, \quad h_i = \max \{1, \text{coeff } P_i(x)\} (i = 1, \ldots, r).

**Corollary 5.** Assume that \(I = \mathcal{Q}\), and let the assumptions of Corollary 4 be valid. There then exist positive constants \(C_i, \lambda_i, \delta_i\) depending only on \(s, C_i, \lambda_i, \delta_i\), such that for all natural numbers \(q\) satisfying \(q > C_i\), \(|q|_p < q^{-1+\delta_i}\) we have
\[
\left| x_s + P_i \left( \log(1 + a_i q) \right) + \cdots + P_r \left( \log(1 + a_s q) \right) \right|_p > q^{-\lambda_i H^{(1+s)/(s+1)}},
\]
where \(H_1 = h_0 H, \quad h_0 = \max \{x_0, h_0\}\).

In the special case \(r = 1\) our Corollary 4 is analogous to Baker's [1] Theorem 1. Baker gives the constants more explicitly, however.

Using Corollary 3 we also obtain the following result.

**Corollary 6.** Let \(I = \mathcal{Q}\), and let \(\varepsilon, 0 < \varepsilon < 1, \beta\) be as in Corollary 4. Let \(P(x, a_1, \ldots, a_s) \neq 0\) be any polynomial in \(\mathcal{Q}[x, a_1, \ldots, a_s]\) of degree \(\leq s\) and height \(\leq H\). There then exist positive constants \(c_1, c_2, c_3, c_4\) depending only on \(s, r, \varepsilon, a_1, \ldots, a_s\), such that
\[
\left| P \left( \log(1 + a_1 q), \ldots, \log(1 + a_s q) \right) \right|_p > q^{-c_4 H^{(1+s)/(s+1)}}
\]
for all natural numbers \(q\) satisfying \(q > c_4\), \(|q|_p < q^{-1+\delta_i}\).

Let us now consider the functions
\[
\varphi_j(x) = \sum_{n=1}^{\infty} x^n q_j, \quad j = 1, \ldots, s,
\]
satisfying
\[
\varphi_j(x) = \frac{1}{1-x}, \quad \varphi_j(x) = \frac{1}{q_j} \varphi_{j-1}(x), \quad j = 2, \ldots, s.
\]
We have
\[ q_{j1}^{(n)}(z) = \text{for (1 - z)^n}, \quad q_{j2}^{(n)}(z) = (n - 1)! \sum_{n_1 + n_2 = n - 1} \frac{j_{n1}^{(n)}(z)}{n_1!} \frac{j_{n2}^{(n)}(z)}{n_2!}, \]
where \( j = 2, \ldots, s, \quad n = 1, 2, \ldots, \)
which may be established by induction that the condition (iii) of our definition of the class \( G(Q, A, B, D) \) is valid when we choose \( T(z) = z(1 - z), \quad a_n = d_0, \quad a_0 = 1 \text{ c.m.} \{1, 2, \ldots, n\}. \)
From paper [14] we have \( d_0 \leq e^{n}. \) The conditions (ii), (iii) of the definition are obviously valid, and thus the functions \( q_j(z) (j = 1, \ldots, s) \) belong to some class \( G(Q, A, B, D). \)

We now prove that the functions
\[ f_{ij}(z) = q_j(a, z), \quad i = 1, \ldots, r, \quad j = 1, \ldots, s, \]
where \( a_1, \ldots, a_r \) are distinct non-zero numbers in \( I, \) together with the function \( f_{bb}(z) = 1 \) are linearly independent over \( C(z), \) since
\[ f_{ij}(z) = \sum_{n=1}^{\infty} n^{-1}(a, z)^n = -\log(1 - a, z), \quad i = 1, \ldots, r, \]
our assertion is valid for \( j = 1. \) Suppose now that the functions 1, \( f_{ij}(z) \) \( (i = 1, \ldots, r, \quad j = 1, \ldots, m - 1, \quad m \geq 2) \) are linearly independent, but the functions 1, \( f_{ij}(z) \) \( (i = 1, \ldots, r, \quad j = 1, \ldots, m) \) are linearly dependent. Then there exist the smallest suffix \( n \) such that
\[ P_{mn}(z) + \sum_{i=1}^{r} \sum_{j=1}^{m-1} P_{ij}(z)(a, z) + P_{im}(z)(a, z) + \ldots \]
\[ \ldots + P_{nm}(z)(a, z) = 0, \]
where \( P_{ij}(z) \) are polynomials with no common factors, \( P_{nm}(z) \neq 0. \) We then get
\[ P_{mn}(z) + \sum_{i=1}^{r} \left\{ \sum_{j=1}^{m-1} P_{ij}(z)(a, z) + \sum_{j=1}^{m-1} \frac{1}{z} P_{ij}(z)P_{j-1}(z)(a, z) + \right. \]
\[ \left. + \frac{1}{1 - a, z} P_{ij}(z) \right\} + \sum_{i=1}^{r} \left\{ P_{im}(z)(a, z) + \frac{1}{z} P_{im}(z)P_{m-1}(z)(a, z) \right\} = 0. \]

Suppose that \( P_{mn}(z) \neq 0. \) By multiplying the left-hand side of (45) by \( T(z) = z(1 - a, z) \ldots (1 - a, z) \) we obtain a polynomial in \( z, f_{ij} \) \( (i = 1, \ldots, r, \quad j = 1, \ldots, m) \), which must be divisible by the left-hand side of (44), since otherwise we obtain a contradiction by eliminating \( f_{nm} \) from the

equations (44) and (45). Thus there exists a polynomial
\[ A(z) = A_0 + A_1 z + \ldots + A_s z^s \neq 0 \]
such that
\[ T(z) \left\{ \sum_{i=1}^{r} \sum_{j=1}^{m-1} P_{ij}(z)f_{ij} + \sum_{j=1}^{m-1} \frac{1}{z} P_{ij}(z)f_{j-1} + \frac{a, z}{1 - a, z} P_{ij}(z) \right\} + \sum_{i=1}^{r} \left\{ P_{im}(z)f_{im} + \frac{1}{z} P_{im}(z)f_{m-1} \right\} \]
identically in \( z, f_{ij}. \) This means that we have
\[ T(z)P_{mn}(z) = A(z)P_{nm}(z), \]
\[ T(z) \left( P_{mn}(z) + \sum_{i=1}^{r} \sum_{j=1}^{m-1} P_{ij}(z)f_{ij} + P_{im}(z)f_{im} + \ldots + P_{nm}(z)f_{nm} \right) \]
identically in \( z. \) Using our assumptions on \( a, \) we obtain from the first equation (46)
\[ P_{mn}(z) \frac{A(z)}{T(z)} = \frac{a, z}{z} \sum_{i=1}^{r} \frac{a, z}{1 - a, z}, \]
where \( a = A_0 \) and \( a, \) \( (i = 1, \ldots, r) \) are certain constants. Thus
\[ P_{mn}(z) = cz \prod_{i=1}^{r} (1 - a, z)^{a, z}, \quad c \neq 0. \]
Since \( P_{nm}(z) \) is a polynomial in \( z, \) the numbers \( a, \) \( a, \) are non-negative integers. Let
\[ P_{nm-1}(z) = p_1 z + p_{i-1} z + \ldots, \quad p_i \neq 0. \]
(If \( P_{nm-1}(z) \equiv 0, \) then a contradiction follows from (46).) From the second equation in (46) we get \( a \geq 1 \) and
\[ p_i + a, z = a p, \]
If \( a = 1, \) then we have a contradiction \( c = 0. \) If \( a > i, \) then \( p_i = a p, \)
which is impossible, because \( p_i \neq 0. \) Thus \( P_{mn}(z) \equiv 0, \) whence \( P_{nm}(z) \equiv c, \) \( c \neq 0, \) a constant. In this case we use the definition of \( n \) and (45),
by which \( P_{nm-1}(z) = c, /z. \) This is impossible, and thus we have proved the linear independence of the functions (43) and \( f_{bb}(z) \equiv 1. \)
The use of the above results and Lemma 7 of [7] together with Corollaries 1 and 2 leads to the following corollaries.

**Corollary 6.** Let $0 < s < 1$, be given. Let $a_1, \ldots, a_r$ be distinct non-zero numbers in $I$, and let $L(z)$ be a linear form (4) with $f_j(z)$ given by (43). Then there exist positive constants $C, \lambda$, depending only on $s, S = s_1 + \ldots + s_r, a_1, \ldots, a_r$, such that

$$|L(z)|_q > q^{-1}H^{-1-s}$$

for all natural numbers $q$ satisfying $q > C$.

**Corollary 7.** Let the assumptions of Corollary 6 with $I = Q$ be valid. Then there exist positive constants $C_1, \lambda_1, \delta$, depending only on $s, S, a_1, \ldots, a_r$, such that

$$|L(q)|_p > q^{-1}H^{-1-s}$$

for all natural numbers $q$ satisfying $q > C_1$, $|q|_p < q^{-1+\delta}$.

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References


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