A note on some polynomial identities

by

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1. Hirschhorn [2] has proved the polynomial identities

\[ \prod_{r=1}^{2n} (1 - ax^r) = (1 - ax^{2n}) \ldots (1 - ax^n) + \sum_{r=1}^{n} (-1)^r (ax^{(2r-1)/2} + ax^{(2r+1)/2}) \times (1 - ax^{2n-2r+1}) \ldots (1 - ax^{2n-2r+2}) \ldots (1 - ax^n) \]

and

\[ \prod_{r=1}^{n} (1 - ax^r)^3 = \sum_{r=0}^{n} (-1)^r (2r+1) ax^{(r+1)/2} \times (1 - ax^{n-r+1}) \ldots (1 - ax^n) (1 - ax^{n-r+3}) \ldots (1 - ax^{2n+1}) \]

He showed also that (1.1) and (1.2) imply

\[ \prod_{r=1}^{\infty} (1 - ax^r) = 1 + \sum_{r=1}^{\infty} (-1)^r (ax^{(2r-1)/2} + ax^{(2r+1)/2}) \]

and

\[ \prod_{r=1}^{\infty} (1 - ax^r)^3 = \sum_{r=0}^{\infty} (-1)^r (2r+1) ax^{(r+1)/2}, \]

the identities of Euler and Jacobi, respectively.

In this note we show that

\[ \prod_{r=1}^{2n} (1 - ax^{(2r)/n}) = (a_n a_1 a_{n-1}) \sum_{r=1}^{n} (-1)^r \binom{n-r}{n-r} ax^{(r-1)/2 + r/2} \]

and

\[ \prod_{r=1}^{n} (1 - ax^r)^2 = \sum_{r=0}^{n} (-1)^r (2r+1) \binom{2n+1}{2n-r-1} ax^{(r+1)/2}, \]

where

\[ (a_n) = (1 - ax)(1 - ax^2) \ldots (1 - ax^n) \]
and
\[ \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{n-k}, \quad 0 \leq k \leq n. \]

The formulas (1.5) and (1.6) are equivalent to (1.1) and (1.2), respectively, and are easily proved by using the well known formula ([1], p. 280)

(1.7) \[ \prod_{k=0}^{n-1} (1 - a \omega^k) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^k \omega^{k(n-1)/2}. \]

2. We have
\[ \sum_{r=0}^{n-1} (-1)^r \binom{2n}{r} a^{n-r}(1 + \omega^{r-1})^3 = \sum_{r=0}^{2n} (-1)^{n-r} \binom{2n}{r} a^{n-r}(r-1)^3/(n-r)^3, \]
\[ = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n}{r} a^{n-r}(1 + \omega^{r-1})^3. \]

Hence, by (1.7) with \( a = \omega^{n+1/2} \), we get
\[ (-1)^n \omega^{n(n-1)/2+2n} \sum_{r=0}^{n-1} \omega^{n-r}(1 + \omega^{r-1/2}) = \sum_{r=0}^{n} \omega^{n-r}(1 + \omega^{r+1/2}) \sum_{r=0}^{n} (-1)^{n-r} \binom{2n}{r} a^{n-r}(1 + \omega^{r-1})^3. \]

This evidently proves (1.5).

In the next place, by (1.7),

(2.1) \[ \sum_{r=0}^{2n+1} (-1)^r \binom{2n+1}{r} a^r \omega^{n-r}(n-r+1)^2 = \omega^{n(n+1)/2} \sum_{r=0}^{2n+1} (-1)^r \binom{2n+1}{r} a^r \omega^{(n-1)/2} = \omega^{n(n+1)/2} \prod_{r=0}^{n} (1 - a \omega^{-r}). \]

On the other hand
\[ \sum_{r=0}^{2n+1} (-1)^r \binom{2n+1}{r} a^r \omega^{n-r}(n-r+1)^2 = \sum_{r=0}^{2n} (-1)^{n-r} \binom{2n+1}{n-r} a^{r+1} \omega^{(r+1)/2}, \]
\[ = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} a^{n-r} \omega^{r+1} \omega^{(r+1)/2} = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} a^{n-r} \omega^{r+1} \omega^{(r+1)/2} \]
\[ = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} a^{n-r} \omega^{(r+1)/2} = \omega^{n(n+1)/2} \prod_{r=0}^{n} (1 - a \omega^{-r}). \]

Thus, by (2.1), we have

(2.2) \[ a^{n(n+1)/2} \prod_{r=0}^{2n} (1 - a \omega^{-r}) = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} a^{n-r} \omega^{r+1} \omega^{(r+1)/2}, \]

and therefore

(2.3) \[ \prod_{r=0}^{2n} (1 - a \omega^{-r}) = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} a^{n-r} \omega^{(r+1)/2}. \]

For \( a = 1 \), (2.3) evidently yields (1.6).

3. In (1.7) replace \( n \) by \( 2n \) and \( a \) by \(-\frac{1}{2} a \omega^n\). Thus

(3.1) \[ \prod_{k=0}^{2n} (1 + a \omega^{k+1/2}) = \sum_{k=0}^{2n} \omega^{2k} \omega^{k^2}. \]

If we now replace \( a \) by \( a \omega^{-n} \), we get, after some manipulation,

(3.2) \[ \prod_{k=0}^{n-1} (1 + a \omega^{k+1})(1 + a^{-1} \omega^{k+1/2}) = \sum_{k=0}^{n} \omega^{2n} \omega^{k^2}. \]

Clearly (1.5) is a special case of (3.1).

Replacing \( a \) by \( a^n \), (3.1) becomes

(3.2) \[ \prod_{k=0}^{n-1} (1 + ax^{k+1})(1 + ax^{-1}x^{k+1/2}) = \sum_{k=0}^{n} \omega^{2n} \omega^{k^2}, \]

where
\[ \left[ \begin{array}{l}
2n, \n-k
\end{array} \right] = \frac{(1 - a^{-k})(1 - a^{4-k}) \cdots (1 - a^{4n-k+2})}{(1 - a^{-k})(1 - a^{k+1}) \cdots (1 - a^{4n-k+2})}.
\]

For \( n \to \infty \), (3.2) yields the Jacobi theta formula ([1], p. 282)

(3.2) \[ \prod_{k=1}^{\infty} (1 + ax^k)(1 + ax^{k-1})(1 + ax^{k-2}) = \sum_{k=0}^{\infty} \omega^{k^2} x^{2k}. \]

References


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