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On sums of powers and a related problem

by

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1. Introduction. K. F. Roth [6] showed that all sufficiently large integers N are representable in the form

$$N = \sum_{s=1}^{50} x_s^{s+1} \quad (x\text{'s being non-negative integers}). \tag{1}$$

In [7], I improved this to $N = \sum_{s=1}^{35} x_s^{s+1}$.

R. C. Vaughan [10] and [11] improved on this further, showing that

$$N = \sum_{s=1}^{26} x_s^{s+1}. \tag{2}$$

Torleiv Kløve [9] found by computations for $N \leqslant 250\,000$ that $N = \sum_{s=1}^{6} x_s^{s+1}$ (for $N \leqslant 250\,000$), and conjectured that for large N, $N = \sum_{s=1}^{4} x_s^{s+1}$.

In this paper, we improve further on (2), and prove the following:

THEOREM 1. *All sufficiently large integers N are representable in the form*

$$N = \sum_{s=1}^{22} x_s^{s+1} \tag{3}$$

where the x's are non-negative integers.

The methods used in [6], [7], [10] or [11] are insufficient to prove (3), and so, we indicate all the necessary changes.

The method in this paper, can also be used to prove

THEOREM 2. *All sufficiently large odd integers N_1, and even integers N_2 are representable in the forms*

$$N_1 = \sum_{s=1}^{23} p_s^{s+1}, \quad N_2 = \sum_{s=1}^{24} p_s^{s+1}, \tag{4}$$

where the p's are primes.

Theorem 2 is an improvement on the corresponding result of R. C. Vaughan [11] and [12] where it is shown that

(5) $$N_2 = \sum_{s=1}^{30} p_s^{s+1}, \quad N_1 = \sum_{s=1}^{31} p_s^{s+1}.$$

2. Preliminary results. Some of the auxiliary results used in the proofs of Theorems 1 and 2 seem to be of interest in themselves, and are more precise than corresponding earlier results. Before formulating these results, we make the following definitions.

DEFINITION A. Given natural numbers $k_1, \ldots, k_s$ with $2 \leqslant k_s \leqslant \ldots \leqslant k_2 \leqslant k_1$ $(s \geqslant 2)$ and real numbers $\lambda_1, \ldots, \lambda_s$ with $0 < \lambda_i \leqslant 1$ $(i = 1, \ldots, s)$, the pairs $(k_1, \lambda_1), (k_2, \lambda_2), \ldots, (k_s, \lambda_s)$ are said to form *admissible exponents*, if for (every) large positive M and every $\varepsilon > 0$, the number of solutions of the equation

(6) $$x_1^{k_1} + x_2^{k_2} + \ldots + x_s^{k_s} = y_1^{k_1} + y_2^{k_2} + \ldots + y_s^{k_s}$$

subject to

(7) $$M^{\lambda_i/k_i} \leqslant x_i \leqslant 2M^{\lambda_i/k_i}, \quad M^{\lambda_i/k_i} \leqslant y_i \leqslant 2M^{\lambda_i/k_i} \quad (i = 1, \ldots, s)$$

is

(8) $$\ll M^{(\sum_{i=1}^{s} \lambda_i/k_i)+\varepsilon}.$$

(Note that this is a generalization of the definition in [3].)

DEFINITION B. With the k's as in Definition A, let $U_s(k_1, \ldots, k_s; M)$ denote the number of distinct integers of the form $u = \sum_{i=1}^{s} x_i^{k_i}$ (the x_i's being non-negative integers), with $u \leqslant M$.

THEOREM 3. *Let the natural numbers $k_1, \ldots, k_s$ satisfy*

(9) $$2 \leqslant k_s < k_{s-1} < \ldots < k_2 \leqslant k_1,$$

and θ_i, δ_i $(i = 1, \ldots, s)$ be defined by

(10) $$\theta_1 = \theta_2 = 1; \quad \theta_{i+1} = \left(1 - \frac{1}{k_i}\right)\theta_i \quad (i \geqslant 2);$$

(11) $$\delta_1 = \delta_2 = 0; \quad \frac{1}{k_1} + \delta_3\theta_3 = \frac{(1+\delta_3)\theta_3}{k_3};$$

and

(12) $$\frac{\theta_i(1+\delta_i)}{k_i} + (\delta_{i+1} - \delta_i)\theta_{i+1} = \frac{(1+\delta_{i+1})}{k_{i+1}}\,\theta_{i+1} \quad (i \geqslant 3).$$

Further let

(13) $$\lambda_i = \theta_i(1+\delta_i) \quad \text{and} \quad \lambda_{i+1} \leqslant \lambda_i \leqslant 1 \quad (i \geqslant 1).$$

Then, the pairs $(k_1, \lambda_1), \ldots, (k_s, \lambda_s)$ form admissible exponents.

[This is an improvement on Theorem 3 of R. C. Vaughan [11] since when the k's are not consecutive integers it is not necessary to take $\delta_3 = \delta_4 = \ldots = \delta_s = \delta$ (as in [11]). The improvement becomes substantial when the differences between consecutive k's get large.]

Proof. The proof is similar to that of Theorem 3 of [11] but must use the fact that in the equation

$$x_1^{k_1} + \ldots + x_s^{k_s} = y_1^{k_1} + \ldots + y_s^{k_s}$$

(subject to (7)), for given $x_1, y_1, x_2, y_2, \ldots, x_{i-1}, y_{i-1}, x_i$ there are at most $O(M^{(\delta_{i+1}-\delta_i)\theta_{i+1}})$ choices for y_i $(i = 3, \ldots, s-1)$.

Let $K_1 = \{9, 11, 16, 17, 20, 23\}$ and $K_2 = \{7, 8, 10, 12, 13, 14, 15, 18, 19, 21, 22\}$.

I. Applying Theorem 3 for the elements of K_1 (i.e. with $k_1 = 23, \ldots, k_6 = 9$), we see that $(k_1, \lambda_1), \ldots, (k_6, \lambda_6)$ form pairs of admissible exponents with

(14) $$\alpha_1 = \sum_{i=1}^{6} \frac{\lambda_i}{k_i} > 0.375349.$$

II. Similarly with the elements of $K_2 \cup \{5\}$ (taking $k_1 = 22, \ldots, k_{11} = 7$, $k_{12} = 5$) by Theorem 3, $(k_1, \lambda_1), \ldots, (k_{12}, \lambda_{12})$ form pairs of admissible exponents with

(15) $$\alpha_2 = \sum_{i=1}^{12} \frac{\lambda_i}{k_i} > 0.72579.$$

Now by an argument similar to that of Theorem 1 in [1], we have

III (taking $k = 6$, $h = 2$, $a = \alpha_1$ in Theorem 2 of [1]) with the k's as in I, $(k_1, \lambda_1'), \ldots, (k_6, \lambda_6')$, $(6, 1)$ form pairs of admissible exponents where

$$\lambda_i' = \lambda\lambda_i \ (i = 1, \ldots, 6),$$

(16) $$\beta_1 = \frac{1}{6} + \left(\sum_{i=1}^{6} \frac{\lambda_i'}{k_i}\right) = \frac{1}{6}\left\{1 + \frac{18\alpha_1}{3+\alpha_1}\right\} > 0.50027$$

and

(17) $$\lambda = \frac{(\beta_1 - 1/6)}{\alpha_1} \quad (0.8888 > \lambda > 0.88879).$$

IV (taking $k_s = 4$, $h = 2$, $\alpha = \alpha_2$ in Theorem 2 of [1]) with the k's as in II, $(k_1, \lambda_1'), \ldots, (k_{12}, \lambda_{12}')$, $(4, 1)$ form pairs of admissible exponents where

$$\lambda_i' = \lambda' \lambda_i \quad (i = 1, \ldots, 12),$$

(18) $$\beta_2 = \frac{1}{4} + \left(\sum_{i=1}^{12} \frac{\lambda_i'}{k_i}\right) = \frac{1}{4}\left\{1 + \frac{12\alpha_2}{3+\alpha_2}\right\} > \frac{5}{6} + \frac{1}{10^3}$$

and

(19) $$\lambda' = \frac{(\beta_2 - 1/4)}{\alpha_2} \quad (0.805197 > \lambda' > 0.805196).$$

For convenience of notation, the conclusions in III and IV are stated as follows:

LEMMA 1. *For $4 \leqslant k \leqslant 23$, there exist μ_k (with $0 < \mu_k \leqslant 1$) with*

(20) $$\mu_4 = \mu_6 = 1; \quad \sigma_1 = \frac{1}{6} + \left(\sum_{k\in K_1} \frac{\mu_k}{k}\right) > 0.50027;$$
$$\frac{1}{4} + \frac{\mu_5}{5} + \left(\sum_{k\in K_2} \frac{\mu_k}{k}\right) > \frac{5}{6} + \frac{1}{10^3}$$

such that

(A) *$\{k, \mu_k\}$ with $k \in K_1 \cup \{6\}$ form pairs of admissible exponents;*

(B) *$\{k, \mu_k\}$ with $k \in K_2 \cup \{4, 5\}$ form pairs of admissible exponents.*

The next two lemmas can be proved in the same way as Theorem 3 using

(21) $$\mu_{23} = \mu_{20} = \lambda; \quad \mu_{17} = \lambda\left(1 - \frac{1}{20}\right)(1+\delta_3); \quad 0.0139 > \delta_3 > 0.0138$$

(λ being defined by (17)).

LEMMA 2. *Letting $\mu_2 = 1$, the number of solutions of the equation*

(22) $$x_2^2 + x_6^6 + \left(\sum_{k\in K_1} x_k^k\right) = y_2^2 + y_6^6 + \left(\sum_{k\in K_1} y_k^k\right)$$

subject to

(23) $$M^{\mu_k/k} \leqslant x_k \leqslant 2M^{\mu_k/k}, \quad M^{\mu_k/k} \leqslant y_k \leqslant 2M^{\mu_k/k} \quad (\textit{for each } k)$$

is $\ll M^{2\sigma_2 - 1 + \varepsilon}$, where

(24) $$\sigma_2 = \tfrac{1}{2} + \sigma_1 \quad (\text{cf. (20)}).$$

LEMMA 3. *The number of solutions of*

$$x_{20}^{20} + x_{17}^{17} + x_5^5 + \left(\sum_{k\in K_2} x_k^k\right) = y_{20}^{20} + y_{17}^{17} + y_5^5 + \left(\sum_{k\in K_2} y_k^k\right)$$

subject to (23) *is*

$$\ll M^{\sigma_3 + \left(\frac{19}{20}\right)\lambda\delta_3 + \varepsilon} \ll M^{2\sigma_3 - \frac{2}{3} - \frac{7}{10^4}},$$

where

(25) $$\sigma_3 = \frac{\mu_{20}}{20} + \frac{\mu_{17}}{17} + \frac{\mu_5}{5} + \left(\sum_{k\in K_2} \frac{\mu_k}{k}\right).$$

3. Notation. Let N denote a large positive integer and δ a small positive constant; μ_k $(4 \leqslant k \leqslant 23)$ be defined as in § 2, and

(26) $$\mu_2 = \mu_3 = 1.$$

Recall that

(27) $$\mu_4 = \mu_6 = 1; \quad 0 < \mu_k < 1 \quad \text{for } k = 5 \text{ and } 7 \leqslant k \leqslant 23.$$

For $2 \leqslant k \leqslant 23$, we define (with $a \leqslant q$ and $(a, q) = 1$)

(28) $$2P_k = N^{\mu_k/k}, \quad f_k = f_k(\alpha) = \sum_{P_k \leqslant x \leqslant 2P_k} e(\alpha x^k),$$

$$J_k = J_k(\beta) = \sum_{(P_k)^k \leqslant y \leqslant (2P_k)^k} \frac{1}{k} y^{\frac{1}{k}-1} e(\beta y), \quad S_k = S_k(a, q) = \sum_{x=1}^{q} e_q(ax^k),$$

$$g_k = g_k(\alpha, a, q) = q^{-1} S_k(a, q) J_k\left(\alpha - \frac{a}{q}\right)$$

(in the rest of the paper, we often abbreviate for the above functions by f_k, J_k, S_k and g_k).

Write

(29) $$F(\alpha) = f_2 f_3 f_4 f_6, \quad F_1(\alpha) = f_5\left(\prod_{k=7}^{23} f_k\right), \quad G(\alpha, a, q) = g_2 g_3 g_4 g_6,$$

(30) $$F_2(\alpha) = F(\alpha)F_1(\alpha) = f_2\left(\prod_{k=2}^{23} f_k\right), \quad F_3(\alpha) = f_2 f_6\left(\prod_{k\in K_1} f_k\right),$$
$$F_4(\alpha) = f_4 f_5\left(\prod_{k\in K_2} f_k\right),$$

(31) $$F_5(\alpha) = f_5 f_{17} f_{20}\left(\prod_{k\in K_2} f_k\right), \quad \mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{5}{4}.$$

Let

(32) $$Q = N^{3/4+\delta}, \quad \tau = 1/4 - 2\delta,$$

and subdivide the interval

$$Q^{-1} \leqslant \alpha \leqslant 1+Q^{-1} \tag{33}$$

as follows:

For $q \leqslant N^{\tau}$, let $\mathfrak{M}_{a,q}$ denote the interval $\alpha = \dfrac{a}{q}+\beta$, $|\beta| \leqslant (qQ)^{-1}$, and denote the aggregate of all $\mathfrak{M}_{a,q}$'s by $\mathfrak{m}$. It can be proved in the standard way that any two $\mathfrak{M}_{a,q}$'s are disjoint. Let $\overline{\mathfrak{M}}$ denote the complement of $\mathfrak{M}$ in (33). Also denote the complement of $\mathfrak{M}_{a,q}$ in (33) by $\overline{\mathfrak{M}}_{a,q}$ $(q \leqslant N^{\tau})$. Writing

$$\begin{aligned} r(N) &= \int_{Q^{-1}}^{1+Q^{-1}} F_2(\alpha)e(-N\alpha)\,d\alpha \\ &= \int_{\mathfrak{M}} F_2(\alpha)e(-N\alpha)\,d\alpha + \int_{\overline{\mathfrak{M}}} F_2(\alpha)e(-N\alpha)\,d\alpha, \end{aligned} \tag{34}$$

we see that $r(N)$ does not exceed the number of representations of N in the form $N = \sum_{s=1}^{22} x_s^{s+1}$ (x's being non-negative integers); so that in order to prove Theorem 1, it suffices to show that $r(N) > 0$ for large N.

We also denote

$$A_1(q) = \sum_{a} q^{-4-\delta}|S_3S_4|^2; \qquad A_2(q) = \sum_{a} q^{-4-\delta}|S_2S_3S_4S_6|; \tag{35}$$

$$A_3(q) = \sum_{a} q^{-4-\delta}|S_2S_3|^2; \qquad z_1 = z_1(\alpha, a, q) = (f_2-g_2)f_3f_4f_6; \tag{36}$$

$$z_2 = (f_3-g_3)g_2f_4f_6; \qquad z_3 = (f_4-g_4)g_2g_3f_6; \qquad z_4 = (f_6-g_6)g_2g_3g_4. \tag{37}$$

4. Some auxiliary lemmas. The next lemma follows from Lemmas 1, 2 and 3 (cf. (20)).

LEMMA 4.

$$\int_0^1 |F_3(\alpha)|^2 d\alpha \ll N^{-1+\varepsilon}\{F_3(0)\}^2, \qquad \int_0^1 |F_4(\alpha)|^2 d\alpha \ll N^{-\frac{5}{6}-10\delta}\{F_4(0)\}^2$$

and

$$\int_0^1 |F_5(\alpha)|^2 d\alpha \ll N^{-\frac{2}{3}-10\delta}\{F_5(0)\}^2.$$

The next lemma follows from Lemma 4 in [2] and Hilfssatz 7.11 in [4].

LEMMA 5. *For* $k = 2, 3, 4$ *and* 6 *(if* $|\beta| \leqslant 1/2$*)*

$$\begin{aligned} J_k(\beta) &\ll \min(N^{1/k}, N^{1/k-1}|\beta|^{-1}); \\ g_k(\alpha, a, q) &\ll q^{-1/k}N^{1/k}\min(1, N^{-1}|\beta|^{-1}) \end{aligned} \tag{38}$$

and

$$f_k(\alpha)-g_k(\alpha, a, q) \ll q^{1-1/k+\varepsilon} \quad (\textit{if } q \leqslant N^{1/k-\delta},\ |\beta| \leqslant q^{-1}N^{1/k-1-\delta}). \tag{39}$$

The next lemma is the main theorem in [5].

LEMMA 6.

$$\sum_{1\leqslant x\leqslant P} e_q(ax^k) - \frac{P}{q}S_k(a, q) \ll q^{1/2+\varepsilon}.$$

LEMMA 7. *For* $k = 2, 3, 4$ *and* 6,

$$f_k(\alpha) - g_k(\alpha, a, q) \ll q^{1/2+\varepsilon}\{\max(1, N|\beta|)\}.$$

Proof. This follows by a partial summation with Lemma 6.

The next lemma is proved in the same manner as Section 10 of [10].

LEMMA 8.

$$\sum_{q\leqslant N^{\tau}} A_1(q) \ll 1; \qquad \sum_{q\leqslant N^{\tau}} A_2(q) \ll 1; \qquad \sum_{q\leqslant N^{\tau}} A_3(q) \ll 1.$$

The next lemma is deduced from Lemmas 5 and 7.

LEMMA 9. *On* $\mathfrak{M}_{a,q}$,

$$z_1 \ll N^{\mu-1/2}; \qquad z_2 \ll N^{\mu-1/3}(q^{-2/3}N^{1/12})\{\min(1, N^{-1}|\beta|^{-1}) + q^{1+\varepsilon}N^{-1/4}\};$$

$$z_3 \ll q^{-2}|S_2S_3|N^{\mu-1/6+\varepsilon}\min(1, N^{-1}|\beta|^{-1});$$

$$z_4 \ll q^{-2}|S_3S_4|N^{\mu-1/6+\varepsilon}\min(1, N^{-1}|\beta|^{-1})$$

(cf. (31), (36) and (37)).

LEMMA 10.

$$\sum_{q\leqslant N^{\tau}} \sum_{a} \int_{\mathfrak{M}_{a,q}} \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\}\,d\alpha \ll N^{2\mu-4/3+\delta}.$$

Proof. This is deduced in a standard way using Lemmas 8 nad 9.

LEMMA 11. *On* m,

$$f_3(\alpha) \ll N^{\frac{1}{3}(1-1/4+2\delta)}.$$

Proof. This is deduced from Weyl's inequality, and is essentially Lemma 8.2 of [10].

5. Integral over m.

LEMMA 12.

$$\int_{m} F_2(\alpha)e(-N\alpha)\,d\alpha \ll N^{-1-\delta}F_2(0) \quad \text{(cf. (30) and (31)).}$$

Proof. Since $F_2 = F_3F_4f_3$, we have by Schwarz's inequality,

$$\int_{m} F_2(\alpha)e(-N\alpha)d\alpha \ll \int_{m} |F_3(\alpha)F_4(\alpha)f_3(\alpha)|d\alpha$$

$$\ll \{\max_{\alpha\in m}|f_3(\alpha)|\}\Big\{\int_0^1 |F_3(\alpha)|^2 d\alpha\Big\}^{1/2}\Big\{\int_0^1 |F_4(\alpha)|^2 d\alpha\Big\}^{1/2}.$$

Result now follows from Lemmas 4 and 11 (on noting that $N^{1/3} \ll f_3(0)$).

6. Integral over $\mathfrak{M}$.

Lemma 13. *With $F(\alpha)$, $G(\alpha, a, q)$ defined by* (29)

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |F(\alpha)-G(\alpha, a, q)|^2 d\alpha \ll N^{-4/3+\delta}\{F(0)\}^2.$$

Proof. From (29), (36) and (37),

$$F(\alpha)-G(\alpha, a, q) = z_1+z_2+z_3+z_4;$$

so that

$$|F(\alpha)-G(\alpha, a, q)|^2 \ll |z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2.$$

Hence, by Lemma 10,

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |F(\alpha)-G(\alpha, a, q)|^2 d\alpha \ll N^{2\mu-4/3+\delta} \ll N^{-4/3+\delta}\{F(0)\}^2 \quad \text{(cf. (31))}.$$

Lemma 14.

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |F(\alpha)-G(\alpha, a, q)|\,|F_1(\alpha)|d\alpha \ll N^{-1-\delta}\{F_2(0)\} \quad \text{(cf. (29))}.$$

Proof. Write

$$A(\alpha) = \begin{cases} F(\alpha)-G(\alpha, a, q) & \text{if } \alpha\in\mathfrak{M}_{a,q},\\ 0 & \text{if } \alpha\in m.\end{cases}$$

Since $\mathfrak{M}_{a,q}$'s (for $q\le N^\tau$) are disjoint, and their union is $\mathfrak{M}$, Lemma 13 is equivalent to

$$\int_{\mathfrak{M}} |A(\alpha)|^2 d\alpha \ll N^{-4/3+\delta}\{F(0)\}^2.$$

Hence, since $\mathfrak{M}$ is contained in the unit interval $Q^{-1}\le\alpha\le 1+Q^{-1}$, we have by Schwarz's inequality,

(40)
$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |F(\alpha)-G(\alpha, a, q)|\,|F_1(\alpha)|d\alpha$$

$$= \int_{\mathfrak{M}} |A(\alpha)|\,|F_1(\alpha)|d\alpha \le \Big\{\int_{\mathfrak{M}} |A(\alpha)|^2 d\alpha\Big\}^{1/2}\Big\{\int_{Q^{-1}}^{1+Q^{-1}} |F_1(\alpha)|^2 d\alpha\Big\}^{1/2}$$

$$\ll \{N^{-4/3+\delta}F^2(0)\}^{1/2}\Big\{\int_0^1 |F_1(\alpha)|^2 d\alpha\Big\}^{1/2}.$$

Now, by (29) and (31), $F_1(\alpha) = \{f_9f_{11}f_{16}f_{23}\}F_5(\alpha)$.

Hence by Lemma 4 and trivial estimates for $f_9, f_{11}, f_{16}, f_{23}$,

(41)
$$\int_0^1 |F_1(\alpha)|^2 d\alpha \ll N^{-2/3-10\delta}\{F_1(0)\}^2.$$

The lemma now follows from Lemma 13, (40) and (41) (since $F_2(0) = F(0)F_1(0)$).

Lemma 15.

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |G(\alpha, a, q)|\,|F_1(\alpha)|d\alpha \ll N^{-1-\delta}F_2(0).$$

Proof. Using Lemma 5, it can be shown in a standard way that

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} |G(\alpha, a, q)|d\alpha \ll N^{\mu-1-2\delta}\sum_{q\le N^\tau} A_2(q).$$

Result now follows from Lemma 8 and the trivial estimate for $F_1(\alpha)$ (using $F(0)F_1(0) = F_2(0)$).

The next lemma follows from Lemmas 14 and 15 in a standard way.

Lemma 16.

$$\sum_{q\le N^\tau}\sum_{a}\int_{\mathfrak{M}_{a,q}} F_2(\alpha)e(-N\alpha)d\alpha -$$

$$-\sum_{q\le N^\tau}\sum_{a}\int_{Q^{-1}}^{1+Q^{-1}} G(\alpha, a, q)F_1(\alpha)e(-N\alpha)d\alpha \ll N^{-1-\delta}F_2(0).$$

Lemma 17.

$$r(N)-\sum_{q\le N^\tau}\sum_{a}\int_{Q^{-1}}^{1+Q^{-1}} G(\alpha, a, q)F_1(\alpha)e(-N\alpha)d\alpha \ll N^{-1-\delta}F_2(0).$$

Proof. Follows from Lemmas 12 and 16 (cf. (34)).

Lemma 18. *Write $u = x_5^5+(\sum_{k=7}^{23} x_k^k)$ with $P_k\le x_k\le 2P_k$ for each k, and $N' = N-u$. Then, each u satisfies*

(42)
$$u = o(N),$$

and

$$\int_{Q^{-1}}^{1+Q^{-1}} G(\alpha, a, q)F_1(\alpha)e(-N\alpha)d\alpha = \{q^{-4}S_2S_3S_4S_6\}\sum_{u} e_q(-N'a)\psi(N'),$$

where

$$\psi(N') = \int_{Q^{-1}}^{1+Q^{-1}} J_2(\beta)J_3(\beta)J_4(\beta)J_6(\beta)e(-N'\beta)d\beta.$$

Also $\psi(N')$ *is real, positive and satisfies*

$$N^{-1}F(0) \ll \psi(N') \ll N^{-1}F(0) \quad \text{(cf. (29))}. \tag{43}$$

Proof. This is a standard type of result proved in the usual way.

7. The singular series. Let

$$A(n,q) = \sum_{a} \{q^{-4}S_2S_3S_4S_6\}e_q(-an),$$

$$\mathfrak{S}(X,n) = \sum_{q\leqslant X} A(n,q), \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q).$$

The treatment of singular series is essentially the same as in [6], but differs in some details (since we replace the 5th power in [6] by 6th power). In place of Lemma 21 in [6], we have the estimate $\sum_{l=2}^{\infty}|A(n,p^l)| \ll p^{-5/4}$. All the other lemmas in the treatment of singular series remain valid including the solubility of the congruences

$$x_2^2+x_3^3+x_6^6 \equiv n \pmod p \quad \text{and} \quad x_2^2+x_3^3+x_6^6 \equiv n-1 \pmod{2^4}$$

(the second congruence requiring a numerical verification). Accordingly, corresponding to Lemmas 28 and 29 in [6], we have

Lemma 19. $\mathfrak{S}(n)$ *is absolutely convergent and* $\mathfrak{S}(n) \gg (\log\log n)^{-C_1}$.

Lemma 20. $\sum_{q\geqslant X} A(n,q) \ll X^{-C_2}n^{\varepsilon}$.

(C_1, C_2 being positive constants.)

8. Proof of Theorem 1. If U denotes the number of u's in Lemma 18, then (cf. (29)) $F_1(0) \ll U \ll F_1(0)$; so that by Lemmas 19, 20 and (43),

$$\sum_{u}\mathfrak{S}(N')\psi(N') \gg N^{-1}F_2(0)(\log\log N)^{-C_1} \qquad \text{(cf. (30))}$$

and

$$\sum_{q>N^{\tau}}\sum_{u} A(N',q)\psi(N') \ll N^{-1}F_2(0)N^{-C_2\tau}N^{\varepsilon} \ll N^{-1-C_3}F_2(0) \qquad (C_3>0).$$

Thus

$$\sum_{u}\mathfrak{S}(N',N^{\tau})\psi(N') \gg N^{-1}F_2(0)(\log\log N)^{-C_1}. \tag{44}$$

Also, by Lemma 18,

$$\sum_{q\leqslant N^{\tau}}\sum_{a}\int_{Q^{-1}}^{1+Q^{-1}} G(\alpha,a,q)F_1(\alpha)e(-N\alpha)\,d\alpha = \sum_{u}\mathfrak{S}(N',N^{\tau})\psi(N').$$

Hence, by Lemma 17 and (44),

$$r(N) \gg N^{-1}F_2(0)(\log\log N)^{-C_1}.$$

Thus $r(N) > 0$ for large N, proving Theorem 1.

9. Outline of proof of Theorem 2. We indicate the main changes required to be made in [8] in order to prove Theorem 2.

I. Theorem A.

$$\int_0^1 |F_3(\alpha)|^2\,d\alpha \ll N^{-1}(\log N)^{C_4}\{F_3(0)\}^2 \qquad \text{(cf. (30))},$$

where C_4 *is a positive constant.*

To prove Theorem A, we need the following lemma which is similar to Satz 3 in [4] but differs in some details.

Lemma 21. *Let* $K_3 = K_1 \cup \{6\}$ *and* $S = \sum_{k\in K_3}(x_k^k - y_k^k)$ *with*

$$P_k \leqslant x_k \leqslant 2P_k, \quad P_k \leqslant y_k \leqslant 2P_k \quad \text{for } k\in K_3 \qquad \text{(cf. (28))}. \tag{45}$$

Then

$$\sum_{S\neq 0} d(|S|) \ll N^{2\sigma_1}(\log N)^{C_4},$$

where $d(n)$ *denotes the divisor function and* $\sigma_1 = \sum_{k\in K_3}\mu_k/k$ (cf. (20)).

Lemma 22. *The number of solutions of*

$$x_2^2 + \Big(\sum_{k\in K_3} x_k^k\Big) = y_2^2 + \Big(\sum_{k\in K_3} y_k^k\Big) \tag{46}$$

with the x_k*'s,* y_k*'s subject to* (45) *for* $k\in K_3$ *and* $k=2$ *is*

$$\ll N^{2\sigma_2-1}(\log N)^{C_4} \qquad \text{(cf. (24))}.$$

Proof. Writing (46) in the form $y_2^2 - x_2^2 = S$, we see that for a given $S\neq 0$, $|y_2-x_2|$ is a divisor of $|S|$ for every pair (x_2,y_2) satisfying (46). Hence, it follows from Lemma 21 that the number of solutions of (46) with $S\neq 0$ is $\ll N^{2\sigma_1}(\log N)^{C_4}$.

Also, by Lemma 1, the number of solutions of (46) with $S=0$, $x_2=y_2$ is

$$\ll N^{\sigma_1+\varepsilon}\cdot N^{1/2} = N^{\sigma_2+\varepsilon} \ll N^{2\sigma_2-1} \qquad \Big(\text{since } \sigma_2 > 1+\frac{2}{10^4}\Big).$$

The lemma follows from these (since by (24), $2\sigma_1 = 2\sigma_2-1$).

Since the integral in Theorem A is the number of solutions of (46), and $N^{2\sigma_2} \ll \{F_3(0)\}^2$ (cf. (30)), Theorem A follows from Lemma 22.

II. With the same k's and λ's as in (15) (but excluding $\frac{\lambda_{12}}{k_{12}} = \frac{\lambda_{12}}{5}$ in the sum)

(47) $$\alpha_3 = \sum_{i=1}^{11}{}' \frac{\lambda_i}{k_i} > 0.634395.$$

Theorem 2 of [1] with $k_s = 5$, $h = 3$, $\alpha = \alpha_3$ gives

(48) $$\beta_3 = \frac{1}{5}\left(\frac{7+33\alpha_3}{7+\alpha_3}\right) > 0.731821.$$

Again taking $k = 4$, $h = 2$, $\alpha = \beta_3$, we have

(49) $$\beta_4 = \frac{1}{4}\left(\frac{3+13\beta_3}{3+\beta_3}\right) > \frac{5}{6} + \frac{4}{10^3}.$$

Also

(50) $$0.80389 < \lambda'' < 0.8039 \quad \text{where} \quad \lambda'' = \frac{(\beta_4 - \frac{1}{4})}{\beta_3}.$$

Letting $K_4 = K_2 \cup \{4, 5\}$, the next lemma follows from (47), (48), (49) and Theorem 2 of [1].

LEMMA 23. *There exist numbers* μ_k' $(k \in K_4)$ *satisfying*

(51) $$0 < \mu_k' \leqslant 1, \quad \mu_4' = 1, \quad \mu_5' = \lambda'', \quad \sum_{k\in K_4}{}' \frac{\mu_k'}{k} = \beta_4 > \frac{5}{6} + \frac{4}{10^3}$$

such that $\{(k, \mu_k')\}$ $(k \in K_4)$ *form pairs of admissible exponents.*

We define (for $k \in K_4$)

(52) $$v_k = v_k(\alpha) = \sum_{P_k' \leqslant x \leqslant 2P_k'} e(\alpha x^k) \quad \text{with } 2P_k' = N^{\mu_k'/k},$$

so that by (28), $v_4(\alpha) = f_4(\alpha)$ (since $\mu_4' = \mu_4 = 1$);

(53) $$F_6(\alpha) = \prod_{k\in K_3} v_k(\alpha) = f_4(\alpha)v_5(\alpha)\Big(\prod_{k\in K_2} v_k(\alpha)\Big); \quad F_7(\alpha) = f_3(\alpha)F_6(\alpha).$$

Then, by Lemma 23,

(54) $$\int_0^1 |F_6(\alpha)|^2 d\alpha \ll N^{\beta_4+\varepsilon} \ll N^{-\beta_4+\varepsilon}\{F_6(0)\}^2 \quad (\text{since } N^{\beta_4} \ll F_6(0)).$$

THEOREM B.
$$\int_0^1 |F_7(\alpha)|^2 d\alpha \ll N^{-1}\{F_7(0)\}^2.$$

Proof. With Q defined by (32), we make the same subdivision of the interval $Q^{-1} \leqslant \alpha \leqslant 1+Q^{-1}$ (into $\mathfrak{M}$ and m) as in the proof of Theorem 1, and show that
$$\int_{Q^{-1}}^{1+Q^{-1}} |F_7(\alpha)|^2 d\alpha \ll N^{-1}\{F_7(0)\}^2,$$
which is equivalent to Theorem B.

As in the proof of Lemma 12, we have from Lemma 11, (49) and (53)

(55) $$\int_m |F_7(\alpha)|^2 d\alpha \ll N^{\frac{2}{3}(1-1/4+2\delta)} \int_0^1 |F_6(\alpha)|^2 d\alpha$$
$$\ll N^{-1+2/3}\{F_6(0)\}^2 \ll N^{-1}\{F_7(0)\}^2 \quad (\text{since } N^{1/3} \ll f_3(0)).$$

LEMMA 24. *On* $\mathfrak{M}_{a,q}$, $v_5(\alpha) \ll q^{-1/7} N^{\mu_5'}$.

Proof. If $W_5(\alpha, a, q)$ is the approximating function corresponding to $v_5(\alpha)$, we have (corresponding to Lemmas 5 and 7),
$$v_5(\alpha) - W_5(\alpha, a, q) \ll q^{1/2+\varepsilon}\max(1, N^{\mu_5'}|\beta|)$$
and
$$W_5(\alpha, a, q) \ll q^{-1/5} N^{\mu_5'/5}\min(1, N^{-\mu_5'}|\beta|^{-1}).$$
The lemma is deduced from these together with (50) and (51).

The next lemma is deduced from Lemmas 5 and 7.

LEMMA 25. *On* $\mathfrak{M}_{a,q}$,
$$f_k - g_k \ll q^{-1/2}N^{1/4}; \quad g_k \ll N^{1/k}q^{-1}|S_k|\min(1, N^{-1}|\beta|^{-1}) \quad (k = 3, 4);$$
$$f_4^2 \ll |g_4|^2 + q^{3/2+2\varepsilon}; \quad f_4 + g_4 \ll q^{-1/4}N^{1/4}.$$

LEMMA 26. *On* $\mathfrak{M}_{a,q}$,
$$(f_3^2 f_4^2 - g_3^2 g_4^2)v_5^2 \ll q^{-2/7}N^{2\mu_5'/5}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5),$$
where
$$\Phi_1 = |f_3 - g_3|\,|g_3|\,|g_4|^2 \ll N^{2(1/3+1/4)}q^{-13/6}\min(1, N^{-1}|\beta|^{-1})^2;$$
$$\Phi_2 = |f_3 - g_3|\,|g_3|\,q^{3/2+2\varepsilon} \ll N^{2/3+1/4+\delta}q^{-4/3}|S_3|;$$
$$\Phi_3 = |f_3 - g_3|\,|g_4|^2 q^{-1/2}N^{1/4} \ll N_1^{2(1/3+1/4)}q^{-13/6}\min(1, N^{-1}|\beta|^{-1})^2;$$
$$\Phi_4 = |f_3 - g_3|(q^{-1/2}N^{1/4})q^{3/2+2\varepsilon} \ll N^{2/3+1/4+\delta}q^{-7/6};$$
$$\Phi_5 = |f_4 - g_4|\,|f_4 + g_4|\,|g_3|^2 \ll N^{2(1/3+1/4)}q^{-3+1/4}|S_3|^2\min(1, N^{-1}|\beta|^{-1})^2.$$

Proof. This is deduced in a standard way from Lemmas 24 and 25.

The next lemma is proved in the same way as Lemma 8.

LEMMA 27.

$$\sum_{q\leqslant N^\tau}\sum_a q^{-7/3-2/7}|S_3| \ll 1; \qquad \sum_{q\leqslant N^\tau}\sum_a q^{-3-2/7+1/4}|S_3|^2 \ll 1;$$

$$\sum_{q\leqslant N^\tau}\sum_a q^{-4-2/7}|S_3|^2|S_4|^2 \ll 1.$$

LEMMA 28.

$$(56)\qquad \sum_{q\leqslant N^\tau}\sum_a \int_{\mathfrak{M}_{a,q}} |(f_3^2f_4^2-g_3^2g_4^2)v_5^2|\,d\alpha \ll N^{2\varrho-1}$$

and

$$(57)\qquad \sum_{q\leqslant N^\tau}\sum_a \int_{\mathfrak{M}_{a,q}} |g_3^2g_4^2v_5^2|\,d\alpha \ll N^{2\varrho-1},$$

where $\varrho = 1/3+1/4+\mu_5'/5$.

Proof. (56) follows in a standard way from Lemmas 26 and 27, (57) is again deduced in a standard way from Lemmas 24, 25, and 27 using the estimate

$$g_3^2g_4^2v_5^2 \ll N^{2\varrho}q^{-4-2/7}|S_3|^2|S_4|^2 \min(1, N^{-1}|\beta|^{-1})^2.$$

It now follows from Lemma 28 that

$$(58)\qquad \int_{\mathfrak{M}} |f_3^2f_4^2v_5^2|\,d\alpha \ll N^{-1}\{f_3(0)f_4(0)v_5(0)\}^2$$

since $N^\varrho \ll f_3(0)f_4(0)v_5(0)$.

Now, using the trivial estimate

$$\prod_{k\in K_2} v_k(\alpha) \ll \prod_{k\in K_2} v_k(0) \qquad (\text{cf. } (53)),$$

we have from (58)

$$(59)\qquad \int_{\mathfrak{M}} |F_7(\alpha)|^2 d\alpha \ll N^{-1}\{F_7(0)\}^2.$$

Theorem B now follows from (55) and (59).

III. Let

$$(60)\qquad 2P_{24} = N^{1/24}, \qquad Q_k = \begin{cases} P_k & \text{if } k = 2, 3, 4, 24 \text{ or } k\in K_3, \\ P_k' & \text{if } k\in K_2, \end{cases}$$

where P_k, P_k' are defined by (28) and (52).

Write

$$(61)\qquad f_k^* = f_k^*(\alpha) = \sum_{Q_k\leqslant p\leqslant 2Q_k} e(\alpha p^k) \quad \text{for } 2\leqslant k\leqslant 24,$$

and (with the C_4 occurring in Theorem A)

$$(62)\qquad C_5 = \frac{C_4}{2}, \quad C_6 = 2^{6\times 24}(C_5+25), \quad L = \log N, \quad F^*(\alpha) = \prod_{k=2}^{24} f_k^*(\alpha).$$

Subdivide the interval

$$(63)\qquad N^{-1}L^{C_6} \leqslant \alpha \leqslant 1+N^{-1}L^{C_6}$$

into basic intervals $\mathfrak{m}_{a,q}^*$ for $q\leqslant L^{C_6}$ with $\alpha = a/q+\beta$, $|\beta|\leqslant q^{-1}N^{-1}L^{C_6}$ and denote the union of $\mathfrak{m}_{a,q}^*$'s (these being disjoint) by $\mathfrak{m}^*$; the supplementary intervals m^* denotes the complement of $\mathfrak{m}^*$ in (63).

Let

$$(64)\qquad r^*(N) = \int_{N^{-1}L^{C_6}}^{1+N^{-1}L^{C_6}} F^*(\alpha)e(-N\alpha)\,d\alpha = \int_{\mathfrak{m}^*} F^*(\alpha)e(-N\alpha)\,d\alpha + \int_{m^*} F^*(\alpha)e(-N\alpha)\,d\alpha.$$

As in Lemma 7 and its corollary in [8] (with slight modifications), we have on m^*,

$$(65)\qquad f_{24}^*(\alpha) \ll NL^{-(C_5+24)}.$$

Replacing (33) and (34) in [8] by Theorems A and B, and arguing as in § 8 of [8], we have from (62) and (65)

$$(66)\qquad \int_{m^*} |F^*(\alpha)|\,d\alpha \ll N^{-1}\{F_3(0)F_7(0)\}L^{-24}.$$

Also, by (28), (52) and (60),

$$L^{C_6} \ll (\log Q_k)^{C_6} \ll L^{C_6} \quad \text{and} \quad N^{-1}L^{C_6} \ll Q_k^{-k}(\log Q_k)^{C_6} \quad \text{for } 2\leqslant k\leqslant 24.$$

Hence, we have by Lemma 8 of [8], on $\mathfrak{m}^*$,

$$(67)\qquad f_k^*(\alpha)-g_k^*(\alpha, a, q) \ll Ne^{-C_7\sqrt{L}} \quad \text{for } 2\leqslant k\leqslant 24 \quad (C_7>0)$$

where g_k^* is the approximating function corresponding to f_k^* given by

$$g_k^*\left(\frac{a}{q}+\beta, a, q\right) = \{\Phi(q)\}^{-1}\Big\{\sum_{\substack{z=1\\(z,q)=1}}^{q} e_q(az^k)\Big\}\Big\{\sum_{Q_k^k\leqslant y\leqslant(2Q_k)^k} y^{1/k-1}(\log y)^{-1}e(\beta y)\Big\}.$$

Also γ as defined in § 9 of [8] is equal to 1 for each prime p if we take $k_1 = 2$, $k_2 = 3, \ldots, k_{23} = 24$. Hence if $M_{23}(p, N) > 0$ for each prime p (noting that the premises of Lemmas 16, 19, 20 in [8] are satisfied), it would follow as in [8] that

$$\text{Re}\Big(\int_{m^*} F^*(\alpha)e(-N\alpha)\,d\alpha\Big) \geqslant N^{-1}\{F_3(0)F_7(0)\}L^{-23}. \tag{68}$$

Then, from (64), (66) and (68) we have $r^*(N) > 0$ for large N provided $M(p, N) = M_{23}(p, N) > 0$ for each prime p.

IV. The argument is completed as follows: Let N be odd. $M(p, N)$ denotes the number of solutions of the congruence

$$\Big(\sum_{k=2}^{24} x_k^k\Big) \equiv N(\text{mod } p), \quad 0 < x_k < p \quad \text{for } 2 \leqslant k \leqslant 24. \tag{69}$$

By Hilfssatz 8.4 of [4], for each prime p, $x_k^k\ (0 < x_k < p)$ has $\dfrac{p-1}{(k, p-1)}$ distinct residue classes mod$p$ ($2 \leqslant k \leqslant 24$). Hence, applying Hilfssatz 8.7 of [4] repeatedly (22 times) we see that the number $n(p)$ of distinct residue classes mod$p$ of $(\sum_{k=2}^{24} x_k^k)$ $(0 < x_k < p)$ satisfies

$$n(p) \geqslant \min\Big(\sum_{k=2}^{24}\Big\{\frac{p-1}{(k, p-1)}\Big\} - 22, p\Big). \tag{70}$$

Now

$$\frac{p-1}{(k, p-1)} \geqslant \frac{p-1}{k} \quad \text{and} \quad \sum_{k=2}^{24}\frac{1}{k} > \frac{5}{2}.$$

Hence, from (70),

$$n(p) \geqslant \min\Big(\frac{5(p-1)}{2} - 22, p\Big). \tag{71}$$

Also $\frac{5}{2}(p-1) - 22 > p$ if $p \geqslant 17$; so that by (71),

$$M(p, N) > 0 \quad \text{for } p \geqslant 17. \tag{72}$$

For $p = 2, 3, 5, 7, 11, 13$ it is an easy verification (by use of Hilfssätze 8.4 and 8.7 of [4]) that $M(p, N) > 0$ (N being odd).

Thus, sufficiently large odd integers N_1 are representable in the form

$$N_1 = \sum_{s=1}^{23} p_s^{s+1} \quad (p\text{'s being primes}), \ (\text{and } N_2 \text{ (even)} = \sum_{s=1}^{24} p_s^{s+1}),$$

completing the proof of Theorem 2.

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