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Rosser's sieve

by

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1. Introduction. Statement of the results. Let there be given a finite integer sequence $\mathscr{A}$ and a sequence P of primes. A basic problem of the sieve is to estimate, for any real number $z \geqslant 2$ the sum (sifting function)

$$S(\mathscr{A}, P, z) = \sum_{\substack{a \in \mathscr{A} \\ (a, P(z))=1}} 1$$

where $P(z) = \prod_{p<z,\, p \in P} p$. The sequence $\mathscr{A}$ can be almost arbitrary. The only knowledge we need about $\mathscr{A}$ is a good approximation formula (in an average sense) for the number of those elements from $\mathscr{A}$ which are divisible by the squarefree number $d | P(z)$;

$$\mathscr{A}_d = \{a \in \mathscr{A};\ a \equiv 0 \,(\mathrm{mod}\, d)\}.$$

We assume that $|\mathscr{A}_d|$ may be written in the form

$$|\mathscr{A}_d| = \frac{\omega(d)}{d} X + R(\mathscr{A}, d), \tag{1.1}$$

where $\frac{\omega(d)}{d} X$ is considered as a main term and $R(\mathscr{A}, d)$ is an error term. The arithmetic function $\omega(d)$ is multiplicative and for each prime number $p \in P$ it satisfies

$$0 < \omega(p) < p. \tag{1.2}$$

Since we need the formula (1.1) only for $d | P(z)$ we are free to define $\omega(p) = 0$ for $p \notin P$.

Our next assumption is about dimension. There exists a parameter $\varkappa \geqslant 0$ (dimension) and a constant $K \geqslant 2$ such that for all $z > w \geqslant 2$ we have

$$\prod_{w \leqslant p < z} \left(1 - \frac{\omega(p)}{p}\right)^{-1} < \left(\frac{\log z}{\log w}\right)^{\varkappa} \left(1 + \frac{K}{\log w}\right). \tag{1.3}$$

Remarks. Note that $\varkappa$ is not uniquely defined, but the smaller it is, the better the results we shall obtain. The advantage that we assumed a one-sided estimation is that one can derive results for $\varkappa$ from those for $\varkappa_1 > \varkappa$. This fact, that sieve results depend essentially only on an upper bound of the type (1.3) was first observed by Halberstam and Richert [3].

The dimension contains two kind of informations; estimation on average of $\omega(p)$, and the distribution of primes from P. It appears in practice that the mean value of $\omega(p)$ is often integer (for example when $\omega(p)$ is the number of solutions of a congruence $\bmod p$), but the set P can be of arbitrary density $\leqslant 1$, so the dimension is not necessarily an integer.

For all $y \geqslant 2$ and $z \geqslant 2$ define $s = \log y/\log z$ and

$$V(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right).$$

All constants implied in the symbols O, $\ll$ will at most depend on $\varkappa$ and we shall not mention this throughout the paper.

THEOREM 1. *For all $y \geqslant z \geqslant 2$ we have*

$$S(\mathscr{A}, z) < XV(z)\{F(s)+e^{\sqrt{K}}Q(s)(\log y)^{-1/3}\}+\sum_{\substack{d<y\\ d|P(z)}}|R(\mathscr{A}, d)|, \tag{1.4}$$

$$S(\mathscr{A}, z) > XV(z)\{f(s)-e^{\sqrt{K}}Q(s)(\log y)^{-1/3}\}-\sum_{\substack{d<y\\ d|P(z)}}|R(\mathscr{A}, d)|. \tag{1.5}$$

For all $s \geqslant 1$ we have

$$Q(s) < \exp\{-s\log s+s\log\log 3s+O(s)\} \tag{1.6}$$

and for $s^{50} < \log z$, we even have

$$Q(s) \leqslant \exp\left\{-s\log s-s\log\log 3s+s\log e\varkappa+O\left(s\frac{\log\log 3s}{\log s}\right)\right\}. \tag{1.7}$$

The functions $F(s)$ and $f(s)$ are the continuous solution of the following system of linear differential-difference equations

$$\begin{cases} s^\varkappa F(s) = A & \text{for } s \leqslant \beta+1,\\ s^\varkappa f(s) = B & \text{for } s \leqslant \beta,\end{cases} \tag{1.8}$$

$$\begin{cases} \big(s^\varkappa F(s)\big)' = \varkappa s^{\varkappa-1}f(s-1) & \text{for } s > \beta+1,\\ \big(s^\varkappa f(s)\big)' = \varkappa s^{\varkappa-1}F(s-1) & \text{for } s > \beta.\end{cases} \tag{1.9}$$

The definition of the numbers A, B and β (sieving limit) requires some results about the solutions $g(s)$ and $h(s)$ of

$$\begin{aligned} \big(sg(s)\big)' &= \varkappa g(s)+\varkappa g(s+1),\\ \big(sh(s)\big)' &= \varkappa h(s)-\varkappa h(s+1),\end{aligned} \tag{1.10}$$

so we give here only a few properties, referring for details to Section 5. We have

$$\begin{aligned} &A > 1, && B > 0, && \beta = 1 && \text{if } \varkappa < 1/2,\\ &A = 2(e^\gamma/\pi)^{1/2}, && B = 0, && \beta = 1 && \text{if } \varkappa = 1/2,\\ &A > 1, && B = 0, && \beta > 1 && \text{if } \varkappa > 1/2.\end{aligned}$$

Moreover, $F(s), f(s)$ are monotonic (decreasing and increasing respectively) and such that

$$0 < f(s) < 1 < F(s) \quad \text{for } s > \beta,$$

$$F(s) = 1+O(e^{-s}), \quad f(s) = 1+O(e^{-s}) \quad \text{as } s\to\infty.$$

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2. Corollaries and comments

2.1. The sieve of dimension $\varkappa < 1/2$. We have $\beta = 1$,

$$A = \frac{e^{\varkappa\gamma}}{\Gamma(1-\varkappa)}\,\frac{\int_0^\infty e^{-z}z^{-\varkappa}\exp\left(\varkappa\int_z^\infty e^{-u}\frac{du}{u}\right)dz}{\int_0^\infty e^{-z}z^{-\varkappa}\cosh\left(\varkappa\int_z^\infty e^{-u}\frac{du}{u}\right)dz} \tag{2.1}$$

and

$$B = \frac{e^{\varkappa\gamma}}{\int_0^\infty e^{-z}z^{-\varkappa}\cosh\left(\varkappa\int_z^\infty e^{-u}\frac{du}{u}\right)dz}. \tag{2.2}$$

THEOREM 2. *Suppose that*

$$|R(\mathscr{A}, d)| \leqslant \omega(d). \tag{2.3}$$

For sufficiently large X and all $z \leqslant X$ the sifting function $S(\mathscr{A}, z)$ is positive. More precisely we have

$$XV(z)\{B-ce^{\sqrt{K}}(\log X)^{-\delta}\} < S(\mathscr{A}, z) < XV(z)\{A+ce^{\sqrt{K}}(\log X)^{-\delta}\} \tag{2.4}$$

uniformly for all $X \geqslant z \geqslant 2$, where $\delta = \min(1-2\varkappa, 1/3)$ and c is a constant depending only on $\varkappa$.

Proof. We have $F(s) \leqslant A$ and $f(s) \geqslant B$, so we must estimate the error term

$$R = \sum_{d<X,\, d|P(z)}\omega(d).$$

In the study of sums of multiplicative functions, it is often convenient to define the generalized von Mangoldt function λ by

$$\omega(d)\log d = \sum_{n|d}\omega(n)\lambda\left(\frac{d}{n}\right).$$

Since ω is multiplicative the support of λ is contained in the set of powers of primes. It is easily seen that $\lambda(p) = \omega(p)\log p$, so

$$\sum_{\substack{d\leqslant X\\ d|P(z)}}\omega(d)\log d = \sum_{\substack{n\leqslant X\\ n|P(z)}}\omega(n)\sum_{\substack{m\leqslant X/n\\ mn|P(z)}}\lambda(m)\leqslant\sum_{\substack{n\leqslant X\\ n|P(z)}}\omega(n)\sum_{p\leqslant X/n}\omega(p)\log p.$$

From (1.3), by partial summation, we obtain

$$\sum_{p\leqslant z}\omega(p)\log p \ll Kz$$

and hence

$$\sum_{\substack{d\leqslant X\\ d|P(z)}}\omega(d)\log d \ll KX\sum_{\substack{n\leqslant X\\ n|P(z)}}\omega(n)/n \leqslant KX/V(z) \ll K^2X(\log X)^{\varkappa}.$$

Using partial summation again, we obtain

$$R \ll K^2X(\log X)^{\varkappa-1} \ll K^2V(z)(\log X)^{2\varkappa-1}.$$

This completes the proof of Theorem 2.

THEOREM 3. *Suppose* (2.3) *holds and, in place of* (1.3), *it holds*

$$\sum_{p<z}\frac{\omega(p)}{p}\log p = o(\log z). \tag{2.5}$$

As $X\to\infty$ *we have*

$$S(\mathscr{A}, z) \sim XV(z)$$

uniformly for all $z\leqslant X$.

Proof. Assumption (2.5) implies (1.3) for arbitrarily small $\varkappa>0$ with the constant K depending on $\varkappa$. Therefore, by Theorem 2 it is sufficient to prove that

$$\lim_{\varkappa=0}A = 1 \quad\text{and}\quad \lim_{\varkappa=0}B = 1.$$

But this follows easily from (2.1) and (2.2).

Remarks. Theorem 3 was first proved by Sullivan (unpublished) by a different method based mainly on the Fundamental Lemma [5] of Halberstam and Richert.

We do not know whether the estimates (1.4) and (1.5) are the best possible (so far as the main terms and general sequences are considered).

We should mention that the Eratosthenes–Legendre sieve (see [10]) yields the following asymptotic formula:

Assume that the elements of $\mathscr{A}$ are not too large; that is

$$\max_{a\in\mathscr{A}}|a| \ll X$$

and the error terms $R(\mathscr{A}, d)$ satisfy (2.3). Next, instead of (1.2) and (1.3), assume $0<\omega(p)<\left(1-\frac{1}{M}\right)p$ and

$$-L<\sum_{w\leqslant p<z}\frac{\omega(p)}{p}\log p-\varkappa\log\frac{z}{w}<K$$

for some constants K, L and $M\geqslant 2$. Then for all $X\geqslant z\geqslant 2$ we have

$$S(\mathscr{A}, z) = XV(z)\{E(s)+O(s^2(\log X)^{2\varkappa-1})\},$$

where $s = \log X/\log z$ and $E(s)$ is the continuous solution of the following equation:

$$s^{\varkappa}E(s) = e^{\gamma\varkappa}/\Gamma(1-\varkappa) \qquad \text{for } 0<s\leqslant 1,$$
$$(s^{\varkappa}E(s))' = \varkappa s^{\varkappa-1}E(s-1) \qquad \text{for } s>1.$$

The function $E(s)$ satisfies

$$f(s)<E(s)<F(s)$$

and $E(s)-1$ changes sign in each interval of length one. The latter fact follows from

$$sh(s)(E(s)-1) = -\varkappa\int_{s-1}^{s}h(x+1)(E(x)-1)\,dx \qquad \text{for } s\geqslant 1$$

and

$$h(s) = \int_0^\infty\exp\left(-sz-\varkappa\int_0^z\frac{1-e^{-u}}{u}\,du\right)dz>0. \tag{2.6}$$

2.2. The sieve of dimension $1/2\leqslant\varkappa\leqslant 1$. The cases $\varkappa = 1/2$ and $\varkappa = 1$ have been considered in [9] and [12] respectively and the results obtained are best possible. The examples showing optimality of (1.4) for $\varkappa = 1/2$ (see [9], [16]) and for $\varkappa = 1$ (see [15], [16]) are quite different and none of them can be adapted for $1/2<\varkappa<1$. The feature of these extreme cases is that both $\varkappa$ and β are simultaneously rational. For $1/2<\varkappa<1$, $\beta-1$ is the zero of

$$g(s) = s^{2\varkappa-1}+\frac{1}{\Gamma(1-2\varkappa)}\int_0^\infty e^{-sz}\left\{\exp\left(\varkappa\int_0^z\frac{1-e^{-u}}{u}\,du\right)-1\right\}z^{-2\varkappa}dz \tag{2.7}$$

so, one can expect that for rational $\varkappa$ with $1/2 < \varkappa < 1$, the sieving limit β is transcendental. It seems to be very difficult to construct $\mathscr{A}$ and P transforming the estimates (1.4) and (1.5) into asymptotic formulae.

Some special values of $\varkappa$ from the interval $1/2 \leqslant \varkappa \leqslant 1$ are very important for applications. For instance, if K/Q is a Galois extension of degree $n \geqslant 2$ then the number of those elements from $\mathscr{A}$ which are norms from K is expressible by the sifting function $S(\mathscr{A}, P, z)$ and the density of P is $1-1/n$. As an example we quote from [11] the following result (2/3 dimensional sieve);

Let K/Q be a cubic normal extension with conductor $\mathfrak{f}$. Each sufficiently large number n satisfying the congruence

$$n \equiv N\mathfrak{a}+N\mathfrak{b} (\operatorname{mod} 2\mathfrak{f})$$

with $(\mathfrak{ab}, 2\mathfrak{f}) = 1$ is expressible as a sum of two norms of integral ideals.

Recall that $B = 0$ and $\beta-1$ is the root of $g(s)$. For A we have the formula

$$A = 2\frac{(\beta-1)^{\varkappa-1}}{h(\beta-1)}. \tag{2.8}$$

Using (2.6) and (2.7) computer calculations give the following approximations.

Table 1

$\varkappa$	β	A
0.55	1.0340	1.6066
0.6	1.1042	1.7254
0.65	1.1922	1.8631
0.7	1.2912	2.0212
0.75	1.3981	2.2020
0.8	1.5107	2.4082
0.85	1.6279	2.6431
0.9	1.7489	2.9106
0.95	1.8731	3.2152

Table 2

$\varkappa$	β	A
1/2	1	$2(e^{\gamma}/\pi)^{1/2}$
2/3	1.2242	1.9134
3/4	1.3981	2.2020
4/5	1.5107	2.4082
5/6	1.5884	2.5614
1	2	$2e^{\gamma}$

We have had the opportunity to see the unpublished computer calculations of Diamond and Jurkat for the sieving limit of the Buchstab iteration method. Their results are very close to those given in Table 1, thus suggesting that the iteration method leads in the limit to the same functions $F(s)$ and $f(s)$. Note that the upper bound is better for all s (even for $s = 2$) than that given by Selberg's method, except in the case $\varkappa = 1$, $s \leqslant 2$ when the results co-incide.

2.3. The sieve of dimension $\varkappa > 1$. In this case the Buchstab iteration method leads (in the limit) to better results if it starts from the Selberg upper bound

$$S(\mathscr{A}, z) < XV(z)\left\{\frac{1}{\sigma(s)}+\frac{Q(s, K)}{\log y}\right\}+\sum_{\substack{d<y\\ d|P(z)}} 3^{\nu(d)}|R(\mathscr{A}, d)| \tag{2.9}$$

which for small s is better than (1.4) (for details see [7]). Here the function $\sigma(s)$ is the continuous solution of

$$\frac{s^{\varkappa}}{\sigma(s)} = 2^{\varkappa}e^{\gamma\varkappa}\Gamma(\varkappa+1) \qquad \text{for } 0 < s \leqslant 2,$$
$$\left(s^{-\varkappa}\sigma(s)\right)' = -\varkappa s^{-\varkappa-1}\sigma(s-2) \qquad \text{for } s > 2.$$

The first iteration step (for all dimensions) was done by Ankeny and Onishi [1]. Their sieving limit satisfies

$$\nu_{\varkappa} \sim \eta\varkappa \quad \text{as } \varkappa \to \infty,$$

where

$$\eta = 2\exp\left\{\int_0^{\log 2}(e^u-1)\frac{du}{u}-1-\log\log 2\right\} = 2.44\ldots$$

For further iterations see [2], [13] and [14].

We shall show that

$$\beta = c\varkappa+O(\varkappa^{2/3}) \quad \text{and} \quad 2(\beta-1)^{\varkappa}\left(1+\frac{\varkappa}{\beta}\right) < A < 2(\beta-1)^{\varkappa}\left(1+\frac{\varkappa}{\beta-1}\right) \tag{2.10}$$

where c is the solution of $c\log c = c+1$ $(c = 3.5911\ldots)$.

2.4. A Fundamental Lemma. It is evident from the above that Selberg's sieve is very strong for large dimensions and small s (at least for $s = 2$). However the function $F(s)$ tends to 1 much faster than $1/\sigma(s)$;

$$\begin{aligned} F(s) &= 1+\exp\left\{-s\log s - s\log\log s + s\log e\varkappa + O\left(s\frac{\log\log s}{\log s}\right)\right\}, \\ \sigma(s) &= 1-\exp\{-\tfrac{1}{2}s\log s - \tfrac{1}{2}s\log\log s + O(s)\}, \end{aligned} \tag{2.11}$$

and thus, for sufficiently large s, (1.4) is sharper than (2.9).

From (1.4), (1.5) and (2.11) we obtain

THEOREM 4 (Fundamental Lemma). *Assume* (1.2) *and* (1.3). *For all $y \geqslant z \geqslant 2$ we have*

$$S(\mathscr{A}, z) = XV(z)\{1+\theta_1 e^{\sqrt{K}}Q(s)\}+\theta_2\sum_{\substack{d<y\\ d|P(z)}}|R(\mathscr{A}, d)|, \tag{2.12}$$

where $|\theta_i| \leqslant 1$, $s = \log y/\log z$ and $Q(s)$ satisfies (1.6) *and* (1.7).

Remarks. Asymptotic formulae of this type were investigated in greater detail in a series of papers by Halberstam and Richert [4], [5], [6]. Their result is essentially (1.6). We should mention, that if $|R(\mathscr{A}, d)| \leqslant \omega(d)$ then the remainder term can be easily estimated as follows:

$$\sum_{\substack{d<y\\ d|P(z)}} \omega(d) \leqslant y \sum_{d|P(z)} \frac{\omega(d)}{d} \leqslant \frac{y}{V(z)} \leqslant yV(z)\, 4K^2(\log z)^{2\varkappa}.$$

3. The weights. We precede the construction of the weights ϱ_d (truncated Möbius function) by some combinatorial identities for the sifting function $S(\mathscr{A}, z)$.

For a given sequence of real numbers $\{\lambda_d\}_{d|P(z)}$ such that $\lambda_1 = 1$ and $\lambda_d = 0$ for $d \geqslant y$, we construct two sequences $\{\varrho_d\}_{d|P(z)}$ and $\{\sigma_d\}_{d|P(z)}$ as follows:

Let $\varrho_1 = 1$ and $\sigma_1 = 0$. For $d > 1$, $d = p_1 \dots p_r$, $p_1 > \dots > p_r$, let

$$\varrho_d = \prod_{1\leqslant l\leqslant r} \lambda_{p_1\dots p_l} \quad \text{and} \quad \sigma_d = (1-\lambda_{p_1\dots p_r}) \prod_{1\leqslant l<r} \lambda_{p_1\dots p_l}.$$

Let $p(d)$ stand for the smallest prime divisor of d if $d > 1$, and anything you like if $d = 1$.

Lemma 1.

$$S(\mathscr{A}, z) = \sum_{d|P(z)} \mu(d)\varrho_d |\mathscr{A}_d| + \sum_{d|P(z)} \mu(d)\sigma_d S(\mathscr{A}_d, p(d)). \tag{3.1}$$

Proof. This follows from a simple application of the well-known Legendre formula

$$S(\mathscr{A}, z) = \sum_{d|P(z)} \mu(d)|\mathscr{A}_d|$$

to $S(\mathscr{A}, z)$ and $S(\mathscr{A}_d, p(d))$.

The identity (3.1) may be found in a slightly different form in [7] (see formula 2.1.8).

We are not going to use any estimates for $S(\mathscr{A}_d, p(d))$ except the trivial one $S(\mathscr{A}_d, p(d)) \geqslant 0$. Thus, in order to get an upper bound for $S(\mathscr{A}, z)$, we must impose

$$\mu(d)\sigma_d \leqslant 0 \quad \text{for all } d|P(z) \tag{3.2}$$

and to get a lower bound we must impose

$$\mu(d)\sigma_d \geqslant 0 \quad \text{for all } d|P(z). \tag{3.3}$$

Writing

$$S^{\pm} = \sum_{d|P(z)} \mu(d)\varrho_d^{\pm} \frac{\omega(d)}{d} \quad \text{and} \quad R^{\pm} = \sum_{d|P(z)} \varrho_d^{\pm} |R(\mathscr{A}, d)|,$$

where the signs $+$, $-$ stress the conditions (3.2) and (3.3) respectively, we obtain

$$S(\mathscr{A}, z) \leqslant XS^{+} + R^{+}, \tag{3.4}$$

$$S(\mathscr{A}, z) \geqslant XS^{-} - R^{-}. \tag{3.5}$$

It would be very interesting to find the weights λ_d^+ and λ_d^- optimizing (3.4) and (3.5) but this seems to be very difficult. A natural reasoning based on the concept of the sieving limit (for the definition see [16]) suggests the choice

$$\lambda_d^{\pm} = \begin{cases} 0 & \text{if } \mu(d) = \mp 1 \text{ and } p(d) \geqslant (y/d)^{1/\beta}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.6}$$

where β ($\geqslant 1$) and y ($\geqslant z^\beta$) are parameters at our disposal. We have $(1 \pm \mu(d))\sigma_d^{\pm} = 0$ so, the conditions (3.2) and (3.3) are satisfied. We have also $|\varrho_d^{\pm}| \leqslant 1$ and $\varrho_d^{\pm} = 0$ if $d \geqslant y$. Hence

$$R^{\pm} \leqslant \sum_{\substack{d<y\\ d|P(z)}} |R(\mathscr{A}, d)|. \tag{3.7}$$

4. Recurrence formulae. For $V(z)$ there holds an identity similar to (3.1), namely

$$V(z) = \sum_{d|P(z)} \mu(d)\varrho_d \frac{\omega(d)}{d} + \sum_{d|P(z)} \mu(d)\sigma_d \frac{\omega(d)}{d} V(p(d)).$$

Hence

$$S^{+} = V(z) + \sum_{r=0}^{\infty} \sum_{\substack{p_{2r+1}<\dots<p_1<z\\ p_{2l+1}^{\beta} p_{2l+1}\dots p_1<y,\, 0\leqslant l<r\\ p_{2r+1}^{\beta} p_{2r+1}\dots p_1 \geqslant y}} \frac{\omega(p_1\dots p_{2r+1})}{p_1\dots p_{2r+1}} V(p_{2r+1}) \tag{4.1}$$

$$= V(z) + \sum_{r=0}^{\infty} S_{r,z}^{+}(s), \text{ say},$$

and

$$S^{-} = V(z) - \sum_{r=1}^{\infty} \sum_{\substack{p_{2r}<\dots<p_1<z\\ p_{2l}^{\beta} p_{2l}\dots p_1<y,\, 0\leqslant l<r\\ p_{2r}^{\beta} p_{2r}\dots p_1 \geqslant y}} \frac{\omega(p_1\dots p_{2r})}{p_1\dots p_{2r}} V(p_{2r}) \tag{4.2}$$

$$= V(z) - \sum_{r=1}^{\infty} S_{r,z}^{-}(s), \text{ say}.$$

Note, that in both cases, S^+ and S^-, we need only upper bounds for $S^{\pm}_{r,z}(s)$ and in consequence only one-sided estimates such as (1.3).

Let us denote

$$T^+_{R,z}(s) = S^+_{0,z}(s) + \ldots + S^+_{R,z}(s)$$

for all $y \geqslant 2$, $z \geqslant 2$, $s = \log y/\log z > \beta - 1$, $R \geqslant 0$ and

$$T^-_{R,z}(s) = S^-_{1,z}(s) + \ldots + S^-_{R,z}(s)$$

for all $y \geqslant 2$, $z \geqslant 2$, $s = \log y/\log z \geqslant \beta$, $R \geqslant 1$.

For $s \geqslant \beta + 2r + \dfrac{1 \pm 1}{2}$, we have $S^{\pm}_{r,z}(s) = 0$.

For $\beta - 1 < s \leqslant \beta + 1$, we have

$$T^+_{0,z}(s) = S^+_{0,z}(s) = \sum_{y^{1/(\beta+1)} \leqslant p < z} \frac{\omega(p)}{p} V(p) = V(y^{1/(\beta+1)}) - V(z). \tag{4.3}$$

It is easy to see that

$$T^-_{R,z}(s) = \sum_{y^{1/(\beta+2R)} \leqslant p < z} \frac{\omega(p)}{p} T^+_{R-1,p}\left(\frac{\log y/p}{\log p}\right), \quad \text{for } s \geqslant \beta,\ R \geqslant 1, \tag{4.4}$$

$$T^+_{R,z}(s) = \sum_{y^{1/(\beta+2R+1)} \leqslant p < z} \frac{\omega(p)}{p} T^-_{R,p}\left(\frac{\log y/p}{\log p}\right), \quad \text{for } s \geqslant \beta+1,\ R \geqslant 1 \tag{4.5}$$

and

$$T^+_{R,z}(s) = T^+_{0,z}(s) + T^+_{R,z}(\beta+1) \qquad \text{for } \beta - 1 < s \leqslant \beta + 1,\ R \geqslant 1. \tag{4.6}$$

5. Some differential-difference equations

5.1. The conjugate equation. In the study of the differential-difference equation

$$sp'(s) = -ap(s) - bp(s-1) \tag{5.1}$$

it is useful to consider the "conjugate" equation

$$\big(sq(s)\big)' = aq(s) + bq(s+1). \tag{5.2}$$

A justification is offered by the integral formula

$$sp(s)q(s) = b\int_{s-1}^{s} p(x)q(x+1)\,dx + C \tag{5.3}$$

which is valid for all sufficiently large s. Although $p(s)$ can be a very wild function, $q(s)$ has always a very smooth behaviour; $q(s) \in C^\infty(0, \infty)$

and $q(s) \sim cs^{a+b-1}$ as $s \to \infty$. For our purpose we need (5.1) with $a = -\varkappa$, $-\varkappa - 1$ and $b = -\varkappa, \varkappa$. Hopefully, other cases will find application in different sieves, so we consider (5.2) with arbitrary coefficients a, b.

It is easy to see that if $a + b = n + 1$ is a positive integer, then $q(s)$ is a polynomial of degree n,

$$q(s) = \sum_{0 \leqslant l \leqslant n} \binom{n}{l} a_l s^{n-l},$$

where $a_0 = 1$ and, for all $l > 0$, we have the recurrence formula

$$la_l = -b \sum_{0 \leqslant i < l} \binom{l}{i} a_i.$$

Example 1. For $a + b = 1, 2, 3$ we have

$$q(s) = 1, \quad q(s) = s - b, \quad q(s) = (s-b)^2 - b/2,$$

respectively.

Example 2. For $a = b = \varkappa = 1/2, 1, 3/2, 2, 5/2, 3$ we have

$$\begin{aligned}
q(s) &= 1,\\
q(s) &= s - 1,\\
q(s) &= s^2 - 3s + \tfrac{3}{2},\\
q(s) &= s^3 - 6s^2 + 9s - \tfrac{8}{3},\\
q(s) &= s^4 - 10s^3 + 30s^2 - \tfrac{85}{3}s + \tfrac{55}{12},\\
q(s) &= s^5 - 15s^4 + 75s^3 - 145s^2 + 90s - \tfrac{18}{5}.
\end{aligned}$$

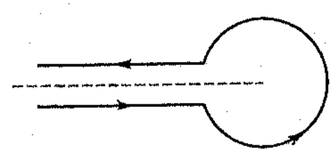
If $s > 0$, the function

$$z^{-a-b}\exp\left(sz + b\int_0^z \frac{1-e^u}{u}\,du\right)$$

tends exponentially to zero as $\operatorname{Re} z \to -\infty$ and $\operatorname{Im} z$ remains bounded. Thus, using integration by parts one can check that

$$q(s) = \frac{\Gamma(a+b)}{2\pi i}\oint_{\mathscr{C}} z^{-a-b}\exp\left(sz + b\int_0^z \frac{1-e^u}{u}\,du\right)dz \tag{5.4}$$

satisfies (5.2), where $\mathscr{C}$ is any curve of the shape



and the power z^{-a-b} is defined by $\exp(-(a+b)\operatorname{Log} z)$, where $\operatorname{Log} z$ is the principal branch of the logarithm. We have

$$\begin{aligned} sq'(s) &= \frac{\Gamma(a+b)}{2\pi i}\oint z^{1-a-b}\exp\left(b\int_0^z \frac{1-e^u}{u}\,du\right)de^{sz} \\ &= -\frac{\Gamma(a+b)}{2\pi i}\oint e^{sz}dz^{1-a-b}\exp\left(b\int_0^z \frac{1-e^u}{u}\,du\right) \\ &= -\frac{\Gamma(a+b)}{2\pi i}\oint (1-a-be^z)\exp\left(sz+b\int_0^z \frac{1-e^u}{u}\,du\right)\frac{dz}{z^{a+b}} \\ &= (a-1)q(s)+bq(s+1) \end{aligned}$$

and hence (5.2) holds.

Expand the function

$$R(z) = \exp\left(b\int_0^z \frac{1-e^u}{u}\,du\right)$$

into the Taylor series

$$R(z) = \sum_{0\leqslant l\leqslant N} R^{(l)}(0)\frac{z^l}{l!}+R_N(z), \text{ say}$$

For $s>0$, we have

$$\frac{1}{2\pi i}\oint_{\mathscr{C}} e^{sz}z^{\nu}\,dz = \frac{s^{1-\nu}}{\Gamma(-\nu)}$$

and, for any integer $l\geqslant 0$,

$$\frac{\Gamma(a+b)}{l!\,\Gamma(a+b-l)} = \binom{a+b-1}{l}.$$

Hence, from (5.4) we get

$$q(s) = \sum_{0\leqslant l\leqslant N}\binom{a+b-1}{l}R^{(l)}(0)s^{a+b-1-l}+\frac{\Gamma(a+b)}{2\pi i}\oint_{\mathscr{C}} e^{sz}R_N(z)\frac{dz}{z^{a+b}}.$$

We have $R_N(z)\ll |z|^{N+1}$ as $z\to 0$, so, if $N>a+b-2$ one can change the contour $\mathscr{C}$ into two negative half-lines with opposite orientations. On the lower half-line we have

$$z^{-a-b} = \exp\{-(a+b)(\log|z|-\pi i)\}$$

and one the upper half-line

$$z^{-a-b} = \exp\{-(a+b)(\log|z|+\pi i)\}.$$

It follows that

$$\frac{\Gamma(a+b)}{2\pi i}\oint_{\mathscr{C}} e^{sz}z^{-a-b}R_N(z)\,dz = \frac{\Gamma(a+b)}{\pi}\sin\pi(a+b)\int_0^\infty e^{-sz}z^{-a-b}R_N(-z)\,dz.$$

Since $\Gamma(a+b)\Gamma(1-a-b)\sin\pi(a+b) = \pi$ we have proved the following lemma:

LEMMA 2. *For any integer $N>a+b-2$ and all $s>0$ we have*

$$(5.5)\qquad q(s) = \sum_{0\leqslant l\leqslant N}\binom{a+b-1}{l}R^{(l)}(0)s^{a+b-1-l}+ \\ +\frac{1}{\Gamma(1-a-b)}\int_0^\infty e^{-sz}R_N(-z)\frac{dz}{z^{a+b}}.$$

COROLLARY 1.

$$(5.6)\qquad q(s)\sim s^{a+b-1} \quad \textit{as } s\to\infty.$$

COROLLARY 2. *If $a<1$ and $a+b<1$ then*

$$(5.7)\qquad q(s)\sim e^{b\gamma}\frac{\Gamma(1-a)}{\Gamma(1-a-b)}s^{a-1} \quad \textit{as } s\to 0+.$$

Proof. It is well know that

$$\lim_{x\to\infty} x^{-1}\exp\left(\int_0^x \frac{1-e^{-u}}{u}\,du\right) = e^{\gamma}.$$

Hence, from (5.5) we obtain

$$\begin{aligned} \lim_{s\to 0+} s^{1-a}q(s) &= \lim_{s\to 0+}\frac{1}{\Gamma(1-a-b)}\int_0^\infty e^{-t}t^{-a}\left(\frac{t}{s}\right)^{-b}\exp\left(b\int_0^{t/s}\frac{1-e^{-u}}{u}\,du\right)dt \\ &= \frac{e^{b\gamma}}{\Gamma(1-a-b)}\int_0^\infty e^{-t}t^{-a}\,dt = e^{b\gamma}\frac{\Gamma(1-a)}{\Gamma(1-a-b)}. \end{aligned}$$

5.2. The zeros of $q(s)$. It follows from (5.6) that $q(s)$ can have only finitely many zeros.

LEMMA 3. *If $b\leqslant 0$ then $q(s)$ is positive.*

Proof. By (5.6) $q(s)$ is positive for all sufficiently large s. Letting α be the largest root of $q(s)$, then $q(\alpha+1)>0$ and $q'(\alpha)\geqslant 0$. If $b<0$ this is in contradiction with the differential equation which gives

$$aq'(\alpha) = bq(\alpha+1).$$

If $b=0$ then $q(s) = s^{a-1}$ and the lemma remains true.

LEMMA 4. *If $a+b<1$ then $q(s)$ is positive.*

Proof. This follows from

$$(5.8)\qquad q(s)=\frac{1}{\Gamma(1-a-b)}\int_0^\infty \exp\left(-sz+b\int_0^z\frac{1-e^{-u}}{u}\,du\right)\frac{dz}{z^{a+b}}.$$

Remark. The derivative $q^{(n)}(s)$ satisfies (5.2) with first coeffi-cient $a-n$ instead of a and it differs from the corresponding q-functic by the factor $(a+b-1)\ldots(a+b-n)$.

COROLLARY. *The number of zeros of $q(s)$ is less than $a+b$.*

LEMMA 5. *If $b>0$ and $1<a+b<2$ then $q(s)$ is increasing and h a zero $\alpha<b$.*

Proof. By Lemma 4 $q'(s)>0$, so $q(s)$ is increasing. We have

$$(5.9)\quad q(s)=s^{a+b-1}+\frac{1}{\Gamma(1-a-b)}\int_0^\infty e^{-sz}\left\{\exp\left(b\int_0^z\frac{1-e^{-u}}{u}\,du\right)-1\right\}\frac{dz}{z^{a+}}$$

whence $\lim\limits_{s=0+}q(s)<0$ and $\lim\limits_{s=\infty}q(s)=\infty$. Therefore $q(s)$ has exactly o zero. By Lemma 4 $q''(s)<0$, so the differential equation gives

$$(5.10)\qquad sq'(s)=(a-1)q(s)+bq(s+1)<(a+b-1)q(s)+bq'(s).$$

Inserting $s=\alpha$ we obtain $\alpha<b$.

LEMMA 6. *Let $b>0$ and $a+b>2$. If α and α_1 are the largest zeros $q(s)$ and $q'(s)$ respectively then $\alpha>\alpha_1$.*

Proof. Since, for $s>\alpha_1$, $q(s)$ is increasing, we obtain

$$0=\alpha_1q'(\alpha_1)=(a-1)q(\alpha_1)+bq(\alpha_1+1)>(a+b-1)q(\alpha_1),$$

which gives the result.

LEMMA 7. *If $b>0$ and $2<a+b<3$ then the largest zero α of $q($ lies in the interval*

$$(5.11)\qquad b<\alpha<b+\left(\tfrac{1}{2}b(a+b-2)\right)^{1/2}.$$

Proof. Since $q''(s)>0$, we have inequalities opposite to (5.1(In particular $\alpha>b$. We have also

$$sq'(s)=(a-1)q(s)+bq(s+1)<(a+b-1)q(s)+bq'(s)+\tfrac{1}{2}bq''(s),$$

and, since $q'''(s)<0$,

$$sq''(s)=(a-2)q'(s)+bq'(s+1)<(a+b-2)q'(s)+bq''(s).$$

Hence, for $s>b$,

$$(5.12)\qquad \left((s-b)^2-\tfrac{1}{2}b(a+b-2)\right)q'(s)<(a+b-1)(s-b)q(s).$$

Substituting $s=\alpha$, the lemma follows.

LEMMA 8. *If $b>0$ and $a+b>3$, then the largest zero α of $q(s)$ satisfies*

$$(5.13)\qquad \alpha>b+\left(\tfrac{1}{2}b(a+b-2)\right)^{1/2}.$$

Proof. If α_1, α_2 stand for the largest zeros of $q'(s)$ and $q''(s)$ respectively then $\alpha>\alpha_1>\alpha_2$ and we have the inequality opposite to (5.12). In particular, we have (5.13).

Remark. The largest zero of $q(s)$ is simple and is a continuous function of a, b when $b>0$ and $a+b>1$.

5.3. The largest zero of $g(s)$. Asymptotic formulae. In this special case ($a=b=\varkappa$) Lemmata 5, 7 and 8 give

$$0<\alpha<\varkappa\qquad\text{for}\quad \tfrac{1}{2}<\varkappa<1,$$

$$\varkappa<\alpha<\varkappa+\sqrt{\varkappa(\varkappa-1)}\qquad\text{for}\quad 1<\varkappa<\tfrac{3}{2},$$

$$\alpha>\varkappa+\sqrt{\varkappa(\varkappa-1)}\qquad\text{for}\quad \varkappa>\tfrac{3}{2}.$$

For $\varkappa=1$ and $\varkappa=3/2$ we have $\alpha=1$ and $\alpha=(3+\sqrt{3})/2$ respectively. Generally, if $2\varkappa$ is a positive integer then $g(s)$ is a polynomial of degree $2\varkappa-1$ and with rational coefficients. Therefore α is an algebraic number (see [16] and [8]).

In the next two theorems we shall show that α tends very rapidly to zero as $\varkappa\to\frac{1}{2}+$ and it tends linearly to infinity as $\varkappa\to\infty$.

THEOREM 5. *As $\varkappa\to\frac{1}{2}+$,*

$$\alpha\sim\pi e^{\gamma}(2\varkappa-1)^2.$$

Proof. For

$$r(z)=z^{-\varkappa}\left\{\exp\left(\varkappa\int_0^z\frac{1-e^{-u}}{u}\,du\right)-1\right\}$$

we have $r(z)\to e^{\gamma\varkappa}$ as $z\to\infty$, whence we obtain

$$\alpha^{\varkappa}=-\frac{1}{\Gamma(1-2\varkappa)}\int_0^\infty e^{-u}u^{-\varkappa}r\left(\frac{u}{\alpha}\right)du\sim -e^{\gamma\varkappa}\frac{\Gamma(1-\varkappa)}{\Gamma(1-2\varkappa)}\sim e^{\gamma/2}\Gamma(\tfrac{1}{2})(2\varkappa-1)$$

as $\varkappa\to\frac{1}{2}+$. This completes the proof.

THEOREM 6. *We have*

$$\alpha=c\varkappa+O(\varkappa^{2/3})$$

where c is the solution of $c\log c=c+1$ ($c=3.5911\ldots$).

It appears that the two largest real zeros of $g(s)$ are very far apart and this makes it possible to localize the largest zero by estimating $g(s)$

in vicinity of $c\varkappa$. Using formula (5.4) we shall show that $g(s) < 0$ for all $s > c\varkappa + c_1\varkappa^{2/3}$ and $g(s') < 0$ for some $s' = c\varkappa + O(\varkappa^{2/3})$.

It will be shown that the part of the integral in (5.4) corresponding to a small arc of $\mathscr{C}$ near $z = r$ dominates if the radius r is a suitably chosen function of s.

Let us denote

$$l(z) = sz + \varkappa \int_0^z \frac{1-e^u}{u}\,du.$$

The part of $g(s)$ corresponding to the half-lines $(-\infty, -r)$ can be majorized by

$$(5.14)\qquad 2\frac{\Gamma(2\varkappa)}{2\pi}\int_{-\infty}^{-r}(-z)^{-2\varkappa}e^{l(z)}\,dz < \frac{\Gamma(2\varkappa)}{\pi(2\varkappa-1)}r^{1-2\varkappa}e^{l(-r)}.$$

The part of $g(s)$ corresponding to the circle $z = re^{i\varphi}$, $-\pi < \varphi \leqslant \pi$ is equal to

$$(5.15)\qquad \frac{\Gamma(2\varkappa)}{2}r^{1-2\varkappa}\int_{-\pi}^{\pi}\exp\bigl(l(re^{i\varphi})-(2\varkappa-1)i\varphi\bigr)\,d\varphi.$$

Writing $L(x) = l(re^x)$ we obtain

$$L_0 = L(0) = rs + \varkappa\int_0^r \frac{1-e^u}{u}\,du,$$

$$L_1 = L'(0) = rs + \varkappa(1-e^r),$$
$$L_2 = L''(0) = rs - \varkappa re^r,$$
$$L_3 = L'''(0) = rs - \varkappa r(r+1)e^r.$$

We shall choose r and s to satisfy

$$(5.16)\qquad 1 < r < 2, \quad s > 2\varkappa \quad \text{and} \quad L_2 > 1.$$

We can then write

$$(5.17)\qquad L(i\varphi) = L_0 + i\varphi L_1 + \frac{(i\varphi)^2}{2}L_2 + \frac{(i\varphi)^3}{6}L_3 + O\bigl((\varkappa+s)\varphi^4\bigr)$$

and

$$(5.18)\qquad L(i\varphi) = L_0 + i\varphi L_1 + \frac{(i\varphi)^2}{2}L_2 + O\bigl((\varkappa+s)\varphi^3\bigr).$$

LEMMA 9. *For $-\pi < \varphi \leqslant \pi$ we have*

$$(5.19)\qquad \operatorname{Re}L(i\varphi) \leqslant L_0 - 2\sin^2(\varphi/2)L_2.$$

Proof. Since

$$l(z) = sz - \varkappa\sum_{n=1}^{\infty}\frac{z^n}{n\cdot n!}$$

we have

$$\begin{aligned}\operatorname{Re}L(i\varphi) &= rs\cos\varphi - \varkappa\sum_{n=1}^{\infty} r^n\frac{\cos n\varphi}{n\cdot n!}\\ &= l(r) - 2rs\sin^2(\varphi/2) + 2\varkappa\sum_{n=1}^{\infty}\frac{r^n}{n\cdot n!}\sin^2(n\varphi/2)\\ &\leqslant l(r) - 2rs\sin^2(\varphi/2) + 2\varkappa\sum_{n=1}^{\infty}\frac{r^n}{n\cdot n!}\sin^2(\varphi/2)\\ &= L_0 - 2\sin^2(\varphi/2)L_2.\end{aligned}$$

LEMMA 10. *We have*

$$(5.20)\qquad l(-r) \leqslant L_0 - 2L_2.$$

Proof. Choose $a = \pi$ in Lemma 9.

LEMMA 11. *There exists $c_1 > 1$ such that for all $s > c\varkappa + c_1\varkappa^{2/3}$, $g(s)$ is positive.*

Proof. Split up the integral (5.15) as follows

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{-(\varkappa+s)^{-1/3}} + \int_{-(\varkappa+s)^{-1/3}}^{(\varkappa+s)^{-1/3}} + \int_{(\varkappa+s)^{-1/3}}^{\pi} = S_1 + S_2 + S_3, \text{ say.}$$

Inserting (5.18) into S_2 and estimating S_1, S_3 trivially by means of (5.19), we arrive at

$$\begin{aligned}\int_{-\pi}^{\pi} &= \int_{-\pi}^{\pi}\exp\Bigl\{L_0 + i\varphi(L_1 - 2\varkappa + 1) - \frac{\varphi^2}{2}L_2\Bigr\}\,d\varphi +\\ &\qquad + O\Bigl(\int_{-\pi}^{\pi}(\varkappa+s)\varphi^3\exp\Bigl(L_0 - 2\sin^2\Bigl(\frac{\varphi}{2}\Bigr)L_2\Bigr)\,d\varphi\Bigr)\\ &= \int_{-\infty}^{\infty}\exp\Bigl\{L_0 + i\varphi(L_1 - 2\varkappa + 1) - \frac{\varphi^2}{2}L_2\Bigr\}\,d\varphi + O\bigl((\varkappa+s)L_2^{-2}e^{L_0}\bigr).\end{aligned}$$

Hence, if we define r by

$$(5.21)\qquad L_1 = 2\varkappa - 1$$

and combine (5.14), (5.15) and (5.20) we obtain

$$(5.22)\quad g(s) = 2r^{1-2\varkappa}\Gamma(2\varkappa)\{\sqrt{2}\,\Gamma(\tfrac{1}{2})\,e^{L_0}L_2^{-1/2}\}+O\big((\varkappa+s)L_2^{-2}e^{L_0}+\varkappa^{-1}e^{L_0-2L_2}\big).$$

It is easy to see that, for $s > c\varkappa+c_1\varkappa^{2/3}$ and r defined by (5.21) the conditions (5.16) are satisfied and the error term in (5.12) is small compared to the main term. This completes the proof of Lemma 11.

Lemma 12. *There exists $s = c\varkappa+O(\varkappa^{2/3})$ such that $g(s) < 0$.*

Proof. As in the proof of Lemma 11, by (5.14), (5.15), (5.17), (5.19) and (5.20) we have

$$(5.23)\qquad g(s)$$

$$= \frac{\Gamma(2\varkappa)}{2}r^{1-2\varkappa}\int_{-\pi}^{\pi}\exp\Big\{L_0+i\varphi(L_1-2\varkappa+1)-\frac{\varphi^2}{2}L_2-\frac{i\varphi^3}{6}L_3\Big\}d\varphi+$$

$$+O\Big(\int_{-\pi}^{\pi}(\varkappa+s)\varphi^4\exp\Big\{L_0-2\sin^2\Big(\frac{\varphi}{2}\Big)L_2\Big\}d\varphi+\varkappa^{-1}e^{L_0-2L_2}\Big)$$

$$= \frac{\Gamma(2\varkappa)}{2}r^{1-2\varkappa}\Big[\int_{-\infty}^{\infty}\exp\Big\{L_0+i\varphi(L_1-2\varkappa+1)-\frac{\varphi^2}{2}L_2-\frac{i\varphi^3}{6}L_3\Big\}d\varphi+$$

$$+O\big((\varkappa+s)L_2^{-5/2}e^{L_0}\big)\Big].$$

We shall choose r and s in such a way to make the integral negative. The following result ensures that we can do this.

Lemma 13. *There exists a cubic polynomial $w(x) = c_3x^3+c_2x^2+c_1x$ with positive coefficients such that*

$$\lambda = \int_{-\infty}^{\infty}e^{w(ix)}dx < 0.$$

Proof. One can prove that

$$\int_0^{\infty}e^{-x^2/16}\cos\frac{\pi}{2}(x^3-3x)\,dx < 0$$

and Lemma 13 follows.

Now, returning to the proof of Lemma 12, we define r and s as the solution of

$$(5.24)\qquad L_2 = 2c_2\Big(\frac{L_3}{6c_3}\Big)^{2/3},\qquad L_1 = 2\varkappa-1+c_1\Big(\frac{L_3}{6c_3}\Big)^{1/3}.$$

One can easily check that the numbers r and s, so defined satisfy

$$r = \nu+O(\varkappa^{-1/3})\quad\text{and}\quad s = \varkappa e^{\nu}+O(\varkappa^{2/3})$$

where $(\nu-1)e^{\nu} = 1$. Hence $\varkappa \ll L_2 \ll \varkappa$ and the conditions (5.16) are satisfied.

From (5.23) we obtain

$$g(s) = \frac{\Gamma(2\varkappa)}{2}r^{1-2\varkappa}e^{L_0}\Big\{\lambda\Big(\frac{6c_3}{L_3}\Big)^{1/3}+O(\varkappa L_2^{-5/2})\Big\}$$

and this is negative. This completes the proof of Lemma 12 and of Theorem 6.

6. The functions $Q^{\pm}(s)$. Assume $\varkappa > \frac{1}{2}$ and let $\beta-1$ be the largest root of $g(s)$. Let $Q^+(s)$ and $Q^-(s)$ be the continuous solution of

$$(6.1)\qquad \begin{aligned} s^{\varkappa+1}Q^+(s) &= \beta^{-\varkappa} &&\text{for } s\leqslant\beta+1,\\ s^{\varkappa+1}Q^-(s) &= (\beta-1)^{-\varkappa} &&\text{for } s\leqslant\beta,\end{aligned}$$

$$(6.2)\qquad \begin{aligned} \big(s^{\varkappa+1}Q^+(s)\big)' &= -\varkappa s^{\varkappa}Q^-(s-1) &&\text{for } s>\beta+1,\\ \big(s^{\varkappa+1}Q^-(s)\big)' &= -\varkappa s^{\varkappa}Q^+(s-1) &&\text{for } s>\beta.\end{aligned}$$

This section is concerned with the proof of the following result.

Lemma 13. *The functions $Q^{\pm}(s)$ are positive and, for $s\geqslant 2$, satisfy the conditions*

$$(6.3)\qquad Q^+(s) \ll Q^-(s) \ll Q^+(s),$$

$$(6.4)\qquad Q^+(s)s\log s \ll Q^+(s-1) \ll s^2Q^+(s),$$

$$(6.5)\qquad Q^+(s) = \exp\Big\{-s\log s-s\log\log s+s\log e\varkappa+O\Big(s\frac{\log\log 2s}{\log s}\Big)\Big\}.$$

$Q^{\pm}(s)$ will be used for the majorization of the error term $Q(s)$ and of the functions $F(s)-1$ and $1-f(s)$ as well.

It is convenient to introduce the functions

$$a(s) = s^{-\varkappa-1}\big(Q^+(s)+Q^-(s)\big),$$

$$b(s) = s^{-\varkappa-1}\big(Q^+(s)-Q^-(s)\big).$$

For $\beta\leqslant s\leqslant\beta+1$ we have

$$(6.6)\qquad s^{\varkappa+1}a(s) = \beta^{-\varkappa}+(\beta-1)^{-\varkappa}-\varkappa\beta^{-\varkappa}\int_{\beta}^{s}t^{\varkappa}(t-1)^{-\varkappa-1}dt,$$

$$(6.7)\qquad s^{\varkappa+1}b(s) = \beta^{-\varkappa}-(\beta-1)^{-\varkappa}+\varkappa\beta^{-\varkappa}\int_{\beta}^{s}t^{\varkappa}(t-1)^{-\varkappa-1}dt$$

and, for $s>\beta+1$

$$(6.8)\qquad sa'(s) = -(\varkappa+1)a(s)-\varkappa a(s-1),$$

$$(6.9)\qquad sb'(s) = -(\varkappa+1)b(s)+\varkappa b(s-1).$$

The conjugate equations are

$$(6.10)\qquad \begin{aligned}(sG(s))' &= (\varkappa+1)G(s)+\varkappa G(s+1),\\ (sH(s))' &= (\varkappa+1)H(s)-\varkappa H(s+1),\end{aligned}$$

so

$$(6.11)\qquad G'(s) = 2\varkappa g(s) \quad \text{and} \quad H(s) = 1.$$

Hence, there exist two constants c_1 and c_2 such that

$$(6.12)\qquad sa(s)G(s) = \varkappa\int_{s-1}^{s} a(x)G(x+1)\,dx + c_1,$$

$$(6.13)\qquad sb(s) = -\varkappa\int_{s-1}^{s} b(x)\,dx + c_2$$

for all $s \geqslant \beta+1$. If we substitute $s = \beta+1$ and utilize (6.6) and (6.7) we find that $c_1 = c_2 = 0$.

To show that $Q^{\pm}(s)$ are positive and have the same order of magnitude (see (6.3)) it suffices to prove:

LEMMA 14. *There exists a constant $\eta < 1$ such that*

$$(6.14)\qquad |b(s)| < \eta a(s).$$

Proof. First we shall prove that

$$(6.15)\qquad |b(s)| < a(s) \quad \text{for all } s \leqslant \beta+1.$$

The inequality $-b(s) < a(s)$ is evident. To show $b(s) < a(s)$ it suffices to consider the worst case $s = \beta+1$. This is equivalent to

$$(6.16)\qquad \varkappa\int_{\beta}^{\beta+1} t^{\varkappa}(t-1)^{-\varkappa-1}\,dt < \left(\frac{\beta}{\beta-1}\right)^{\varkappa}.$$

Integrating of

$$(s^{-\varkappa}G(s))' = \varkappa s^{-\varkappa-1}G(s+1) \quad \text{and} \quad (s^{-\varkappa})' = -\varkappa s^{-\varkappa-1}$$

from $\beta-1$ to β we obtain

$$(6.17)\qquad \beta^{-\varkappa}G(\beta)-(\beta-1)^{-\varkappa}G(\beta-1) = \varkappa\int_{\beta}^{\beta+1}(x-1)^{-\varkappa-1}G(x)\,dx$$

and

$$(6.18)\qquad \beta^{-\varkappa}-(\beta-1)^{-\varkappa} = -\varkappa\int_{\beta}^{\beta+1}(x-1)^{-\varkappa-1}\,dx.$$

Since $\beta-1$ is the largest zero of $G'(s)$, $G(s)$ is increasing for $s > \beta-1$ and, by (6.10), $G(\beta) = -G(\beta-1)$. In particular, the function

$$C(t) = \left(\frac{\beta}{t}\right)^{\varkappa}\left(1+\frac{G(t)}{G(\beta)}\right)$$

is increasing for $t > \beta$ since

$$C'(t) = \frac{\varkappa}{t}\left(\frac{\beta}{t}\right)^{\varkappa}\left(\frac{G(t+1)}{G(\beta)}-1\right) > 0.$$

Hence $C(t) > C(\beta) = 2$. By (6.17) and (6.18) we obtain

$$\begin{aligned}2\left(\frac{\beta}{\beta-1}\right)^{\varkappa} &= \left(\frac{\beta}{\beta-1}\right)^{\varkappa}\left(1-\frac{G(\beta-1)}{G(\beta)}\right)\\ &= \varkappa\int_{\beta}^{\beta+1} t^{\varkappa}(t-1)^{-\varkappa-1}C(t)\,dt > 2\varkappa\int_{\beta}^{\beta+1} t^{\varkappa}(t-1)^{-\varkappa-1}\,dt.\end{aligned}$$

This completes the proof of (6.16) and (6.15).

Since $a(s)$ and $b(s)$ are continuous there exists $\eta < 1$ such that (6.14) holds for all $s \leqslant \beta+1$. Now, we shall prove this is true for all s. Suppose that the set $\{s;\ |b(s)| \geqslant \eta a(s)\}$ is not empty and denote by u its infimum. We have $u > \beta+1$ and $|b(u)| = \eta a(u)$. From (6.12) and (6.13) we obtain

$$|ub(u)| \leqslant \varkappa\int_{u-1}^{u}|b(x)|\,dx < \eta\varkappa\int_{u-1}^{u}a(x)\,dx < \eta\varkappa\int_{u-1}^{u}a(x)\frac{G(x+1)}{G(u)}\,dx = \eta u a(u),$$

which is a contradiction. This completes the proof of Lemma 14.

LEMMA 15. *For $s \geqslant 2$ we have*

$$a(s-1) \ll s^2 a(s).$$

Proof. It is sufficient to prove this for $s \geqslant \beta+\frac{3}{2}$. From (6.8) $a(s)$ is increasing and by (6.12) we obtain

$$sa(s)G(s) > \tfrac{1}{2}\varkappa a(s-\tfrac{1}{2})G(s)$$

for $s \geqslant \beta+1$. This completes the proof.

LEMMA 16. *There exists $\varepsilon > 0$ such that $(\varepsilon s)^s a(s)s^{\varkappa+1}$ is decreasing for all $s \geqslant \beta+1$.*

COROLLARY. *For $s \geqslant 2$ we have*

$$a(s-1) \gg a(s)s\log s.$$

Proof of Corollary. For $s \geqslant \beta+1$ we have

$$(6.19)\qquad (s^{\varkappa+1}a(s))' = -\varkappa s^{\varkappa}a(s-1)$$

and hence writing $r(s) = (\varepsilon s)^s$ and $A(s) = a(s)r(s)s^{\varkappa+1}$ we obtain

$$(6.20)\qquad \frac{A'}{A}(s) = \frac{r'}{r}(s) - \frac{\varkappa a(s-1)}{sa(s)}.$$

Since $\frac{r'}{r}(s) = \log(\varepsilon e s)$ and $\frac{A'}{A}(s) < 0$, it follows that

$$\frac{\varkappa a(s-1)}{sa(s)} > \log(\varepsilon e s) \quad \text{for all } s \geqslant \beta+1.$$

Proof of Lemma 16. Suppose u is the infimum of those numbers $s \geqslant \beta+1$ for which $A(s)$ is not decreasing. Thus, by (6.20) we have

$$\frac{r'}{r}(u) = \frac{\varkappa a(u-1)}{ua(u)} \tag{6.21}$$

and, since ε can be chosen arbitrarily small, we may assume u is sufficiently large. From (6.12) we obtain

$$uG(u)a(u) = \varkappa \int_{u-1}^{u} a(x)G(x+1)\,dx < \varkappa a(u-1)G(u+1) < 2\varkappa a(u-1)G(u),$$

i.e. $\frac{r'}{r}(u) > 1/2$. Hence

$$\frac{r'}{r}(u-1) = \frac{r'}{r}(u) + \log\left(1-\frac{1}{u}\right) > \frac{r'}{r}(u)\frac{u-3}{u-1}.$$

Using again (6.12) we obtain

$$uG(u)a(u) = \varkappa \int_{u-1}^{u} a(x)G(x+1)\,dx < \varkappa A(u-1)G(u+1)\int_{u-1}^{u} \frac{x^{-\varkappa+1}}{r(x)}\,dx.$$

We have

$$\left(\frac{-1}{r(x)}\right)' = \frac{1}{r(x)}\frac{r'}{r}(x) > \frac{1}{r(x)}\frac{r'}{r}(u-1) > \frac{1}{r(x)}\frac{r'}{r}(u)\frac{u-3}{u-1}\left(\frac{u-1}{x}\right)^{\varkappa+}$$

whence

$$\frac{r'}{r}(u)\frac{ua(u)}{\varkappa a(u-1)} < \frac{u-1}{u-3}\,\frac{G(u+1)}{G(u)}\left(1-\frac{r(u-1)}{r(u)}\right) < \frac{u-1}{u-3}\,\frac{G(u+1)}{G(u)}\left(1-\frac{1}{\varepsilon e u}\right) < 1$$

provided ε is sufficiently small. This is in contradiction with (6.21) and the proof of Lemma 16 is completed.

Remark. Similar, but more precise calculations lead to the following result: There exists a constant c such that for all $s \geqslant \beta+1$ the function

$$s^{\varkappa+1}a(s)\left(\frac{s\log s}{e\varkappa}\right)^{s}(\log s)^{-cs/\log s}$$

is decreasing.

To complete the proof of Lemma 13 it remains to show (6.5) or the same for the function $a(s)$. From (6.12) we obtain

$$\varkappa\int_{s-1}^{s} a(x)\,dx = \big(s+O(1)\big)a(s)$$

provided s is sufficiently large. Hence, a standard argument shows that

$$a(s) = \exp\left\{-s\log s - s\log\log s + s\log e\varkappa + O\left(s\frac{\log\log s}{\log s}\right)\right\}.$$

7. The functions $T^{\pm}(s)$.

Let $\varkappa \geqslant 0$ and

$$\beta-1 = \begin{cases} 0 & \text{if } \varkappa \leqslant 1/2, \\ \text{the largest zero of } g(s) & \text{if } \varkappa > 1/2. \end{cases}$$

Define

$$S_0^{+}(s) = \begin{cases} (\beta+1)^{\varkappa} - s^{\varkappa} & \text{for } \beta-1 < s \leqslant \beta+1, \\ 0 & \text{for } s > \beta+1, \end{cases}$$

$$S_R^{-}(s) = \int_s^{\infty} (t-1)^{-\varkappa} S_{R-1}^{+}(t-1)\,dt^{\varkappa} \quad \text{for } s \geqslant \beta,\ R \geqslant 1,$$

$$S_R^{+}(s) = \int_s^{\infty} (t-1)^{-\varkappa} S_R^{-}(t-1)\,dt^{\varkappa} \quad \text{for } s \geqslant \beta+1,\ R \geqslant 1,$$

$$S_R^{+}(s) = S_0^{+}(\beta+1) \quad \text{for } \beta-1 < s \leqslant \beta+1,$$

and

$$T_R^{+}(s) = \sum_{r=0}^{R} S_r^{+}(s) \quad \text{for } s > \beta-1, \tag{7.1}$$

$$T_R^{-}(s) = \sum_{r=1}^{R} S_r^{-}(s) \quad \text{for } s \geqslant \beta. \tag{7.2}$$

It is easy to see that $T_R^{\pm}(s)$ are continuous with the compact support $\left[\beta-\frac{1\pm 1}{2},\ \beta+\frac{1\pm 1}{2}+2R\right]$ and satisfy

$$T_R^{-}(s) = \varkappa\int_s^{\infty}\left(1-\frac{1}{t}\right)^{-\varkappa} T_{R-1}^{+}(t-1)\frac{dt}{t}, \quad s \geqslant \beta,$$

$$T_R^{+}(s) = \varkappa\int_s^{\infty}\left(1-\frac{1}{t}\right)^{-\varkappa} T_R^{-}(t-1)\frac{dt}{t}, \quad s \geqslant \beta+1, \tag{7.3}$$

$$T_R^{+}(s) + s^{\varkappa} = (\beta+1)^{\varkappa} + T_R^{+}(\beta+1), \quad \beta-1 < s \leqslant \beta+1.$$

In this section we prove that the series (7.1) and (7.2) converge and examine the limit functions;

$$T^{\pm}(s) = \lim_{R=\infty} T_R^{\pm}(s) \quad \text{for } s > \beta - \frac{1\pm 1}{2}.$$

From the previous section we recall that, for $\varkappa > 1/2$, there exist two positive functions $q^{\pm}(s) = s^{\varkappa+1}Q^{\pm}(s)$ such that

$$q^{\pm}(s) = \text{constant} \quad \text{for } 0 < s \leqslant \beta + \frac{1\pm 1}{2},$$

$$(7.4)\qquad q^{\pm}(s) = \varkappa \int_s^{\infty} \left(1-\frac{1}{t}\right)^{-\varkappa-1} q^{\mp}(t-1)\frac{dt}{t} \quad \text{for } s \geqslant \beta + \frac{1\pm 1}{2}.$$

It will be shown that $q^{\pm}(s)$ majorize $T_R^{\pm}(s)$. Since $q^{\pm}(s)$ is not defined if $\varkappa \leqslant 1/2$ we choose, in this case a $q^{\pm}(s)$ corresponding to some $\varkappa_1$ slightly greater than 1/2. For these we have

$$(7.5)\qquad \begin{aligned} &q^{\pm}(s) = \text{constant} && \text{for } 0 < s \leqslant 1 + \frac{1\pm 1}{2}, \\ &q^{\pm}(s) > \frac{1}{2}\int_s^{\infty} \left(1-\frac{1}{t}\right)^{-9/10} q^{\mp}(t-1)\frac{dt}{t} && \text{for } s \geqslant 1 + \frac{1\pm 1}{2} \end{aligned}$$

instead of (7.4). This is an immediate consequence of the fact that

$$\lim_{\varkappa=\frac{1}{2}+} \beta_{\varkappa} = 1.$$

Having chosen the functions $q^{\pm}(s)$, we shall prove

LEMMA 17. *There exists a constant $c > 0$ such that*

$$(7.6)\qquad T_R^{\pm}(s) < cq^{\pm}(s) \quad \textit{for } s > \beta - \frac{1\pm 1}{2}.$$

Proof. Since $q^{\pm}(s)$ is positive and $T_R^{\pm}(s)$ have compact support, the result is true for any fixed R. We shall show that if (7.6) is true for $T_{R-1}^{+}(s)$ (respectively for $T_R^{-}(s)$) then it is true, provided c is sufficiently large, with the same constant c, for $T_R^{-}(s)$ (respectively for $T_R^{+}(s)$).

From (7.3) and the inductive assumption (7.6) we obtain

$$T_R^{-}(s) < c\varkappa \int_s^{\infty} \left(1-\frac{1}{t}\right)^{-\varkappa} q^{+}(t-1)\frac{dt}{t} < cq^{-}(s)$$

for $s \geqslant \beta$. This follows from (7.4) if $\varkappa > 1/2$ and from (7.5) if $\varkappa \leqslant 1/2$. The same argument applies to $T_R^{+}(s)$, $s \geqslant \beta+1$. For $s = \beta+1$ we must save a little. We have

$$T_R^{+}(\beta+1) < c\varkappa \int_{\beta+1}^{\infty} \left(1-\frac{1}{t}\right)^{-\varkappa} q^{-}(t-1)\frac{dt}{t} < c(1-\delta)q^{+}(\beta+1)$$

with some positive δ. This follows from (7.4) if $\varkappa > 1/2$ and from (7.5) if $\varkappa \leqslant 1/2$ because the kernel $\left(1-\frac{1}{t}\right)^{-\varkappa}$ is less than $\left(1-\frac{1}{t}\right)^{-\varkappa-1}$ and $\left(1-\frac{1}{t}\right)^{-9/10}$ respectively. Hence, by (7.3) we obtain

$$\begin{aligned} T_R^{+}(s) &\leqslant (\beta+1)^{\varkappa} + T_R^{+}(\beta+1) < (\beta+1)^{\varkappa} + c(1-\delta)q^{+}(\beta+1) \\ &< cq^{+}(\beta+1) = cq^{+}(s) \end{aligned}$$

for $\beta-1 < s \leqslant \beta+1$, provided c is sufficiently large. This completes the proof of Lemma 17.

Now, it is a simple matter to deduce the following result.

LEMMA 18. *The series* (7.1) *and* (7.2) *converge and the limit functions $T^{\pm}(s)$ satisfy*

$$0 < T^{\pm}(s) \leqslant s^{\varkappa+1}Q^{\pm}(s),$$

$$(7.7)\qquad \begin{aligned} &T^{\pm}(s) \pm s^{\varkappa} = \text{constant} && \textit{for } s \leqslant \beta + \frac{1\pm 1}{2}, \\ &T^{\pm}(s) = \varkappa \int_s^{\infty} \left(1-\frac{1}{t}\right)^{-\varkappa} T^{\mp}(t-1)\frac{dt}{t} && \textit{for } s \geqslant \beta + \frac{1\pm 1}{2}. \end{aligned}$$

In fact the function $T^{-}(s)$ was not defined for $s < \beta$, so the equality $T^{-}(s) - s^{\varkappa} = \text{constant}$ for $s \leqslant \beta$ is to be considered as a definition.

Our next task is to find the constants

$$A = T^{+}(\beta) + \beta^{\varkappa}, \quad B = T^{-}(\beta) - \beta^{\varkappa}.$$

It is convenient to introduce the functions

$$m(s) = s^{-\varkappa}\big(T^{+}(s) + T^{-}(s)\big),$$
$$n(s) = s^{-\varkappa}\big(T^{+}(s) - T^{-}(s)\big).$$

For $\beta \leqslant s \leqslant \beta+1$, we have

$$(7.8)\qquad sm(s) = A - B - \varkappa A \int_{\beta}^{s} (t-1)^{-\varkappa} t^{\varkappa-1}\,dt,$$

$$(7.9)\qquad sn(s) = A + B + \varkappa A \int_{\beta}^{s} (t-1)^{-\varkappa} t^{\varkappa-1}\,dt$$

and, for $s > \beta+1$,

$$(7.10)\qquad sm'(s) = -\varkappa m(s) - \varkappa m(s-1),$$

$$(7.11)\qquad sn'(s) = -\varkappa n(s) + \varkappa n(s-1).$$

The conjugate equations are

$$(sg(s))' = \varkappa g(s) + \varkappa g(s+1),$$
$$(sh(s))' = \varkappa h(s) - \varkappa h(s+1).$$

Since $g(s) \ll s^{2\varkappa-1}$, $h(s) \ll s^{-1}$, $m(s) \ll e^{-s}$ and $n(s) \ll e^{-s}$ as $s \to \infty$, we obtain

$$sg(s)m(s) = \varkappa \int_{s-1}^{s} g(x+1)m(x)\,dx,$$

$$sh(s)n(s) = -\varkappa \int_{s-1}^{s} h(x+1)n(x)\,dx$$

for all $s \geqslant \beta+1$. If we substitute $s = \beta+1$, integrate by parts, and use (7.8) and (7.9) we arrive at

(7.12) $$\beta g(\beta)m(\beta) = A[\beta^{1-\varkappa}g(\beta) - (\beta-1)^{1-\varkappa}g(\beta-1)],$$

(7.13) $$\beta h(\beta)n(\beta) = A[\beta^{1-\varkappa}h(\beta) - (\beta-1)^{1-\varkappa}h(\beta-1)] + 2\varkappa \int_{\beta}^{\beta+1} h(x)\,dx.$$

If $\beta = 1$, the symbols $0^{1-\varkappa}g(0)$ and $0^{1-\varkappa}h(0)$ mean

$$\lim_{s=0+} s^{1-\varkappa}g(s) = e^{\gamma\varkappa}\frac{\Gamma(1-\varkappa)}{\Gamma(1-2\varkappa)}$$

and

$$\lim_{s=0+} s^{1-\varkappa}h(s) = e^{-\gamma\varkappa}\Gamma(1-\varkappa),$$

respectively (see Lemma 2, Corollary 2). From Lemma 2, Corollary 1 we obtain

$$sh(s) + \varkappa \int_{s}^{s+1} h(x)\,dx = \text{constant} = \lim_{s=\infty} sh(s) = 1.$$

In particular, $\varkappa \int_{\beta}^{\beta+1} h(x)\,dx = 1 - \beta h(\beta)$. Finally, from (7.12) and (7.13) we get two linear equations for A and B;

$$(\beta-1)^{1-\varkappa}g(\beta-1)A - \beta^{1-\varkappa}g(\beta)B = 0,$$
$$(\beta-1)^{1-\varkappa}h(\beta-1)A + \beta^{1-\varkappa}h(\beta)B = 2.$$

For $\varkappa > 1/2$ we have $\beta > 1$ and $g(\beta-1) = 0$. Thus

$$B = 0 \quad \text{and} \quad A = 2\frac{(\beta-1)^{\varkappa-1}}{h(\beta-1)}.$$

For $\varkappa = 1/2$ we have $\beta = 1$, $\lim_{s=0+} s^{1/2}g(s) = 0$ and $\lim_{s=0+} s^{1/2}h(s) = (\pi/e^{\gamma})^{1/2}$. Thus

$$B = 0 \quad \text{and} \quad A = 2(e^{\gamma}/\pi)^{1/2}.$$

or $\varkappa < 1/2$ we have $\beta = 1$ and hence

$$e^{\gamma\varkappa}\frac{\Gamma(1-\varkappa)}{\Gamma(1-2\varkappa)}A - g(1)B = 0,$$
$$e^{-\gamma\varkappa}\Gamma(1-\varkappa)A + h(1)B = 2.$$

rom (5.8) we obtain

$$g(1) = \frac{1}{\Gamma(1-2\varkappa)}\int_0^{\infty} \exp\left(-z + \varkappa \int_0^{z} \frac{1-e^{-u}}{u}\,du\right) z^{-2\varkappa}\,dz$$

nd

$$h(1) = \int_0^{\infty} \exp\left(-z - \varkappa \int_0^{z} \frac{1-e^{-u}}{u}\,du\right) dz.$$

imple calculations lead to (2.1) and (2.2).

Lemma 19. *We have*

$$1 < (s+\varkappa)h(s) < 1 + \frac{\varkappa}{s(s+1+\varkappa)}.$$

Proof. Since $h(s)$ is decreasing, we obtain

$$1 = sh(s) + \varkappa \int_{s}^{s+1} h(x)\,dx < (s+\varkappa)h(s)$$

nd hence

$$1 > sh(s) + \varkappa \int_{s}^{s+1} \frac{dx}{x+\varkappa} > sh(s) + \frac{\varkappa}{s+1+\varkappa}.$$

his completes the proof.

Corollary. *For $\varkappa > 1/2$ we have*

$$2(\beta-1)^{\varkappa}\left(1+\frac{\varkappa}{\beta}\right) < A < 2(\beta-1)^{\varkappa}\left(1+\frac{\varkappa}{\beta-1}\right).$$

8. Estimates of $T^{\pm}_{R,z}(s)$. This section is devoted to estimates of $^{\pm}_{R,z}(s)$. Let $q^{\pm}(s)$ be chosen as in Lemma 17.

Lemma 20. *For $y \geqslant 2$, $z \geqslant 2$, and $s > \beta - \frac{1\pm1}{2}$ we have*

.1) $$T^{\pm}_{R,z}(s) < V(z)s^{-\varkappa}\{T^{\pm}_{R}(s) + G^{\pm}_{z}(s)(\log y)^{-1/3}\},$$

here

$$G^{\pm}_{z}(s) = Ce^{\sqrt{K}}\left(1+\frac{s^{50}}{\log y}\right)^{s} q^{\pm}(s)$$

ıd C is some constant depending at most on $\varkappa$.

Remarks. Assuming two-sided inequalities of the type (1.3) one can prove that the main term $V(z)s^{-\varkappa}T^{\pm}_{R,z}(s)$ is actually equal to the asymptotic value of $T^{\pm}_{R,z}(s)$ as z approaches ∞.

The dependence on K and y can be improved; $e^{\sqrt{K}}$ can be reduced to some power of K and the exponent $1/3$ can be increased to an arbitrary number <1 if $\varkappa > 1/2$ and $<1/2$ if $\varkappa \leqslant 1/2$.

The very artificial factor $\left(1+\frac{s^{50}}{\log y}\right)^s$ is introduced in order to weaken the result in the range where the induction method does not work.

The following lemma follows immediately from (1.3) by partial summation.

LEMMA 21. *Let $B(x)$ be a positive, continuous and increasing function in the interval $w \leqslant x \leqslant z$. We have*

$$\sum_{w\leqslant p<z} \frac{\omega(p)}{p}\frac{V(p)}{V(z)}\left(\frac{\log p}{\log z}\right)^{\varkappa} B(p) \leqslant \varkappa \int_w^z B(x)\,d\log\log x + B(z)\frac{(\varkappa+1)K}{\log w}. \tag{8.2}$$

Proof of Lemma 20. The idea of the proof is similar to that of Lemma 17, but is more complicated in details because of the error term $B(z)(\varkappa+1)K/\log w$ in the estimate of sums of the type (8.2).

For $\beta-1 < s \leqslant \beta+1$ we have

$$T^{+}_{0,z}(s) = V(y^{1/(\beta+1)}) - V(z) < V(z)s^{-\varkappa}\left\{(\beta+1)^{\varkappa} - s^{\varkappa} + \frac{K(\beta+1)^{\varkappa+1}}{\log y}\right\} \tag{8.3}$$

and, for $s > \beta+1$, we have $T^{+}_{0,z}(s) = 0$. This completes the proof of (8.1) for $T^{+}_{0,z}(s)$.

Now, assume that (8.1) is true for $T^{+}_{R-1,z}(s)$ or for $T^{-}_{R,z}(s)$ with some $R \geqslant 1$. We shall show that (8.1) is true, provided C is sufficiently large, with the same constant C, for $T^{-}_{R,z}(s)$ and $T^{+}_{R,z}(s)$ respectively.

Let $s \geqslant \beta + \frac{1\pm1}{2}$. First we eliminate some trivial cases. If $s \geqslant 2r + \beta + \frac{1\pm1}{2}$ then $S^{\pm}_{r,z}(s) = 0$ and so, we have

$$T^{\pm}_{R,z}(s) \leqslant \sum_{r>s-\beta-1} \frac{1}{r!}\left(\sum_{p<z}\frac{\omega(p)}{p}\right)^r \leqslant L^{\beta+1-s}\exp\left(L\sum_{p<z}\frac{\omega(p)}{p}\right) \leqslant L^{\beta+1-s}V(z)^{-L}$$

with any $L \geqslant 1$. Note that for $z \geqslant 2$, $V(z)^{-1} < 2K(\log z)^{\varkappa}$ and for $z<2$, $T^{\pm}_{R,z}(s) = 0$. Therefore, putting $L = 1+\frac{s}{\log Ks}$ we obtain

$$T^{\pm}_{R,z}(s) < V(z)s^{-\varkappa-1}\exp\left\{-s\log s + s\log\log Ks + O\left(s\frac{\log\log y}{\log 2s}\right)\right\}\frac{1}{\log y}. \tag{8.4}$$

But, from (6.5), we have

$$G^{\pm}_z(s) = C\exp\left\{-s\log s - s\log\log 2s + s\log\left(1+\frac{s^{50}}{\log y}\right) + \sqrt{K} + O(s)\right\}. \tag{8.5}$$

Hence, (8.1) is evident if $K^{24} \gg \log y$ or if $s^{50} \geqslant \log y(\log\log y)^3$. Therefore we can assume that

$$K^{-24}\log y \text{ is sufficiently large} \tag{8.6}$$

and

$$s < (\log y)^{1/50}(\log\log y)^{3/50} = s_0, \text{ say}. \tag{8.7}$$

By the recurrence formulae (4.4), (4.5) and the inductive assumption we obtain

$$\begin{aligned} T^{\pm}_{R,z}(s) &< T^{\pm}_{R,z}(s_0) + \sum_{\substack{y^{1/s_0}\leqslant p<y^{1/s}\\ 2p<y}} \frac{\omega(p)}{p}\left(\frac{\log p}{\log(y/p)}\right)^{\varkappa} V(p) + \\ &\quad + \left\{T^{\mp}_{R-1+\frac{1\pm1}{2}}\left(\frac{\log(y/p)}{\log p}\right) + G^{\mp}_p\left(\frac{\log(y/p)}{\log p}\right)(\log(y/p))^{-1/3}\right\} \\ &= T^{\pm}_{R,z}(s_0) + \Sigma_1 + \Sigma_2, \text{ say}. \end{aligned} \tag{8.8}$$

It is clear from the definition of $S^{\pm}_R(s)$ that $T^{\pm}_R(s)$ are decreasing. Hence

$$B(x) = T^{\mp}_{R-1+\frac{1\pm1}{2}}\left(\frac{\log(y/x)}{\log x}\right)(\log(y/x))^{-\varkappa}$$

is increasing and we may apply Lemma 21, getting

$$\Sigma_1 \leqslant V(z)s^{-\varkappa}\left\{T^{\pm}_R(s) + O\left(T^{\mp}(s-1)\left(1-\frac{1}{s_1}\right)^{-\varkappa}\frac{Ks_0}{\log y}\right)\right\},$$

where $s_1 = \max\left(s, \frac{\log 2}{\log y}\right)$. We have

$$\left(1-\frac{1}{s_1}\right)^{-\varkappa} \ll (\log y)^{1/2} \quad\text{and}\quad T^{\mp}(s-1) \ll q^{\mp}(s-1) \ll s^2q^{\pm}(s).$$

Hence, by (8.7) we obtain

$$\Sigma_1 \leqslant V(z)s^{-\varkappa}\left\{T^{\pm}_R(s) + O\left(\frac{K}{s_0}q^{\pm}(s)(\log y)^{-3/8}\right)\right\}. \tag{8.9}$$

Consider the two functions

$$A_\varepsilon(t) = \left(1+\frac{(t+\varepsilon)^{50}}{\log y}\right)^t q^{\pm}(t) \quad (\varepsilon = 0, 1)$$

in the interval $\beta_k+\frac{1\pm 1}{2}<t<s_0$, where $k=\varkappa$ if $\varkappa>1/2$ and $k=\varkappa_1$ if $\varkappa\leqslant 1/2$. From (7.4) and (6.4) we obtain

$$-\Lambda_s'(t)=\left(1+\frac{(t+\varepsilon)^{50}}{\log y}\right)^t kt^{-1}\left(1-\frac{1}{t}\right)^{-k-1}q^{\mp}(t-1)\left\{1+O\left(\frac{\log\left(1+\frac{t^{50}}{\log y}\right)}{\log 2t}\right)\right\}.$$

The term in the bracket $\{\ \}$ is positive and for sufficiently large t (say $t>s_2$) it is greater than $\left(1+\frac{(t+\varepsilon)^{50}}{\log y}\right)^{-1}$. This implies that $\Lambda_1(t)$ is decreasing and for $s_2<s<s_0$,

$$\varkappa\int_s^{s_0}\left(1+\frac{t^{50}}{\log y}\right)^{t-1}\left(1-\frac{1}{t}\right)^{-\varkappa-1/3}q^{\mp}(t-1)\frac{dt}{t}<\left(1-\frac{1}{s_0}\right)^{2/3}\left(1+\frac{s^{50}}{\log y}\right)^s q^{\pm}(s). \tag{8.10}$$

For $\beta+\frac{1\pm 1}{2}\leqslant s\leqslant s_2$ this is also true. For, write $u=s_2+2$ and split up the integral into $\varkappa\int_s^u+\varkappa\int_u^{s_0}$. The first integral, by (7.4) and (7.5), satisfies

$$\varkappa\int_s^u<\left(1-\frac{1}{u}\right)^{1/15}\left(1+\frac{u^{50}}{\log y}\right)^u q^{\pm}(s)$$

and the second integral, by (8.10), satisfies

$$\varkappa\int_u^{s_0}<\left(1-\frac{1}{s_0}\right)^{2/3}\left(1+\frac{u^{50}}{\log y}\right)^u q^{\pm}(u).$$

Since $q^{\pm}(u)\ll u^{-2}q^{\pm}(s)$, (8.10) follows.

For $\beta_k+\frac{1\pm 1}{2}<t<s_0+1$, the function

$$\Lambda(t)=\left(1-\frac{1}{t}\right)^{-\varkappa-1/3}\Lambda_1(t-1)=\left(1-\frac{1}{t}\right)^{-\varkappa-1/3}\left(1+\frac{t^{50}}{\log y}\right)^{t-1}q^{\mp}(t-1) \tag{8.11}$$

is decreasing and this is also true in the interval $\beta+\frac{1\pm 1}{2}\leqslant t\leqslant\beta_k+\frac{1\pm 1}{2}+1$ because $q^{\mp}(t-1)=\text{constant}$ and y is large.

It follows from (8.11) that $B(x)=\Lambda\left(\frac{\log y}{\log x}\right)$ is increasing and we may apply Lemma 21 getting

$$\Sigma_2\leqslant V(z)s^{-\varkappa}\left\{\left(1-\frac{1}{s_0}\right)^{2/3}\left(1+\frac{s^{50}}{\log y}\right)^s q^{\pm}(s)(\log y)^{-1/3}+O\left(q^{\pm}(s-1)\left(1-\frac{1}{s_1}\right)^{-\varkappa-1/3}\left(1+\frac{s^{50}}{\log y}\right)^s\frac{Ks_0}{(\log y)^{4/3}}\right)\right\}.$$

Hence, as for Σ_1, we obtain

$$\Sigma_2\leqslant V(z)s^{-\varkappa}\left(1+\frac{s^{50}}{\log y}\right)^s q^{\pm}(s)\left\{\left(1-\frac{1}{s_0}\right)^{2/3}+O\left(\frac{K}{s_0}(\log y)^{-1/24}\right)\right\}(\log y)^{-1/3}. \tag{8.12}$$

For $T^{\pm}_{R,z}(s_0)$ we apply (8.4) getting

$$T^{\pm}_{R,z}(s_0)<V(z)s_0^{-\varkappa-1}\left(1+\frac{s_0^{50}}{\log y}\right)^{s_0}q^{\pm}(s_0)(\log y)^{-3/8}. \tag{8.13}$$

Collecting together (8.8), (8.9), (8.12) and (8.13) we obtain

$$T^{\pm}_{R,z}(s)<V(z)s^{-\varkappa}\left\{T^{\pm}_R(s)+G^{\pm}_z(s)\left[\left(1-\frac{1}{s_0}\right)^{2/3}+O\left(s_0^{-1}(\log y)^{-1/24}\right)\right](\log y)^{-1/3}\right\}. \tag{8.14}$$

The term in the square bracket is, for sufficiently large y (see (8.6)), less than $1-\frac{1}{2s_0}<1$ and hence (8.1) follows.

Now, it only remains to prove (8.1) for $T^{+}_{R,z}(s)$ with $\beta-1<s\leqslant\beta+1$. To this end we apply the recurrence formula (4.6). If we estimate $T^{+}_{0,z}(s)$ by (8.3) and $T^{+}_{R,z}(\beta+1)$ by (8.14) we arrive again at (8.14) but with an extra term

$$O\left(Ke^{-\sqrt{K}}(\log y)^{-2/3}\right)$$

in the square bracket. The same argument completes the proof.

9. Proof of Theorem 1 (Conclusion). For $s^{50}<\log z$ we have $\left(1+\frac{s^{50}}{\log y}\right)^s<e$ and hence by Lemma 20

$$T^{\pm}_{R,z}(s)<V(z)s^{-\varkappa}\{T^{\pm}_R(s)+Ce^{1+\sqrt{K}}q^{\pm}(s)(\log y)^{-1/3}\}. \tag{9.1}$$

For $s^{50}\geqslant\log z$, by (8.4) we have

$$T^{\pm}_{R,z}(s)<V(z)s^{-\varkappa}\exp\{-s\log s+s\log\log 3s+\sqrt{K}+O(s)\}(\log y)^{-1/3}. \tag{9.2}$$

Writing

$$F(s) = 1+s^{-\varkappa}T^{+}(s) \quad \text{for } s \geqslant \beta+1,$$
$$f(s) = 1-s^{-\varkappa}T^{-}(s) \quad \text{for } s \geqslant \beta$$

and

$$s^{\varkappa}F(s) = (\beta+1)^{\varkappa}F(\beta+1) \quad \text{for } 1 \leqslant s \leqslant \beta+1,$$
$$s^{\varkappa}f(s) = \beta^{\varkappa}f(\beta) \quad \text{for } 1 \leqslant s \leqslant \beta.$$

Theorem 1 follows from (3.4), (3.5), (3.7), (4.1), (4.2), (9.1), (9.2), (6.5) and (7.7).

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