und es gilt

$$M(|k_p|^2) = |e_p|^2 \sum_{r=0}^{\infty} |a^*_{p^r}|^2 \varphi(p^r).$$

Ein Vergleich mit (26) gibt

$$\left(1-\frac{1}{p}\right)\left(1+\frac{|f(p)|^2}{p}+\ldots\right) = |e_p|^2 \sum_{r=0}^{\infty} |a^*_{p^r}|^2 \varphi(p^r).$$

Damit folgt aus (25) und Hilfssatz 6 die Konvergenz von

$$\sum_{q=1}^{\infty} |a_q|^2 \varphi(q) = M(|f|^2),$$

wie in Satz 2 behauptet.

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Definite binary quadratic forms with class number one

by

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Let k be a totally real algebraic number field, $\mathfrak{o}$ its maximal order. Let (V, q) be a totally positive binary quadratic space over k, let G be a primitive free $\mathfrak{o}$-lattice of rank 2 in (V, q) with $q(G) \subseteq \mathfrak{o}$ (primitive means: the $\mathfrak{o}$-module generated by $q(G)$ is the unit ideal). Let $\mathfrak{d} = \mathfrak{d}G$ be the discriminant of G ($4\cdot$ "volume" $\mathfrak{v}G$ in the terminology of [4], § 82); in our situation $\mathfrak{d}G$ is an integral ideal. Let $\mathfrak{s}G$ be the scale of G ([4], § 82).

Let $H_k(\mathfrak{d}, G)$ be the proper class number of the genus of G over k. If $H_k(\mathfrak{d}, G) = 1$ for a G, let us call $(\mathfrak{d}, k)$ a (generalized) *idoneal number*. The $(\mathfrak{d}, Q)$ are — crudely speaking — Euler's idoneal numbers.

We prove the following

THEOREM. (i) *For a fixed k there are only finitely many idoneal numbers $(\mathfrak{d}, k)$. For $n = [k:Q] > 2$ these idoneal numbers can be enumerated effectively.*

(ii) *For fixed n there are only finitely many idoneal numbers.*

(iii) *Assuming the Generalized Riemann Hypothesis, there are altogether only finitely many idoneal numbers.*

Proof. We use the explicit determination of the Minkowski–Siegel–Maßformula for binary quadratic forms in [6]:

$$M(G) = \frac{\frac{\sqrt{d}}{\pi^n}\sqrt{\frac{\mathfrak{N}(\mathfrak{v}G)}{\mathfrak{N}(\mathfrak{s}G)^2}}}{\prod_{\mathfrak{p}|\mathfrak{s}} 2 \prod_{\mathfrak{p}\nmid\mathfrak{s}}\left(1-\frac{\chi(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)\prod_{\mathfrak{p}|2}\mathfrak{N}\mathfrak{p}^{\left[\frac{\sigma}{2}\right]} r_{\mathfrak{p}}} \tag{1}$$

where $\mathfrak{s} = (\mathfrak{s}G)^{-2}\mathfrak{v}G$, $r_{\mathfrak{p}} = \frac{1}{2}, 1, 2$ or $\left(1-\frac{\chi(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)^{-1}$, where χ is the generating character of $k(\sqrt{-\det G})/k$; let

$$G \cong \begin{pmatrix} 1 & \\ & \pi^s a \end{pmatrix},$$

a a unit, then σ is defined by $\mathfrak{p}^\sigma = 4\mathfrak{o} + 2\mathfrak{p}^{\left[\frac{s}{2}\right]} + \partial(-\det G)$, where ∂ denotes the quadratic defect. Writing

$$L(1,\chi) = \frac{1}{\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}}\right)},$$

we use Stark's inequality

$$L(1,\chi) \geq \frac{C}{ng(n)} d^{-(\sigma_1-1)/2}(\sqrt{f}d)^{-1/n}\frac{1}{\zeta_k(\sigma_1)} \tag{2}$$

for every σ_1 with $1+(8\log(\sqrt{f}d))^{-1} \leq \sigma_1 \leq 2$, where d, ζ_k denote the discriminant and the Dedekind Zeta-function for k, and f is defined by $|\operatorname{disc}(k(\sqrt{-\det G}))| = d^2 f$; C is an effectively computable constant, $g(n) = n!$ and $g(n) = n$, if one assumes the Generalized Riemann Hypothesis (see [7], [2], (4.2) ff., cf. also [5], (9) (there it should read $2^n d$ instead of $2d$)).

Assuming the Generalized Riemann Hypothesis, one has

$$d^{1/n} \geq 188 + o(1) \quad \text{for} \quad n\to\infty \tag{3}$$

(see [3]).

If G has only one class, this implies $M(G) \leq 1/2$. Comparing this with (1), (2), (3) and with [2], (4.3) to (4.9) ff., one gets (i) and (iii) of the theorem for k with $n > 2$. The case $k = \boldsymbol{Q}$ is well known. For quadratic fields k one has only finitely many idoneal numbers $(\mathfrak{d}, k)$, as follows using the Brauer–Siegel-Theorem instead of (2). Similarly one gets (ii) of the theorem (for a similar theorem and proof, see [1], Satz 20).

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On linear forms of a certain class of G-functions and p-adic G-functions

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1. Introduction. In the present paper we employ the ideas of Baker ([2], [3], Chapter 10) and the Siegel–Shidlovski theory ([3], [10], [15], [16]) to examine the linear forms of certain G-functions. We have two main aims, firstly to generalize the results of Galochkin [8], and thus obtain for G-functions an analogue of Makarov's [11] result concerning E-functions, and secondly to find p-adic analogues to the results obtained. Our studies have been motivated by a recent paper of Flicker [6], where he obtains p-adic analogues of the results of Galochkin [7] and Nurmagomedov [13]. Here we shall obtain similar p-adic analogues in connection with the papers [2], [4], [5], [8], [11], [17], [18]. In particular we shall give lower bounds in terms of all the coefficients for the p-adic valuations of linear forms in the values of certain G-functions.

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2. Main results. Let $\boldsymbol{I}$ denote the field of rational numbers or an imaginary quadratic field. We consider r systems of G-functions

$$f_{i1}(z), \ldots, f_{is_i}(z), \quad s_i \geq 1, \quad i = 1, \ldots, r \tag{1}$$

(in [8] $s_i = 1$ $(i = 1, \ldots, r)$), and assume that these functions satisfy the corresponding systems of differential equations

$$y'_{ij} = Q_{ij0}(z) + \sum_{\nu=1}^{s_i} Q_{ij\nu}(z) y_{i\nu}, \quad i = 1, \ldots, r,\ j = 1, \ldots, s_i, \tag{2}$$

where all $Q_{ij\nu}(z) \in \boldsymbol{I}(z)$. We thus immediately obtain, for $l = 0, 1, \ldots$,

$$y_{ij}^{(l)} = Q_{ij0l}(z) + \sum_{\nu=1}^{s_i} Q_{ij\nu l}(z) y_{i\nu}, \quad i = 1, \ldots, r,\ j = 1, \ldots, s_i, \tag{3}$$

where all $Q_{ij\nu l}(z) \in \boldsymbol{I}(z)$.

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