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Received on 30. 5. 1977
and in revised form on 21. 6. 1977 (948)

ACTA ARITHMETICA
XXXVI (1980)

A note on some polynomial identities

by

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1. Hirschhorn [2] has proved the polynomial identities

$$\prod_{r=1}^{3n}(1-x^r)=(1-x^{3n+3})\dots(1-x^{6n})+\sum_{r=1}^{n}(-1)^r(x^{r(3r-1)/2}+x^{r(3r+1)/2}\times$$
$$\times(1-x^{3n-3r+3})\dots(1-x^{3n})(1-x^{3n-3r+3})\dots(1-x^{6n}) \tag{1.1}$$

and

$$\prod_{r=1}^{n}(1-x^r)^3=\sum_{r=0}^{n}(-1)^r(2r+1)x^{(r+1)/2}\times$$
$$\times(1-x^{n-r+1})\dots(1-x^{n})(1-x^{n+r+2})\dots(1-x^{2n+1}). \tag{1.2}$$

He showed also that (1.1) and (1.2) imply

$$\prod_{r=1}^{\infty}(1-x^r)=1+\sum_{r=1}^{\infty}(-1)^r(x^{r(3r-1)/2}+x^{r(3r+1)/2}) \tag{1.3}$$

and

$$\prod_{r=1}^{\infty}(1-x^r)^3=\sum_{r=0}^{\infty}(-1)^r(2r+1)x^{r(r+1)/2}, \tag{1.4}$$

the identities of Euler and Jacobi, respectively.

In this note we show that

$$\prod_{r=1}^{3n}(1-x^{r/3})=(x)_n\sum_{r=-n}^{n}(-1)^r\begin{bmatrix}2n\\n-r\end{bmatrix}x^{r(r-1)/2+r/3} \tag{1.5}$$

and

$$\prod_{r=1}^{n}(1-x^r)^2=\sum_{r=0}^{n}(-1)^r(2r+1)\begin{bmatrix}2n+1\\n-r\end{bmatrix}x^{r(r+1)/2}, \tag{1.6}$$

where

$$(x)_n=(1-x)(1-x^2)\dots(1-x^n)$$

and

$$\begin{bmatrix} n\\ k\end{bmatrix} = \frac{(x)_n}{(x)_k(x)_{n-k}} = \begin{bmatrix} n\\ n-k\end{bmatrix}, \qquad 0\leqslant k\leqslant n.$$

The formulas (1.5) and (1.6) are equivalent to (1.1) and (1.2), respectively, and are easily proved by using the well known formula ([1], p. 280)

$$(1.7)\qquad \prod_{k=0}^{n-1}(1-ax^k) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix} a^k x^{k(k-1)/2}.$$

2. We have

$$\sum_{r=-n}^{n}(-1)^r\begin{bmatrix} 2n\\ n-r\end{bmatrix} x^{r(r-1)/2+r/3} = \sum_{r=0}^{2n}(-1)^{n-r}\begin{bmatrix} 2n\\ r\end{bmatrix} x^{(n-r)(n-r-1)/2+(n-r)/3}$$
$$= (-1)^n x^{n(n-1)/2+n/3}\sum_{r=0}^{2n}(-1)^r\begin{bmatrix} 2n\\ r\end{bmatrix} x^{-nr+2r/3}x^{r(r-1)/2}.$$

Hence, by (1.7) with $a = x^{-n+2/3}$, we get

$$(-1)^n x^{n(n-1)/2+n/3}\prod_{k=0}^{2n-1}(1-x^{-n+k+2/3}) = \prod_{k=0}^{n-1}(1-x^{(3n-3k-2)/3})\cdot\prod_{k=0}^{n-1}(1-x^{(3k+2)/3})$$
$$= \prod_{k=0}^{n-1}(1-x^{(3k+1)/3})\cdot\prod_{k=0}^{n-1}(1-x^{(3k+2)/3}).$$

This evidently proves (1.5).

In the next place, by (1.7),

$$(2.1)\qquad \sum_{r=0}^{2n+1}(-1)^r\begin{bmatrix} 2n+1\\ r\end{bmatrix} a^r x^{(n-r)(n-r+1)/2}$$
$$= x^{n(n+1)/2}\sum_{r=0}^{2n+1}(-1)^r\begin{bmatrix} 2n+1\\ r\end{bmatrix}(ax^{-n})^r x^{r(r-1)/2} = x^{n(n+1)/2}\prod_{k=0}^{2n}(1-ax^{-n+k}).$$

On the other hand

$$\sum_{r=0}^{2n+1}(-1)^r\begin{bmatrix} 2n+1\\ r\end{bmatrix} a^r x^{(n-r)(n-r+1)/2}$$
$$= \sum_{r=0}^{n}(-1)^r\begin{bmatrix} 2n+1\\ r\end{bmatrix} a^r x^{(n-r)(n-r+1)/2} + \sum_{r=0}^{n}(-1)^{n+r+1}\begin{bmatrix} 2n+1\\ n+r+1\end{bmatrix} a^{n+r+1}x^{r(r+1)/2}$$
$$= \sum_{r=0}^{n}(-1)^{n-r}\begin{bmatrix} 2n+1\\ n-r\end{bmatrix} a^{n-r}x^{r(r+1)/2} + \sum_{r=0}^{n}(-1)^{n+r+1}\begin{bmatrix} 2n+1\\ n-r\end{bmatrix} a^{n+r+1}x^{r(r+1)/2}$$
$$= \sum_{r=0}^{n}(-1)^{n+r}\begin{bmatrix} 2n+1\\ n-r\end{bmatrix} x^{r(r+1)/2}(a^{n-r}-a^{n+r+1}).$$

Thus, by (2.1), we have

$$(2.2)\qquad x^{n(n+1)/2}\prod_{k=0}^{2n}(1-ax^{-n+k}) = \sum_{r=0}^{n}(-1)^{n+r}\begin{bmatrix} 2n+1\\ n-r\end{bmatrix} x^{r(r+1)/2}(a^{n-r}-a^{n+r+1})$$

and therefore

$$(2.3)\qquad \prod_{k=0}^{n-1}(1-ax^k)(a-x^k) = \sum_{r=0}^{n}(-1)^r\begin{bmatrix} 2n+1\\ n-r\end{bmatrix} x^{r(r+1)/2}\frac{a^{n-r}-a^{n+r+1}}{1-a}.$$

For $a=1$, (2.3) evidently yields (1.6).

3. In (1.7) replace n by $2n$ and a by $-\frac{1}{2}ax$. Thus

$$\prod_{k=0}^{2n-1}(1+ax^{k+1/2}) = \sum_{k=0}^{2n}\begin{bmatrix} 2n\\ k\end{bmatrix} a^k x^{k^2/2}.$$

If we now replace a by ax^{-n}, we get, after some manipulation,

$$(3.1)\qquad \prod_{k=0}^{n-1}(1+ax^{k+1/2})(1+a^{-1}x^{k+1/2}) = \sum_{k=-n}^{n}\begin{bmatrix} 2n\\ n-k\end{bmatrix} a^k x^{k^2/2}.$$

Clearly (1.5) is a special case of (3.1).

Replacing x by x^2, (3.1) becomes

$$(3.2)\qquad \prod_{k=1}^{n-1}(1+ax^{2k-1})(1+a^{-1}x^{2k-1}) = \sum_{k=-n}^{n}\begin{bmatrix} 2n\\ n-k\end{bmatrix}' a^k x^{k^2},$$

where

$$\begin{bmatrix} 2n\\ n-k\end{bmatrix}' = \frac{(1-x^{4n})(1-x^{4n-2})\dots(1-x^{2n+2k+2})}{(1-x^2)(1-x^4)\dots(1-x^{2n-2k})}.$$

For $n\to\infty$, (3.2) yields the Jacobi theta formula ([1], p. 282)

$$\prod_{k=1}^{\infty}(1-x^{2k})(1+ax^{2k-1})(1+a^{-1}x^{2k-1}) = \sum_{k=-\infty}^{\infty}a^k x^{k^2}.$$

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Received on 24. 6. 1977 (956)