$$L(1,\chi) = \frac{1}{\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)},$$

we use Stark's inequality

$$(2)\qquad L(1,\chi) \geqslant \frac{C}{ng(n)}\, d^{-(\sigma_1-1)/2}(\sqrt{f}d)^{-1/n}\frac{1}{\zeta_k(\sigma_1)}$$

for every σ_1 with $1+(8\log(\sqrt{f}d))^{-1} \leqslant \sigma_1 \leqslant 2$, where d, ζ_k denote the discriminant and the Dedekind Zeta-function for k, and f is defined by $|\operatorname{disc}(k(\sqrt{-\det G}))| = d^2 f$; C is an effectively computable constant, $g(n) = n!$ and $g(n) = n$, if one assumes the Generalized Riemann Hypothesis (see [7], [2], (4.2) ff., cf. also [5], (9) (there it should read $2^n d$ instead of $2d$)).

Assuming the Generalized Riemann Hypothesis, one has

$$(3)\qquad d^{1/n} \geqslant 188+o(1) \quad \text{for} \quad n\to\infty$$

(see [3]).

If G has only one class, this implies $M(G) \leqslant 1/2$. Comparing this with (1), (2), (3) and with [2], (4.3) to (4.9) ff., one gets (i) and (iii) of the theorem for k with $n>2$. The case $k = \boldsymbol{Q}$ is well known. For quadratic fields k one has only finitely many idoneal numbers $(\mathfrak{d}, k)$, as follows using the Brauer–Siegel-Theorem instead of (2). Similarly one gets (ii) of the theorem (for a similar theorem and proof, see [1], Satz 20).

References

[1] J. Dzewas, *Quadratsummen in reell-quadratischen Zahlkörpern*, Math. Nachr. 21 (1960), pp. 233–284.

[2] A. M. Odlyzko, *Some analytic estimates of class numbers and discriminants*, Inv. Math. 29 (1975), pp. 275–286.

[3] — *Lower bounds for discriminants of number fields II*, Tôhoku Math. J. 29 (1977), pp. 209–216.

[4] O. T. O'Meara, *Introduction to quadratic forms*, Springer, 1963.

[5] M. Peters, *Einklassige Geschlechter von Einheitsformen in totalreellen algebraischen Zahlkörpern*, Math. Ann. 226 (1977), pp. 117–120.

[6] H. Pfeuffer, *Darstellungsmaße binärer quadratischer Formen über totalreellen algebraischen Zahlkörpern*, Acta Arith. 34 (1978), pp. 103–111.

[7] H. M. Stark, *Some effective cases of the Brauer–Siegel theorem*, Inv. Math. 23 (1974), pp. 135–152.

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On linear forms of a certain class of G-functions and p-adic G-functions

by

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1. Introduction. In the present paper we employ the ideas of Baker ([2], [3], Chapter 10) and the Siegel–Shidlovski theory ([3], [10], [15], [16]) to examine the linear forms of certain G-functions. We have two main aims, firstly to generalize the results of Galochkin [8], and thus obtain for G-functions an analogue of Makarov's [11] result concerning E-functions, and secondly to find p-adic analogues to the results obtained. Our studies have been motivated by a recent paper of Flicker [6], where he obtains p-adic analogues of the results of Galochkin [7] and Nurmagomedov [13]. Here we shall obtain similar p-adic analogues in connection with the papers [2], [4], [5], [8], [11], [17], [18]. In particular we shall give lower bounds in terms of all the coefficients for the p-adic valuations of linear forms in the values of certain G-functions.

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2. Main results. Let $\boldsymbol{I}$ denote the field of rational numbers or an imaginary quadratic field. We consider r systems of G-functions

$$(1)\qquad f_{i1}(z),\ldots,f_{is_i}(z),\quad s_i\geqslant 1,\quad i=1,\ldots,r$$

(in [8] $s_i = 1$ $(i=1,\ldots,r)$), and assume that these functions satisfy the corresponding systems of differential equations

$$(2)\qquad y'_{ij} = Q_{ij0}(z)+\sum_{\nu=1}^{s_i} Q_{ij\nu}(z)y_{i\nu},\quad i=1,\ldots,r,\ j=1,\ldots,s_i,$$

where all $Q_{ij\nu}(z)\in \boldsymbol{I}(z)$. We thus immediately obtain, for $l=0,1,\ldots,$

$$(3)\qquad y_{ij}^{(l)} = Q_{ij0l}(z)+\sum_{\nu=1}^{s_i} Q_{ij\nu l}(z)y_{i\nu},\quad i=1,\ldots,r,\ j=1,\ldots,s_i,$$

where all $Q_{ij\nu l}(z)\in\boldsymbol{I}(z)$.

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We assume that the functions (1) belong to the class $G(\boldsymbol{I}, A, B, D)$ (A, B, D positive constants, $B \geqslant 1$, $D \geqslant 1$), by which we mean that these functions are of the form

$$f_{ij}(z) = \sum_{\nu=0}^{\infty} a_{ij\nu} z^{\nu}, \quad i = 1, \ldots, r,\ j = 1, \ldots, s_i,$$

where all $a_{ij\nu} \in \boldsymbol{I}$, and the following conditions are satisfied:

(i) *There exists a constant γ_1 such that*

$$|a_{ij\nu}| \leqslant \gamma_1 A^{\nu}, \quad i = 1, \ldots, r,\ j = 1, \ldots, s_i,\ \nu = 0, 1, \ldots;$$

(ii) *There exist a constant γ_2 and a sequence $\{b_n\}$ of natural numbers, such that $b_n \leqslant \gamma_2 B^n$ $(n = 0, 1, \ldots)$ and all the numbers $a_{ij\nu} b_n$ $(i = 1, \ldots, r,$ $j = 1, \ldots, s_i,\ \nu = 0, \ldots, n)$ are integers in $\boldsymbol{I}$;*

(iii) *There exist a constant γ_3, a sequence $\{d_n\}$ of natural numbers, and a polynomial $T(z) \in \boldsymbol{I}[z]$ with integer coefficients, not all zero, such that $d_n \leqslant \gamma_3 D^n$ $(n = 0, 1, \ldots)$ and all the functions*

$$\frac{d_n}{l!} \big(T(z)\big)^l Q_{ij\nu l}(z),$$

$$i = 1, \ldots, r, \quad j = 1, \ldots, s_i, \quad \nu = 0, \ldots, s_i, \quad l = 0, \ldots, n,$$

are polynomials in $\boldsymbol{I}[z]$ with integer coefficients.

The following notations are used,

$$S = s_1 + \ldots + s_r, \quad d = \max_{i,j,\nu}\{\deg T(z) - 1, \deg T(z) Q_{ij\nu}(z)\},$$

$$b = \max_{i,j,\nu}\{|\mathrm{coeff}\, T(z)|, |\mathrm{coeff}\, T(z) Q_{ij\nu}(z)|\},$$

where $|\mathrm{coeff}\, P(z)|$ denotes the maximum of the absolute values of the coefficients of the polynomial $P(z)$.

We shall obtain lower bounds for the absolute values of the linear forms

$$L(z) = x_{00} + \sum_{i=1}^{r} \sum_{j=1}^{s_i} x_{ij} f_{ij}(z), \tag{4}$$

where all x_{ij} are integers in $\boldsymbol{I}$, not all zero. Let us denote

$$h_i = \max_{1 \leqslant j \leqslant s_i}\{1, |x_{ij}|\}, \quad i = 1, \ldots, r, \quad h_0 = \max_{1 \leqslant i \leqslant r}\{|x_{00}|, h_i\}.$$

Our main results are given in the following theorems and corollaries.

THEOREM 1. *Let the G-functions (1) belong to the class $G(\boldsymbol{I}, A, B, D)$, $A \geqslant 1$, and assume that these functions and the function $f_{00}(z) \equiv 1$ are linearly independent over $\boldsymbol{C}(z)$. Let $L(z)$ be any linear form given by (4). There then exist positive constants $\gamma_4, \gamma_5, \gamma_6$, depending only on S and the functions (1), such that if $\alpha, \beta, \omega, \tau$ are positive numbers satisfying*

$$\big(\tau - (1 + S\tau)(d+2)\omega^{-1} - \beta\big)\alpha > \gamma_4, \tag{5}$$

then

$$|L(q^{-1})| > q^{-\lambda} H^{-(1+S\tau)(1+(d+1)\omega^{-1}+(\log E)(\log q)^{-1})} \tag{6}$$

for all natural numbers q satisfying

$$q > \max\{F^{(1+S\tau)\beta^{-1}}, 4A\}, \quad T(q^{-1}) \neq 0, \tag{7}$$

where λ, H, E and F are given by

$$\begin{aligned} H &= \prod_{i=1}^{r} h_i^{s_i}, \quad \lambda = \gamma_5 + S(\alpha+1)\big(1 + (S+1)\tau\big), \\ E &= \gamma_6^{2\omega-1}(6beDS\omega)^{\omega^{-1}}(AB)^{(2\omega-1)^2\omega^{-1}S}, \\ F &= E(2-\omega^{-1})^{\omega^{-1}} A^{2-\omega^{-1}}. \end{aligned} \tag{8}$$

This theorem admits the following corollary.

COROLLARY 1. *Let the functions (1) and a linear form $L(z)$ be as in Theorem 1. Let ε, $0 < \varepsilon < 1$, be given. There then exist positive constants λ, C, depending only on ε, S and the functions (1), such that*

$$|L(q^{-1})| > q^{-\lambda} H^{-1-\varepsilon}, \quad H = \prod_{i=1}^{r} h_i^{s_i}, \tag{9}$$

for all natural numbers q satisfying $q \geqslant C$, $T(q^{-1}) \neq 0$.

We note that if ω, τ are positive numbers satisfying

$$\omega > S(d+2), \quad \tau > (d+2)\big(\omega - S(d+2)\big)^{-1},$$

then we can find positive numbers $\alpha = \alpha(\omega, \tau, \gamma_4)$ and $\beta = \beta(\omega, \tau, \gamma_4)$ such that (5) is valid. Thus the special case $s_i = 1$ $(i = 1, \ldots, r)$ of Theorem 1 gives an analogue of Galochkin's [8] result.

Now let p be any fixed prime number, $\boldsymbol{Q}_p$ the p-adic completion of $\boldsymbol{Q}$, $\boldsymbol{C}_p$ a p-adically complete algebraically closed extension of $\boldsymbol{Q}_p$, and let $|\ |_p$ denote the p-adic valuation on $\boldsymbol{C}_p$ satisfying $|p|_p = p^{-1}$. If the functions (1) belong to the class $G(\boldsymbol{Q}, A, B, D)$, then the power series of these functions converge p-adically for all $|z|_p < B^{-1}$, since

$$1 \geqslant |b_\nu a_{ij\nu}|_p = |b_\nu|_p |a_{ij\nu}|_p \geqslant |a_{ij\nu}|_p b_\nu^{-1},$$

which gives, for all $i = 1, \ldots, r$, $j = 1, \ldots, s_i$,

$$|a_{ij\nu}|_p \leqslant b_\nu \leqslant \gamma_2 B^\nu, \quad \nu = 0, 1, \ldots \tag{10}$$

The p-adic G-functions (1) are thus defined at least for all $|z|_p < B^{-1}$. In the following theorem we give a lower bound for $|L(z)|_p$ at some of these points z.

THEOREM 2. *Let the G-functions (1) belong to the class $G(\boldsymbol{Q}, A, B, D)$, $AB \geqslant 1$, and assume that these functions and the function $f_{00}(z) \equiv 1$ are linearly independent over $C_p(z)$. Assume that $L(z)$ is any linear form (4) with $x_{ij} \in \boldsymbol{Z}$. There then exist positive constants $\Gamma_1, \Gamma_2, \Gamma_3$, depending only on S and the functions (1), such that if $\alpha, \beta, \omega > 2$, τ and δ are positive numbers satisfying*

$$\Big(\tau - \big(1+(S+1)\tau\big)\big(d+2+\delta(\omega-2)\big)\omega^{-1} - \beta\Big)\alpha > \Gamma_1, \tag{11}$$

then

$$\left|L\left(\frac{q}{q_1}\right)\right|_p > q^{-\lambda_1} H_1^{\tau-(1+(S+1)\tau)(1+d\omega^{-1}+(\log Kq_1)(\log q)^{-1})} \tag{12}$$

for all natural numbers q, q_1 satisfying

$$\begin{gathered}(q, q_1) = 1, \quad q > (Mq_1)^{(1+(S+1)\tau)\beta^{-1}}, \\ |q|_p < q^{-1+\delta}, \quad T\left(\frac{q}{q_1}\right) \neq 0, \end{gathered} \tag{13}$$

where λ_1, H_1, K and M are given by

$$\begin{gathered} H_1 = h_0 \prod_{i=1}^{r} h_i^{s_i}, \quad \lambda_1 = \Gamma_2 + (S+1)(\alpha+1)\big(1+(S+1)\tau\big), \\ K = \Gamma_3^{\omega-1}\big(2beDS(d+1)^2\omega\big)^{\omega-1}(AB)^{(\omega-1)^2\omega^{-1}S}, \\ M = KB^{(1-\omega^{-1})}. \end{gathered} \tag{14}$$

This theorem implies the following corollary.

COROLLARY 2. *Let the functions (1) and a linear form $L(z)$ be as in Theorem 2. Let ε, $0 < \varepsilon < 1$, be given. There then exist positive constants δ, λ_1, C_1, c, depending only on ε, S and the functions (1), such that*

$$\left|L\left(\frac{q}{q_1}\right)\right|_p > q^{-\lambda_1} H_1^{-1-\varepsilon}, \quad H_1 = h_0 \prod_{i=1}^{r} h_i^{s_i}, \tag{15}$$

for all natural numbers q, q_1 satisfying $(q, q_1) = 1$, $q > (C_1 q_1)^c$, $|q|_p < q^{-1+\delta}$, $T(q/q_1) \neq 0$.

We note that if we wish to find a lower bound depending only on $\max_i \{h_i\}$, then the proof of Theorem 2 can be used with weaker conditions

than (13). We are thus able to improve on a certain special case of Flicker's Theorem [6]. This result will be given in Section 8.

3. Lemmas I. We may now proceed to the lemmas needed in the proof of Theorem 1. We assume in all lemmas that the functions (1) satisfy the conditions given in Theorem 1. Let $[x]$ denote the integer part of x, and let $\omega > S(d+2)$ be a positive number. By $\gamma_7, \gamma_8, \ldots$ we denote positive constants depending only on S and the functions (1).

LEMMA 1. *Let n_i $(i = 1, \ldots, r)$ be natural numbers, and let $n_0 = n = \max_{1 \leqslant i \leqslant r} \{n_i\} \geqslant 3$, $N = s_1 n_1 + \ldots + s_r n_r$. There exist $S+1$ polynomials*

$$C_{ij}(z) = \sum_{\mu=n-n_i}^{n-1} c_{ij\mu} z^\mu, \quad i = 0,\ j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i,$$

not all identically zero, with the following properties:

(1°) *All $c_{ij\mu}$ are integers in $\boldsymbol{I}$, and satisfy*

$$|c_{ij\mu}| \leqslant \gamma_7^{(2\omega-1)\log N} E_0^N, \quad E_0 = (AB)^{(2\omega-1)^2\omega^{-1}};$$

(2°) *We have*

$$R_0(z) = C_{00}(z) + \sum_{i=1}^{r} \sum_{j=1}^{s_i} C_{ij}(z) f_{ij}(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu,$$

where

$$g_\nu = 0, \quad \nu = 0, 1, \ldots, n+N-\left[\frac{N}{\omega}\right]-2,$$

and, for all $|z| < (2A)^{-1}$,

$$|R_0(z)| \leqslant \gamma_8^{(2\omega-1)\log N} F_0^N |z|^{n+N-\left[\frac{N}{\omega}\right]-1}, \quad F_0 = E_0 A^{(2\omega-1)\omega^{-1}}.$$

Proof. We need only refer to the proof of [8], Lemma 2, which uses Siegel's lemma.

We then construct new linear forms from the form $R_0(z)$ as follows,

$$R_k(z) = \frac{d_k}{k!}\big(T(z)\big)^k R_0^{(k)}(z), \quad k = 1, 2, \ldots$$

The use of (3) gives an equality

$$\begin{aligned} R_k(z) = \frac{d_k}{k!}\big(T(z)\big)^k \Big\{ & C_{00}^{(k)}(z) + \sum_{i=1}^{r} \sum_{j=1}^{s_i} \sum_{l=0}^{k} \binom{k}{l} C_{ij}^{(k-l)}(z) Q_{ij0l}(z) + \\ & + \sum_{i=1}^{r} \sum_{j=1}^{s_i} \sum_{l=0}^{k} \sum_{\nu=1}^{s_i} \binom{k}{l} C_{i\nu}^{(k-l)}(z) Q_{i\nu jl}(z) f_{ij}(z) \Big\}, \quad k = 1, 2, \ldots, \end{aligned}$$

from which it follows that

$$R_k(z) = C_{00k}(z) + \sum_{i=1}^{r}\sum_{j=1}^{s_i} C_{ijk}(z) f_{ij}(z), \qquad k = 0, 1, \ldots, \tag{16}$$

where

$$C_{000}(z) = C_{00}(z), \qquad C_{ij0}(z) = C_{ij}(z),$$

$$C_{00k}(z) = \frac{d_k}{k!}(T(z))^k C_{00}^{(k)}(z) + \sum_{i=1}^{r}\sum_{j=1}^{s_i}\sum_{l=0}^{k} \frac{C_{ij}^{(k-l)}(z)}{(k-l)!}(T(z))^{k-l}\frac{d_k}{l!}B_{ij0l}(z), \tag{17}$$

$$C_{ijk}(z) = \sum_{\nu=1}^{s_i}\sum_{l=0}^{k} \frac{C_{i\nu}^{(k-l)}(z)}{(k-l)!}(T(z))^{k-l}\frac{d_k}{l!}B_{i\nu jl}(z),$$

$$i = 1, \ldots, r, \; j = 1, \ldots, s_i, \; k = 1, 2, \ldots,$$

for which we denote $B_{ij\nu l}(z) = (T(z))^l Q_{ij\nu l}(z)$ $(i = 1, \ldots, r,\ j = 1, \ldots, s_i,\ \nu = 0, \ldots, s_i,\ l = 0, 1, \ldots)$. For these i, j, ν, l we have, by (2) and (3),

$$Q_{ij\nu, l+1}(z) = Q'_{ij\nu l}(z) + \sum_{k=1}^{s_i} Q_{ijkl}(z) Q_{ik\nu 1}(z),$$

and thus we obtain the recursive formulae

$$B_{ij\nu, l+1}(z) = T(z)B'_{ij\nu l}(z) - lT'(z)B_{ij\nu l}(z) + \sum_{k=1}^{s_i} B_{ijkl}(z)B_{ik\nu 1}(z), \tag{18}$$

by means of which these functions $B_{ij\nu l}(z)$ can be easily considered. We see that, for all $l = 0, 1, \ldots$, $B_{ij\nu l}(z)$ $(i = 1, \ldots, r,\ j = 1, \ldots, s_i,\ \nu = 0, \ldots, s_i)$ are polynomials of degree $\leqslant ld$. Thus we have the estimates

$$\deg C_{ijk}(z) \leqslant n + kd, \qquad \operatorname{ord} C_{ijk}(z) \geqslant n - n_i - k \tag{19}$$

for the degrees and orders at $z = 0$ of the polynomials $C_{ijk}(z)$ $(i = 0,\ j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i;\ k = 0, 1, \ldots)$.

For the sake of simplicity, let the functions (1) and $f_{00}(z) \equiv 1$, in some order, be $G_0(z), G_1(z), \ldots, G_S(z)$, and for $k = 0, \ldots, S$ let $m_k = n_i$, when $G_k(z)$ is one of the functions $f_{ij}(z)$ $(j = 1, \ldots, s_i;\ j = 0$ if $i = 0)$. We then have, by (16) and (19),

$$R_k(z) = \sum_{i=0}^{S} G_{ki}(z) G_i(z), \qquad k = 0, 1, \ldots, \tag{20}$$

where $G_{ki}(z)$ are polynomials of degree $\leqslant n + kd$, and the order of $G_{ki}(z)$ at $z = 0$ is at least $n - m_i - k$.

The following lemma is valid (the constants γ_9, γ_{10} are given in the proof of the lemma).

LEMMA 2. *If $m_i > [N/\omega] + \gamma_9$ $(i = 0, \ldots, S)$, then the determinant $D(z)$ of the matrix $G = (G_{ki}(z))_{k,i=0,\ldots,S}$ has the form*

$$D(z) = z^{(S+1)n - [N/\omega] - \gamma_{10}} D_1(z),$$

where $D_1(z) \not\equiv 0$ is a polynomial of degree $\leqslant [N/\omega] + \gamma_9$.

Proof. First we prove that $D(z) \not\equiv 0$. For the general theory of this step we refer to [15] and [10], Chapter 4. We follow Galochkin's [8] deduction and write here those parts of the proof which will also be of use in the proof of Lemma 5.

Assume that $D(z) \equiv 0$, and let $l+1$ $(< S+1)$ be the rank of G. By Lemma 1 we have $l \geqslant 0$. The matrix G has a minor $\not\equiv 0$ of order $l+1$. Assume that the functions $G_i(z)$ $(i = 0, \ldots, S)$ are numbered in such a way that this minor is

$$D_0(z) = \det(G_{ki}(z))_{k,i=0,\ldots,l} \not\equiv 0.$$

Further we may assume that $m_l = \max\{m_0, \ldots, m_l\}$. Using the general theory as in [8], p. 547, or [18], pp. 11–14, we obtain an inequality

$$\operatorname{ord} D_0(z) \geqslant \operatorname{ord} \begin{vmatrix} G_{00}(z) & \ldots & G_{0,l-1}(z) & R_0(z) \\ \cdot & \cdot & \cdot & \cdot \\ G_{l0}(z) & \ldots & G_{l,l-1}(z) & R_l(z) \end{vmatrix} - \gamma_{11}. \tag{21}$$

The use of the properties of $G_{ki}(z)$ and $R_k(z)$ gives us

$$\operatorname{ord} D_0(z) \geqslant \sum_{i=0}^{l-1}(n - m_i - l) + n + N - \left[\frac{N}{\omega}\right] - l - 1 - \gamma_{11}$$

$$> ln + \sum_{i=l}^{S} m_i - \left[\frac{N}{\omega}\right] - (S+1)^2 - \gamma_{11},$$

$$\deg D_0(z) < (l+1)n + (S+1)^2 d.$$

Since $\operatorname{ord} D_0(z) \leqslant \deg D_0(z)$, it follows that

$$n + \left[\frac{N}{\omega}\right] + \gamma_9 > \sum_{i=l}^{S} m_i,$$

where we denote $\gamma_9 = \gamma_{11} + (S+1)^2(d+2)$. Now $m_i = n$ for some $i = l, \ldots, S$, and thus the above inequality is impossible if $m_i > [N/\omega] + \gamma_9$ $(i = 0, \ldots, S)$. This gives $D(z) \not\equiv 0$.

By choosing $\gamma_{10} = (S+1)^2$ we can now establish the truth of Lemma 2 (as in [18], p. 15, for instance).

Let, for $k = 0, 1, \ldots,$

$$r_k(z) = q^{n+kd}R_k(z) = c_{00k}(z) + \sum_{i=1}^{r}\sum_{j=1}^{s_i} c_{ijk}(z) f_{ij}(z), \tag{22}$$

where q is a natural number and $c_{ijk}(z) = q^{n+kd}C_{ijk}(z)$. By using the above results we obtain the following lemma, which gives certain important properties of linear forms (22).

LEMMA 3. *Let $n_i > [N/\omega]+\gamma_9$ $(i = 1, \ldots, r)$. Let q be a natural number satisfying $q > 2A$, $T(q^{-1}) \neq 0$, and denote $\theta = q^{-1}$. There then exist $S+1$ numbers $k_0, k_1, \ldots, k_S$, such that $k_0+k_1+\ldots+k_S \leqslant [N/\omega]+\gamma_{12}$ and the linear forms*

$$r_{k_0}(\theta), r_{k_1}(\theta), \ldots, r_{k_S}(\theta)$$

are linearly independent. Further all the numbers $c_{ijk}(0)$ $(i = 0, j = 0;$ $i = 1, \ldots, r,$ $j = 1, \ldots, s_i;$ $k = 0, 1, \ldots, [N/\omega]+\gamma_{12})$ are integers in $\boldsymbol{I}$, and satisfy

$$\max_j\{|c_{ijk}(\theta)|\} \leqslant (1-\theta)^{-2k}\gamma_{13}^{(2\omega-1)\log N}E_0^N(3bD)^k q^{n_i+(d+1)k}\left(\frac{en}{\nu(k)}\right)^{\nu(k)}, \tag{23}$$

where $\nu(k) = \max\{1, \min\{n, k\}\}$. If $N > 4\gamma_{12}$ and $q > 4A$, then

$$|r_k(\theta)| \leqslant (1-\theta)^{-k}\gamma_{14}^{(2\omega-1)\log N}F_0^N(bD)^k \times \\ \times q^{-N+[N/\omega]+1+(d+1)k}\left(\frac{en}{\nu(k)}\right)^{\nu(k)}\left(1+\frac{N}{n}\left(1-\frac{1}{\omega}\right)\right)^k. \tag{24}$$

Proof. The first part can be proved in a manner analogous to the proof of [7], Lemma 4 (Lemma 2 is essentially needed), and from (iii), (17) and (22) it follows immediately that the numbers $c_{ijk}(\theta)$ $(i = 0, j = 0;$ $i = 1, \ldots, r, j = 1, \ldots, s_i;$ $k = 0, 1, \ldots)$ are integers in $\boldsymbol{I}$. Thus it remains to prove the estimates (23) and (24). For this we consider first the polynomials $B_{ij\nu l}(z)$, which satisfy (18).

We use the notation $f(z) \ll g(z)$ if

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad g(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu, \quad b_\nu \geqslant 0,$$

and $|a_\nu| \leqslant b_\nu$ $(\nu = 0, 1, \ldots)$. Since $B_{ij\nu 1}(z) = T(z)Q_{ij\nu}(z)$, we have, for all $|z| < (2A)^{-1}$, $B_{ij\nu 1}(z) \ll b(1-z)^{-1}$ $(i = 1, \ldots, r, j = 1, \ldots, s_i, \nu = 0, \ldots$ $\ldots, s_i)$. Using (18) we obtain, for all these i, j, ν,

$$B_{ij\nu l}(z) \ll b^l(1-z)^{1-2l}\prod_{\mu=0}^{l-1}(3\mu+S), \quad l = 1, 2, \ldots,$$

which gives

$$|B_{ij\nu l}(\theta)| \leqslant (1-\theta)^{1-2l}(3b)^l(l!)\gamma_{15}^{\log N}, \quad l = 1, \ldots, [N/\omega]+\gamma_{12}. \tag{25}$$

Since $B_{ij\nu 0}(z) \equiv \delta_{j\nu}$, (25) is also valid for $l = 0$.

By Lemma 1 we get

$$\max_{0\leqslant l\leqslant k}\left|\frac{C_{ij}^{(k-l)}(\theta)}{(k-l)!}\right| \leqslant n\gamma_7^{(2\omega-1)\log N}E_0^N\theta^{n-n_i-(k-l)}\max_{0\leqslant s\leqslant k}\binom{n}{s},$$

and thus, by (iii), (17), (22), (25) and $\max\limits_{0\leqslant s\leqslant k}\binom{n}{s} \leqslant 5\left(\frac{en}{\nu(k)}\right)^{\nu(k)}$, the inequality (23) immediately follows.

To prove (24) we put $\Gamma_k = k(n+N-N/\omega-k)^{-1}$ (see [8], p. 916; if $k = 0$, then (24) follows from (2°) of Lemma 1). Now $\omega > S(d+2)$, $N > 4\gamma_{12}$ and $q > 4A$, and thus we obtain

$$(1+\Gamma_k)\theta < 2q^{-1} < (2A)^{-1}, \quad k = 1, \ldots, [N/\omega]+\gamma_{12}.$$

Therefore the circles $|z-\theta| \leqslant \Gamma_k\theta$ $(k = 1, \ldots, [N/\omega]+\gamma_{12})$ lie in the circle $|z| < (2A)^{-1}$, and thus the use of Cauchy's integral formula

$$R_0^{(k)}(\theta) = \frac{k!}{2\pi i}\oint_{|z-\theta|=\Gamma_k\theta}\frac{R_0(z)}{(z-\theta)^{k+1}}\,dz$$

together with (2°) of Lemma 1 gives (24) exactly as in [8], p. 916. Thus Lemma 3 is true.

4. Proof of Theorem 1. Let

$$\gamma_4 = \max\{\gamma_9+1, 4\gamma_{12}+1, (d+2)(\gamma_{12}+2)\}, \quad \gamma_5 = (d+1)\gamma_{12}+1. \tag{25}$$

We prove that (6) is valid provided that the conditions of Theorem 1 are satisfied.

First we consider the case $H \geqslant q^a$. Let

$$n_i = \left[\frac{\log(h_iH^\tau)}{\log q}\right], \quad i = 1, \ldots, r, \quad n_0 = \max_{1\leqslant i\leqslant r}\{n_i\}. \tag{26}$$

Since $a\log q \leqslant \log H$, we have, by (5),

$$\left(\tau - \frac{1+S\tau}{\omega}\right)\frac{\log H}{\log q} \geqslant \left(\tau - \frac{1+S\tau}{\omega}\right)a > \gamma_4,$$

which gives, by (25) and (26),

$$n_i > \frac{\log h_i + \tau \log H}{\log q} - 1 \geqslant \left(\frac{1+S\tau}{\omega}\right)\frac{\log H}{\log q} + \gamma_4 - 1 \geqslant \frac{N}{\omega} + \gamma_4 - 1 \geqslant \frac{N}{\omega} + \gamma_9,$$

$$N > \frac{SN}{\omega} + S(\gamma_4 - 1) > 4S\gamma_{12} \geqslant 4\gamma_{12}.$$

Thus we can use Lemma 3, by which we find S linear forms, say $r_{k_i}(\theta)$ $(i = 1, \ldots, S)$, $\theta = q^{-1}$, that together with the form $L(\theta)$ are linearly independent. Then the determinant Δ of these linear forms is different from zero and, by Lemma 3, an integer in $\boldsymbol{I}$, whence $|\Delta| \geqslant 1$.

The determinant Δ can be expressed in the form

(27) $$\Delta = \begin{vmatrix} L(\theta) & x_{11} & \ldots & x_{1s_1} & \ldots & x_{r1} & \ldots & x_{rs_r} \\ r_{k_1}(\theta) & c_{11k_1}(\theta) & \ldots & c_{1s_1k_1}(\theta) & \ldots & c_{r_1k_1}(\theta) & \ldots & c_{rs_rk_1}(\theta) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{k_S}(\theta) & c_{11k_S}(\theta) & \ldots & c_{1s_1k_S}(\theta) & \ldots & c_{r_1k_S}(\theta) & \ldots & c_{rs_rk_S}(\theta) \end{vmatrix},$$

and thus, again using Lemma 3, we obtain the estimates

$$1 \leqslant |\Delta| \leqslant |L(\theta)|\{\gamma_{15}^{(2\omega-1)N^{-1}\log N}((1-\theta)^{-2}3bD)^{\omega^{-1}}E_0^S\}^N \times$$
$$\times \left(\prod_{l=1}^{S}\left(\frac{en}{\nu(k_l)}\right)^{\nu(k_l)}\right) q^{N+(d+1)(N/\omega+\gamma_{12})} +$$
$$+ \{\gamma_{16}^{(2\omega-1)N^{-1}\log N}((1-\theta)^{-2}3bD)^{\omega^{-1}}E_0^{S-1}F_0\}^N \max_{(i,j)\neq(0,0)} \{|x_{ij}|\, q^{-n_i}\} \times$$
$$\times \left(\prod_{l=1}^{S}\left(\frac{en}{\nu(k_l)}\right)^{\nu(k_l)}\right) \max_{1\leqslant m\leqslant S}\left\{\left(1+\frac{N}{n}\left(1-\frac{1}{\omega}\right)\right)^{k_m}\right\} q^{(d+2)(N/\omega+\gamma_{12}+1)}.$$

Since (see [7], p. 411; we have $n > N/\omega + \gamma_4 - 1 \geqslant N/\omega + \gamma_{12}$)

$$\prod_{l=1}^{S}\left(\frac{en}{\nu(k_l)}\right)^{\nu(k_l)} \leqslant \left(\frac{eSn}{N/\omega+\gamma_{12}}\right)^{N/\omega+\gamma_{12}} < \left(eS\omega\frac{n}{N}\right)^{N/\omega+\gamma_{12}},$$

it follows that

(28) $$1 \leqslant |\Delta| \leqslant |L(\theta)|\, E^N q^{N+(d+1)(N/\omega+\gamma_{12})} + F^N q^{(d+2)(N/\omega+\gamma_{12}+1)} \max_{1\leqslant i\leqslant r}\{h_i q^{-n_i}\} = \Delta_1 + \Delta_2.$$

Our next purpose is to prove that $2\Delta_2 < 1$. By (26) we have

$$\max_{1\leqslant i\leqslant r}\{h_i q^{-n_i}\} \leqslant qH^{-\tau},$$

and thus our purpose is achieved provided that

(29) $$N\log F + (d+2)(N/\omega + \gamma_{12} + 2)\log q < \tau \log H.$$

Since $H \geqslant q^a$, we obtain, by (5) and (25),

$$(d+2)(\gamma_{12}+2)\log q \leqslant \gamma_4 \log q < \left(\tau - \frac{(1+S\tau)(d+2)}{\omega} - \beta\right)\log H.$$

With this inequality we use the results obtained from (7) and (26), namely

(30) $$(1+S\tau)\log F < \beta \log q; \qquad N\log q \leqslant (1+S\tau)\log H.$$

This implies

$$\tau\log H > \beta\log H + \frac{(1+S\tau)(d+2)}{\omega}\log H + (d+2)(\gamma_{12}+2)\log q$$
$$\geqslant N\log F + (d+2)(N/\omega + \gamma_{12} + 2)\log q.$$

Thus (29) is valid, which implies, by (28) and (30), that

$$1 < 2\Delta_1 = |L(\theta)|\, 2E^N q^{N+(d+1)(N/\omega+\gamma_{12})}$$
$$\leqslant |L(\theta)|\, 2\exp\left\{(1+S\tau)\left(1+\frac{d+1}{\omega}+\frac{\log E}{\log q}\right)\log H\right\} q^{(d+1)\gamma_{12}}$$
$$\leqslant |L(\theta)|\, q^{\gamma_5} H^{(1+S\tau)\left(1+\frac{d+1}{\omega}+\frac{\log E}{\log q}\right)},$$

which gives (6) in the case $H \geqslant q^a$.

If $H < q^a$, then in considering the linear form $L_1(\theta) = [2q^a]L(\theta)$, we obtain, as above,

$$1 \leqslant |L_1(\theta)|\, q^{\gamma_5}(2^S q^{Sa} H)^{(1+S\tau)\left(1+\frac{d+1}{\omega}+\frac{\log E}{\log q}\right)}.$$

Since $E \leqslant F$, the use of (5), (8) and (30) now easily yields (6), and thus Theorem 1 is proved.

5. Proof of Corollary 1. Let ε, $0 < \varepsilon < 1$, be given. For proving Corollary 1 we denote

$$a = \frac{100S(\gamma_4+1)}{7\varepsilon}, \quad \beta = \frac{\varepsilon}{100S}, \quad \omega = \frac{100S(d+2)}{\varepsilon}, \quad \tau = \frac{\varepsilon}{10S}.$$

We then have

$$\left(\tau - \frac{(1+S\tau)(d+2)}{\omega} - \beta\right)a > \frac{7\varepsilon a}{100S} > \gamma_4.$$

Now define the constants λ, E, F by (8), where a, β, ω, τ have the values given above, and let

(31) $$C = \max\{F^{(1+S\tau)\beta^{-1}}, 4A\}+1.$$

In this way we immediately obtain, for all $q \geqslant C$,

$$S\tau+(1+S\tau)\left(\frac{d+1}{\omega}+\frac{\log E}{\log q}\right) < \varepsilon,$$

and thus Corollary 1 follows from Theorem 1.

6. Lemmas II. We shall now consider lemmas required for the proof of Theorem 2. The functions (1) are assumed to satisfy the conditions of this theorem. Let $\omega > (S+1)(d+2)$ be a positive number, and denote by $\Gamma_4, \Gamma_5, \ldots$ positive constants depending only on S and the functions (1).

Although we shall construct the approximation polynomials in a different manner from that used in Lemma 1, earlier considerations can be used in many parts of this section. We replace Lemma 1 by the following lemma.

LEMMA 4. *Let n_i $(i = 0, \ldots, r)$ be natural numbers, and assume that $n_0 \geqslant \max\limits_{1\leqslant i\leqslant r} \{n_i\} \geqslant 3$. Let $N' = n_0+s_1n_1+\ldots+s_rn_r$. There then exist $S+1$ polynomials*

$$P_{ij}(z) = \sum_{\mu=0}^{n_i-1} p_{ij\mu}z^\mu, \quad i = 0,\ j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i,$$

not all identically zero, with the following properties:

(1°°) *All $p_{ij\mu} \in \mathbf{Z}$, and satisfy*

$$|p_{ij\mu}| \leqslant \Gamma_4^{(\omega-1)\log N'} K_0^{N'}, \quad K_0 = (AB)^{(\omega-1)^2\omega^{-1}};$$

(2°°) *We have*

$$U_0(z) = P_{00}(z)+\sum_{i=1}^{r}\sum_{j=1}^{s_i} P_{ij}(z)f_{ij}(z) = \sum_{\nu=0}^{\infty} t_\nu z^\nu,$$

where

$$t_\nu = 0, \quad \nu = 0, 1, \ldots, N'-[N'/\omega]-2,$$

and, for all $|z|_p < B^{-1}$,

$$|U_0(z)|_p \leqslant \Gamma_5(B|z|_p)^{N'-[N'/\omega]-1}.$$

Proof. Part (1°°) can be proved as in Lemma 1, see [8], Lemma 2. Since (let $p_{ij\mu} = 0$ for all $\mu \geqslant n_i$)

$$t_\nu = p_{00\nu}+\sum_{i=1}^{r}\sum_{j=1}^{s_i}\sum_{\mu=0}^{\nu} p_{ij\mu}a_{ij,\nu-\mu}, \quad \nu = 0, 1, \ldots,$$

and $|a_{ij,\nu-\mu}|_p \leqslant \gamma_2 B^{\nu-\mu}$(p. 276), we obtain (let $\gamma_2 \geqslant 1$)

$$|U_0(z)|_p \leqslant \max_{\nu\geqslant N'-[N'/\omega]-1} \{|t_\nu z^\nu|_p\} \leqslant \gamma_2(B|z|_p)^{N'-[N'/\omega]-1},$$

which proves the last part of Lemma 4.

Again we construct linear forms

$$U_k(z) = \frac{d_k}{k!}(T(z))^k U_0^{(k)}(z), \quad k = 1, 2, \ldots,$$

and thus obtain

(32) $$U_k(z) = P_{00k}(z)+\sum_{i=1}^{r}\sum_{j=1}^{s_i} P_{ijk}(z)f_{ij}(z), \quad k = 0, 1, \ldots,$$

where $P_{ijk}(z)$ are polynomials having exactly the same representation (17) as the polynomials $C_{ijk}(z)$ (we only replace C by P). Since $\deg P_{ij}(z) < n_i$, the degrees of these polynomials satisfy

(33) $$\deg P_{ijk}(z) \leqslant n_i+kd,$$

$$i = 0, j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i;\ k = 0, 1, \ldots$$

As in Section 3, denote now

$$U_k(z) = \sum_{i=0}^{S} G_{ki}(z)G_i(z), \quad k = 0, 1, \ldots,$$

where the functions $G_i(z)$ $(i = 0, \ldots, S)$ are, in some order, the functions (1) and $f_{00}(z) \equiv 1$, and $G_{ki}(z)$ are the corresponding polynomials $P_{ijk}(z)$. Let $m_k = n_i$, when $G_k(z)$ is one of the functions $f_{ij}(z)$ $(j = 1, \ldots, s_i;$ $j = 0$ if $i = 0)$. We obtain the following lemma (the constants Γ_6 and Γ_7 are given in the proof of the lemma).

LEMMA 5. *If $m_i > [N'/\omega]+\Gamma_6$ $(i = 0, \ldots, S)$, then the determinant $D(z)$ of the matrix $G = (G_{ki}(z))_{k,i=0,\ldots,S}$ has the form*

$$D(z) = z^{N'-[N'/\omega]-\Gamma_7}D_1(z),$$

where $D_1(z) \not\equiv 0$ is a polynomial of degree $\leqslant [N'/\omega]+\Gamma_6$.

Proof. If $D(z) \equiv 0$, then, as in the proof of Lemma 2, we come to the inequality (21), which now gives

$$\operatorname{ord} D_0(z) > N'-\left[\frac{N'}{\omega}\right]-(S+1)-\Gamma_3.$$

Since

$$\deg D_0(z) < \sum_{i=0}^{l} m_i+(S+1)^2 d,$$

we obtain

$$\sum_{i=l+1}^{S} m_i < \left[\frac{N'}{\omega}\right] + \Gamma_6,$$

where $\Gamma_6 = (S+1)^2(d+1) + \Gamma_8$. We thus have a contradiction if $m_i > [N'/\omega] + \Gamma_6$ $(i = 0, \ldots, S)$, which means that $D(z) \neq 0$. The truth of Lemma 5 is now easily verified. We can choose $\Gamma_7 = S+1$.

Instead of Lemma 3 we use the following lemma, whose proof utilizes Lemmas 4 and 5.

LEMMA 6. *Let $n_i > [N'/\omega] + \Gamma_6$ $(i = 0, \ldots, r)$. Let q, q_1 be any natural numbers satisfying $q_1 < q$, $(q, q_1) = 1$, $|q|_p < B^{-1}$, $T(q/q_1) \neq 0$, and denote $\theta = q/q_1$. There then exist $S+1$ numbers $k_0, k_1, \ldots, k_S$, such that*

$$k_0 + k_1 + \ldots + k_S \leqslant \left[\frac{N'}{\omega}\right] + \Gamma_9$$

and the linear forms

$$U_{k_0}(\theta),\ U_{k_1}(\theta),\ \ldots,\ U_{k_S}(\theta)$$

are linearly independent. Further all the numbers

$$p_{ijk}(\theta) = q_1^{n_i+kd} P_{ijk}(\theta)$$

$(i = 0,\ j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i;\ k = 0, \ldots, [N'/\omega] + \Gamma_9)$ are rational integers, and for these values of i, j, k we have the estimates

(34) $$\max_j \{|P_{ijk}(\theta)|\} \leqslant \Gamma_{10}^{(\omega-1)\log N'} K_0^{N'} \big(2b(d+1)^2 D\big)^k \theta^{n_i+kd} \left(\frac{en}{\nu(k)}\right)^{\nu(k)}, \quad n = n_0,$$

(35) $$|U_k(\theta)|_p \leqslant \Gamma_5 B^k (B|q|_p)^{N'-[N'/\omega]-k-1}.$$

Proof. The first part follows from Lemma 5 (see the proof of [7], Lemma 4), and it is also clear that the numbers $p_{ijk}(\theta)$ $(i = 0,\ j = 0;\ i = 1, \ldots, r,\ j = 1, \ldots, s_i;\ k = 0, 1, \ldots)$ are integers, because the coefficients of $P_{ijk}(z)$ are integers and (33) is valid. As in the proof of Lemma 3, we obtain (now $|z| > 1$), for all $i = 1, \ldots, r$, $j = 1, \ldots, s_i$, $\nu = 0, \ldots, s_i$,

$$B_{ij\nu l}(z) \ll b^l (1 + z + \ldots + z^d)^l \prod_{\mu=0}^{l-1} \big(2(d+1)\mu + S\big), \quad l = 0, 1, \ldots,$$

by which

$$|B_{ij\nu l}(\theta)| \leqslant \big(2b(d+1)^2\big)^l \theta^{dl} (l!) \Gamma_{11}^{\log N'}, \quad l = 0, \ldots, [N'/\omega] + \Gamma_9.$$

Since, by Lemma 4,

$$\max_{0 \leqslant l \leqslant k} \left| \frac{P_{ij}^{(k-l)}(\theta)}{(k-l)!} \right| \leqslant n_i \Gamma_4^{(\omega-1)\log N'} K_0^{N'} \theta^{n_i-k+l-1} \max_{0 \leqslant s \leqslant k} \binom{n_i}{s},$$

we get, as in Lemma 3, the estimates (34).

Obviously

$$|U_k(\theta)|_p \leqslant |(k!)^{-1} U_0^{(k)}(\theta)|_p, \quad k = 0, 1, \ldots,$$

and thus the use of Lemma 4 gives

$$|U_k(\theta)|_p \leqslant \left| \sum_{\nu = N'-[N'/\omega]-1}^{\infty} t_\nu \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!} \theta^{\nu-k} \right|_p$$
$$\leqslant \max_{\nu \geqslant N'-[N'/\omega]-1} \left\{ \left| \binom{\nu}{k} t_\nu \theta^{\nu-k} \right|_p \right\} \leqslant \max_{\nu \geqslant N'-[N'/\omega]-1} \{\gamma_2 B^k (B|q|_p)^{\nu-k}\}$$
$$\leqslant \Gamma_5 B^k (B|q|_p)^{N'-[N'/\omega]-k-1}.$$

This proves Lemma 6.

7. Proof of Theorem 2. Let

(36) $$\Gamma_1 = \max\{\Gamma_6 + 1, (d+1)\Gamma_9 + 2 + \Gamma_5\}, \quad \Gamma_2 = d\Gamma_9 + 2,$$

and let the assumptions of Theorem 2 be satisfied.

Firstly we consider the case $H_1 \geqslant q^a$. Define the numbers n_i, n, n' by

(37) $$n_i = \left[\frac{\log(h_i H_1^\tau)}{\log q}\right], \quad i = 0, \ldots, r, \quad n = n_0 = \max_i \{n_i\};$$
$$n' = \min_i \{n_i\}.$$

We then have, by (11), (36) and (37), for all $i = 0, \ldots, r$,

$$n_i > \frac{\log h_i + \tau \log H_1}{\log q} - 1 > \frac{1 + (S+1)\tau}{\omega} \frac{\log H_1}{\log q} + \Gamma_1 - 1 \geqslant \frac{N'}{\omega} + \Gamma_6,$$

and thus the first assumption of Lemma 6 is satisfied. The constant Γ_3 in (14) can be assumed to be greater than 1, and therefore it also follows, by (11) and (13), that $|q|_p < B^{-1}$.

By Lemma 6 we find linearly independent linear forms $L(\theta)$ and, say $U_{k_1}(\theta), \ldots, U_{k_S}(\theta)$, where $\theta = q/q_1$ and $k_1 + \ldots + k_S \leqslant [N'/\omega] + \Gamma_9$. If Δ is the determinant of these linear forms, then the number

$$\Delta' = q_1^{N'-n'+(k_1+\ldots+k_S)d} \Delta$$
$$= \begin{vmatrix} x_{00} q_1^{n_0-n'} & x_{11} q_1^{n_1-n'} & \ldots & x_{1s_1} q_1^{n_1-n'} & \ldots & x_{r1} q_1^{n_r-n'} & \ldots & x_{rs_r} q_1^{n_r-n'} \\ p_{00k_1}(\theta) & p_{11k_1}(\theta) & \ldots & p_{1s_1k_1}(\theta) & \ldots & p_{r1k_1}(\theta) & \ldots & p_{rs_rk_1}(\theta) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{00k_S}(\theta) & p_{11k_S}(\theta) & \ldots & p_{1s_1k_S}(\theta) & \ldots & p_{r1k_S}(\theta) & \ldots & p_{rs_rk_S}(\theta) \end{vmatrix}$$

is different from zero and, by Lemma 6, an integer.

By using (34) we obtain the upper bound

$$|\Delta'| \leqslant (S+1)!\,\Gamma_{10}^{(\omega-1)S\log N'} K_0^{SN'} q^{N'} \max_{i,j}\{|x_{ij}|q^{-n_i} q_1^{n_i-n'}\} \times$$
$$\times \prod_{l=1}^{S}\left\{\left(2b(d+1)^2 D\right)^{k_l} q^{dk_l}\left(\frac{en}{\nu(k_l)}\right)^{\nu(k_l)}\right\}.$$

From (37) it follows that $\max_{i,j}\{|x_{ij}|q^{-n_i}\} \leqslant qH_1^{-\tau}$, and thus (see p. 282) we can find Γ_3 such that

$$|\Delta'| \leqslant K^{N'} q_1^{n-n'} q^{N'+d(N'/\omega+\Gamma_9)+1} H_1^{-\tau}.$$

By this inequality we immediately obtain ($|q_1|_p = 1$)

(38) $$|\Delta|_p = |\Delta'|_p \geqslant K^{-N'} q_1^{n'-n} q^{-N'-d(N'/\omega+\Gamma_9)-1} H_1^{\tau}.$$

The determinant Δ can be expressed in a similar form to (27), from which

$$\Delta = XL(\theta) + \sum_{l=1}^{S} X_l U_{k_l}(\theta),$$

where $\max_l\{|X|_p, |X_l|_p\} \leqslant 1$, whence

(39) $$|\Delta|_p \leqslant \max_l\{|L(\theta)|_p, |U_{k_l}(\theta)|_p\}.$$

We next prove that $|U_{k_l}(\theta)|_p < |\Delta|_p$ for all $l = 1, \ldots, S$. By (35) and (38) this is satisfied provided that

$$H_1^{\tau} > M^{N'} q_1^{n-n'} q^{N'+d(N'/\omega+\Gamma_9)+1+\Gamma_5} |q|_p^{N'-2N'/\omega-\Gamma_9-1},$$

and, since $|q|_p < q^{-1+\delta}$, this inequality follows from the inequality

(40) $$H_1^{\tau} > M^{N'} q_1^{n-n'} q^{(N'/\omega)(d+2+\delta(\omega-2))+\Gamma_1}.$$

By (11)

$$\left(\tau - (1+(S+1)\tau)(d+2+\delta(\omega-2))\omega^{-1} - \beta\right)\log H_1 > \Gamma_1 \log q,$$

whence, by (13) and (37)

$$\tau\log H_1 > \beta\log H_1 + (d+2+\delta(\omega-2))(1+(S+1)\tau)\omega^{-1}\log H_1 + \Gamma_1\log q$$
$$\geqslant \frac{\beta\log q}{1+(S+1)\tau} N' + (d+2+\delta(\omega-2))\frac{N'}{\omega}\log q + \Gamma_1\log q$$
$$\geqslant N'\log(Mq_1) + \frac{N'}{\omega}(d+2+\delta(\omega-2))\log q + \Gamma_1\log q.$$

Thus (40) is valid, and then (38) and (39) give

(41) $$|L(\theta)|_p > (Kq_1)^{-N'} q^{-N'-d(N'/\omega+\Gamma_9)-1} H_1^{\tau}.$$

By (37) we have

$$N'\log(Kq_1) + N'\left(1+\frac{d}{\omega}\right)\log q \leqslant (1+(S+1)\tau)\left(1+\frac{d}{\omega}+\frac{\log(Kq_1)}{\log q}\right)\log H_1.$$

By using (36) and (41) we now obtain (12) in the case $H_1 \geqslant q^{\alpha}$. The proof can be completed in a manner analogous to the proof of Theorem 1. Thus Theorem 2 is true.

8. Proof of Corollary 2 and certain additional results. Let ε, $0 < \varepsilon < 1$, be given. To prove Corollary 2 we use Theorem 2, choosing

$$\alpha = \frac{100(S+1)(\Gamma_1+1)}{6\varepsilon}, \quad \beta = \frac{\varepsilon}{100(S+1)}, \quad \omega = \frac{100(S+1)(d+2)}{\varepsilon},$$
$$\tau = \frac{\varepsilon}{10(S+1)}, \quad \delta = \frac{\varepsilon}{100(S+1)}.$$

Further let λ_1, K, M be given by (14), where α, β, ω, τ have the values given above, and denote

$$C_1 = M, \quad c = (1+(S+1)\tau)\beta^{-1}.$$

We now see immediately that (11) is valid, and we also have

$$(S+1)\tau + (1+(S+1)\tau)\left(\frac{d}{\omega}+\frac{\log(Kq_1)}{\log q}\right) < \varepsilon,$$

if q, q_1 are natural numbers satisfying $q > (C_1 q_1)^c$. Thus Corollary 2 follows from Theorem 2.

Our next purpose is to improve slightly on one special case of Flicker's [6] Theorem, namely the case when the algebraic number field $\boldsymbol{K}$ equals $\boldsymbol{Q}$. To do this we follow the proof of Theorem 2, but choose

$$H_1 = h^{S+1}, \quad n = \left[\frac{\log(hH_1^{\tau})}{\log q}\right], \quad n_i = n, \quad i = 0, 1, \ldots, r,$$

where $h = h_0 = \max_{i,j}\{|x_{ij}|\}$. It is important that $n = n' = \min_i\{n_i\}$, which means that q_1 does not appear in (38) or (40). In this way we arrive at the following theorem (where we can assume, without loss of generality, that $r = 1$).

THEOREM 3. *Let the assumptions of Theorem 2 be valid. There then exist positive constants Γ_1, Γ_2, Γ_3, depending only on S and the functions* (1), *such that if α, β, $\omega > 2$, τ and δ are positive numbers satisfying* (11), *then*

$$\left|L\left(\frac{q}{q_1}\right)\right|_p > q^{-\lambda_1} h^{(S+1)(\tau-(1+(S+1)\tau)(1+d\omega^{-1}+(\log K)(\log q)^{-1}))}$$

for all natural numbers q, q_1 satisfying

$$(q, q_1) = 1, \quad q > \max\{M^{(1+(S+1)r)\beta^{-1}}, q_1\}, \quad |q|_p < q^{-1+\delta}, \quad T\left(\frac{q}{q_1}\right) \neq 0,$$

where λ_1, K, M are given by (14), *and* $h = \max_{i,j}\{|x_{ij}|\}$.

Now let $f_j(z)$ $(j = 1, \ldots, m)$ be p-adic G-functions satisfying

$$y_j' = Q_{j0}(z) + \sum_{k=1}^{m} Q_{jk}(z) y_k, \quad j = 1, \ldots, m, \tag{42}$$

where all the functions $Q_{jk}(z) \in \boldsymbol{Q}(z)$, and assume that these functions $f_j(z)$ $(j = 1, \ldots, m)$ belong to the class $G(\boldsymbol{Q}, A, B, D)$, where the positive numbers A, B, D are not smaller than 1. Suppose that our functions do not satisfy any equation of the form

$$P(z, f_1(z), \ldots, f_m(z)) \equiv 0,$$

where $P(x, x_1, \ldots, x_m) \not\equiv 0$ is a polynomial in $\boldsymbol{Q}[x, x_1, \ldots, x_m]$ of degree $\leqslant d_0$ with respect to $x_1, \ldots, x_m$. By applying Lemma 7 of [7] and our Theorem 3 to the functions

$$f_1^{i_1}(z) \ldots f_m^{i_m}(z), \quad 1 \leqslant i_1 + \ldots + i_m \leqslant s, \quad s \leqslant d_0,$$

where i_j $(j = 1, \ldots, m)$ are non-negative integers, we obtain the following corollary giving the special case $\boldsymbol{K} = \boldsymbol{Q}$ of Flicker's [6] Theorem.

COROLLARY 3. *Let the functions $f_j(z)$ $(j = 1, \ldots, m)$ be as above, and let $\varepsilon, 0 < \varepsilon < 1$, be given. Let $P(x_1, \ldots, x_m) \not\equiv 0$ be any polynomial in $\boldsymbol{Z}[x_1, \ldots, x_m]$ of total degree $s \leqslant d_0$, and of height $\leqslant H$. There then exist positive constants c_1, c_2, δ, depending only on ε, m, s and the functions $f_j(z)$ $(j = 1, \ldots, m)$, such that*

$$\left|P\left(f_1\left(\frac{q}{q_1}\right), \ldots, f_m\left(\frac{q}{q_1}\right)\right)\right|_p > q^{-c_1} H^{-\binom{m+s}{s}-\varepsilon}$$

for all natural numbers q, q_1 satisfying $(q, q_1) = 1$, $q > \max\{c_2, q_1\}$, $|q|_p < q^{-1+\delta}$, q/q_1 different from the poles of the functions $Q_{jk}(z)$ $(j = 1, \ldots, m$, $k = 0, \ldots, m)$.

9. Examples. From Lemmas 7–9 of [7] it follows that we can apply our results to the functions

$$f_{ij}(z) = \log^j(1 + a_i z), \quad i = 1, \ldots, r, \; j = 1, \ldots, s_i,$$

where $a_1, \ldots, a_r$ are distinct non-zero numbers in $\boldsymbol{I}$. Corollaries 1 and 2 give the following results concerning logarithms and p-adic logarithms, respectively.

COROLLARY 4. *Let $\varepsilon, 0 < \varepsilon < 1$, be given. Assume that $a_1, \ldots, a_r$ are distinct non-zero numbers in $\boldsymbol{I}$, and let*

$$P_i(z) = \sum_{j=1}^{s_i} x_{ij} z^j, \quad i = 1, \ldots, r,$$

be polynomials with integer coefficients in $\boldsymbol{I}$, not all identically zero. Further let x_0 be any integer in $\boldsymbol{I}$. There then exist positive constants C, λ, depending only on ε, $S = s_1 + \ldots + s_r$, $a_1, \ldots, a_r$, such that for all natural numbers $q > C$ we have

$$\left|x_0 + P_1\left(\log\left(1 + \frac{a_1}{q}\right)\right) + \ldots + P_r\left(\log\left(1 + \frac{a_r}{q}\right)\right)\right| > q^{-\lambda} H^{-1-\varepsilon},$$

where $H = \prod_{i=1}^{r} h_i^{s_i}$, $h_i = \max\{1, |\text{coeff } P_i(z)|\}$ $(i = 1, \ldots, r)$.

COROLLARY 5. *Assume that $\boldsymbol{I} = \boldsymbol{Q}$, and let the assumptions of Corollary 4 be valid. There then exist positive constants C_1, λ_1, δ, depending only on $\varepsilon, S, a_1, \ldots, a_r$, such that for all natural numbers q satisfying $q > C_1$, $|q|_p < q^{-1+\delta}$ we have*

$$|x_0 + P_1(\log(1 + a_1 q)) + \ldots + P_r(\log(1 + a_r q))|_p > q^{-\lambda_1} H_1^{-1-\varepsilon},$$

where $H_1 = h_0 H$, $h_0 = \max_i\{|x_0|, h_i\}$.

In the special case $r = 1$ our Corollary 4 is analogous to Baker's [1] Theorem 1. Baker gives the constants more explicitly, however.

Using Corollary 3 we also obtain the following result.

COROLLARY 6. *Let $\boldsymbol{I} = \boldsymbol{Q}$, and let $\varepsilon, a_1, \ldots, a_r$ be as in Corollary 4. Let $P(x_1, \ldots, x_r) \not\equiv 0$ be any polynomial in $\boldsymbol{Z}[x_1, \ldots, x_r]$ of degree $\leqslant s$ and height $\leqslant H$. There then exist positive constants c_1, c_2, δ, depending only on $\varepsilon, r, s, a_1, \ldots, a_r$, such that*

$$|P(\log(1 + a_1 q), \ldots, \log(1 + a_r q))|_p > q^{-c_1} H^{-\binom{r+s}{s}-\varepsilon}$$

for all natural numbers q satisfying $q > c_2$, $|q|_p < q^{-1+\delta}$.

Let us now consider the functions

$$\varphi_j(z) = \sum_{n=1}^{\infty} n^{-j} z^n, \quad j = 1, \ldots, s,$$

satisfying

$$\varphi_1'(z) = \frac{1}{1-z}, \quad \varphi_j'(z) = \frac{1}{z}\varphi_{j-1}(z), \quad j = 2, \ldots, s.$$

We have

$$\varphi_1^{(n)}(z) = \frac{(n-1)!}{(1-z)^n}, \quad \varphi_j^{(n)}(z) = (n-1)! \sum_{n_1+n_2=n-1} \frac{f^{(n_1)}(z)}{n_1!} \frac{\varphi_{j-1}^{(n_2)}(z)}{n_2!},$$
$$j = 2, \ldots, s, \; n = 1, 2, \ldots,$$

where we denote $f(z) = z^{-1}$. Thus it may be established by induction that the condition (iii) of our definition of the class $G(Q, A, B, D)$ is valid when we choose $T(z) = z(1-z)$, $d_n = d_n'^s$, where $d_n' = \text{l.c.m.}\ \{1, 2, \ldots, n\}$. From paper [14] we have $d_n' \leqslant e^{2n}$. The conditions (i), (ii) of the definition are obviously valid, and thus the functions $\varphi_j(z)$ $(j = 1, \ldots, s)$ belong to some class $G(Q, A, B, D)$.

We now prove that the functions

(43) $$f_{ij}(z) = \varphi_j(\alpha_i z), \quad i = 1, \ldots, r, \; j = 1, \ldots, s,$$

where $\alpha_1, \ldots, \alpha_r$ are distinct non-zero numbers in I, together with the function $f_{00}(z) \equiv 1$ are linearly independent over $C(z)$. Since

$$f_{i1}(z) = \sum_{n=1}^{\infty} n^{-1}(\alpha_i z)^n = -\log(1-\alpha_i z), \quad i = 1, \ldots, r,$$

our assertion is valid for $j = 1$. Suppose now that the functions 1, $f_{ij}(z)$ $(i = 1, \ldots, r,\ j = 1, \ldots, m-1,\ m \geqslant 2)$ are linearly independent, but the functions 1, $f_{ij}(z)$ $(i = 1, \ldots, r,\ j = 1, \ldots, m)$ are linearly dependent. There then exist the smallest suffix n such that

(44) $$P_{00}(z) + \sum_{i=1}^{r} \sum_{j=1}^{m-1} P_{ij}(z)\varphi_j(\alpha_i z) + P_{1m}(z)\varphi_m(\alpha_1 z) + \ldots$$
$$\ldots + P_{nm}(z)\varphi_m(\alpha_n z) = 0,$$

where $P_{ij}(z)$ are polynomials with no common factors, $P_{nm}(z) \not\equiv 0$. We then get

(45) $$P'_{00}(z) + \sum_{i=1}^{r} \left\{ \sum_{j=1}^{m-1} P'_{ij}(z)\varphi_j(\alpha_i z) + \sum_{j=2}^{m-1} \frac{1}{z} P_{ij}(z)\varphi_{j-1}(\alpha_i z) + \right.$$
$$\left. + \frac{\alpha_i}{1-\alpha_i z} P_{i1}(z) \right\} + \sum_{i=1}^{n} \left\{ P'_{im}(z)\varphi_m(\alpha_i z) + \frac{1}{z} P_{im}(z)\varphi_{m-1}(\alpha_i z) \right\} = 0.$$

Suppose that $P'_{nm}(z) \not\equiv 0$. By multiplying the left-hand side of (45) by $T(z) = z(1-\alpha_1 z) \ldots (1-\alpha_r z)$ we obtain a polynomial in z, f_{ij} $(i = 1, \ldots, r,\ j = 1, \ldots, m)$, which must be divisible by the left-hand side of (44), since otherwise we obtain a contradiction by eliminating f_{nm} from the equations (44) and (45). Thus there exists a polynomial

$$A(z) = A_0 + A_1 z + \ldots + A_r z^r \not\equiv 0$$

such that

$$T(z) \left\{ P'_{00}(z) + \sum_{i=1}^{r} \left\{ \sum_{j=1}^{m-1} P'_{ij}(z) f_{ij} + \sum_{j=2}^{m-1} \frac{1}{z} P_{ij}(z) f_{i,j-1} + \frac{\alpha_i}{1-\alpha_i z} P_{i1}(z) \right\} + \right.$$
$$\left. + \sum_{i=1}^{n} \left\{ P'_{im}(z) f_{im} + \frac{1}{z} P_{im}(z) f_{i,m-1} \right\} \right\}$$
$$= A(z) \left\{ P_{00}(z) + \sum_{i=1}^{r} \sum_{j=1}^{m-1} P_{ij}(z) f_{ij} + P_{1m}(z) f_{1m} + \ldots + P_{nm}(z) f_{nm} \right\}$$

identically in z, f_{ij}. This means that we have

$$T(z)P'_{nm}(z) = A(z)P_{nm}(z),$$

(46) $$T(z)\left(P'_{n,m-1}(z) + \frac{1}{z} P_{nm}(z)\right) = A(z)P_{n,m-1}(z)$$

identically in z. Using our assumptions on α_i, we obtain from the first equation (46)

$$\frac{P'_{nm}(z)}{P_{nm}(z)} = \frac{A(z)}{T(z)} = \frac{a}{z} - \sum_{i=1}^{r} \frac{a_i \alpha_i}{1-\alpha_i z},$$

where $a = A_0$ and a_i $(i = 1, \ldots, r)$ are certain constants. Thus

$$P_{nm}(z) = cz^a \prod_{i=1}^{r} (1-\alpha_i z)^{a_i}, \quad c \neq 0.$$

Since $P_{nm}(z)$ is a polynomial in z, the numbers a, a_i are non-negative integers. Let

$$P_{n,m-1}(z) = p_l z^l + p_{l+1} z^{l+1} + \ldots, \quad p_l \neq 0.$$

(If $P_{n,m-1}(z) \equiv 0$, then a contradiction follows from (46).) From the second equation in (46) we get $a \geqslant l$ and

$$lp_l + \delta_{l,a} c = ap_l.$$

If $a = l$, then we have a contradiction $c = 0$. If $a > l$, then $lp_l = ap_l$, which is impossible, because $p_l \neq 0$. Thus $P'_{nm}(z) \equiv 0$, whence $P_{nm}(z) \equiv c' \neq 0$, c' a constant. In this case we use the definition of n and (45), by which $P'_{n,m-1}(z) = -c'/z$. This is impossible, and thus we have proved the linear independence of the functions (43) and $f_{00}(z) \equiv 1$.

The use of the above results and Lemma 7 of [7] together with Corollaries 1 and 2 leads to the following corollaries.

COROLLARY 6. *Let $\varepsilon, 0<\varepsilon<1$, be given. Let $a_1, \ldots, a_r$ be distinct non-zero numbers in* $\boldsymbol{I}$, *and let* $L(z)$ *be a linear form* (4) *with* $f_{ij}(z)$ *given by* (43). *There then exist positive constants* C, λ, *depending only on* ε, $S = s_1 + \ldots + s_r$, $a_1, \ldots, a_r$, *such that*

$$\left|L\left(\frac{1}{q}\right)\right| > q^{-\lambda} H^{-1-\varepsilon}$$

for all natural numbers q satisfying $q > C$.

COROLLARY 7. *Let the assumptions of Corollary 6 with* $\boldsymbol{I} = \boldsymbol{Q}$ *be valid. There then exist positive constants* C_1, λ_1, δ, *depending only on* $\varepsilon, S, a_1, \ldots, a_r$, *such that*

$$|L(q)|_p > q^{-\lambda_1} H_1^{-1-\varepsilon}$$

for all natural numbers q satisfying $q > C_1$, $|q|_p < q^{-1+\delta}$.

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References

[1] A. Baker, *Approximations to the logarithms of certain rational numbers*, Acta Arith. 10 (1964), pp. 315–323.
[2] — *On some diophantine inequalities involving the exponential function*, Canad. J. Math. 17 (1965), pp. 616–626.
[3] — *Transcendental number theory*, Cambridge University Press, Cambridge 1975.
[4] N. I. Fel'dman, *Lower estimates for some linear forms* (Russian), Vestnik Moscov. Univ. Ser. I, Mat. Meh. 22, No. 2 (1967), pp. 63–72.
[5] — *On a linear form*, Acta Arith. 21 (1972), pp. 347–355.
[6] Y. Flicker, *On p-adic G-functions*, J. London Math. Soc. 15 (1977), pp. 395–402.
[7] A. I. Galochkin, *Lower bounds for polynomials of the values of a class of analytic functions* (Russian), Mat. Sbornik 95 (137), No. 3 (1974), pp. 396–417.
[8] — *Lower bounds of linear forms of G-functions* (Russian), Mat. Zametki 18, No. 4 (1975), pp. 541–552.
[9] K. Mahler, *On a paper by A. Baker on the approximation of rational powers of e*, Acta Arith. 27 (1975), pp. 61–87.
[10] — *Lectures on transcendental numbers*, Lecture Notes in Mathematics, 546, Springer-Verlag, Berlin 1976.
[11] Ju. N. Makarov, *On the estimate of the measure of linear independence for the values of E-functions* (Russian), Vestnik Moscov. Univ. Ser. I, Mat. Meh. 33, No 2 (1978), pp. 3–12. (1)

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[12] Ju. V. Nesterenko, *Estimates of the orders of zeros of analytic functions of a certain class and their application to the theory of transcendental numbers* (Russian), Dokl. Akad. Nauk SSSR 205, No. 2 (1972), pp. 292–295.
[13] M. S. Nurmagomedov, *The arithmetic properties of the values of G-functions* (Russian), Vestnik Moscov. Univ. Ser. I, Mat. Meh. 26, No. 6 (1971), pp. 79–86.
[14] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), pp. 64–94.
[15] A. B. Shidlovski, *On a criterion for algebraic independence of the values of a class of integral functions* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 23 (1959), pp. 35–66.
[16] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., No. 1 (1929).
[17] K. Väänänen, *On lower estimates for linear forms involving certain transcendental numbers*, Bull. Austral. Math. Soc. 14 (1976), pp. 161–179.
[18] — *On linear forms of the values of one class of E-functions*, Acta Univ. Oulu A 41, Math. 12 (1976).

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