critical line \( \text{Re} \sigma = \frac{1}{2} \) does not contain an arithmetical progression. It follows that \( f_{\alpha, \beta} \) is not constant.

Case 4. \( \alpha = \frac{1}{2} \). In this case we invoke a theorem of Putnam [2] saying that the set of zeros of \( \zeta(s) \) on the critical line \( \text{Re} \sigma = \frac{1}{2} \) does not contain an arithmetical progression. It follows that also in this case \( f_{\alpha, \beta} \) cannot be constant.

Case 5. \( 0 < \alpha < \frac{1}{2} \). Because of the functional equation for \( \zeta(s) \) this case may be reduced to Case 3.

Summarizing, we have the following

**Theorem.** If \( \alpha \) and \( \beta \) are positive constants and \( f_{\alpha, \beta} : \mathbb{R} \to \mathbb{R} \) is defined by (3) then \( f_{\alpha, \beta} \) is a constant function only in case \( \alpha = \beta = \frac{\log 2}{k} \), where \( k \) is any positive integer.

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**References**


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**Some results in number theory, I**

by

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Dedicated to the memory of Professor Paul Turán

Let \( \varphi(n) \) denote Euler's totient function and \( V(n) \) the number of distinct prime factors of \( n \). In this paper, we shall study the quantity \( V(n, \varphi(n)) \) which arises naturally in group theory. For example, letting \( G(n) \) denote the number of non-isomorphic groups of order \( n \), we have by a classical result of Burnside that \( G(n) = 1 \) if and only if \( V(n, \varphi(n)) = 0 \) (i.e., \( n, \varphi(n) = 1 \)). Erdős [1] showed that the number \( F_k(n) \) of \( n \leq x \) satisfying the latter condition is

\[
F_k(x) = (1 + o(1)) x e^{-\gamma}/\log x
\]

where \( \gamma \) is Euler's constant and we write \( \log x = \log x, \log_2 x = \log (\log x_2 x) \). More generally, we can define \( F_k(n) \) to be the number of \( n \leq x \) for which \( G(n) = k \). The authors [2] have shown that for each \( k \),

\[
F_k(x) \ll x/\log^2 x
\]

The proof depended essentially on a weak form of the following result stated by Erdős in [1]: for each \( \varepsilon > 0 \), the number of \( n \leq x \) that fail to satisfy

\[
(1 - \varepsilon) \log x < V(n, \varphi(n)) < (1 + \varepsilon) \log x
\]

is \( o(x) \). (A proof of this was supplied by the authors in [2].)

It is an interesting number-theoretic problem to estimate the number \( A_k(n) \) of \( n \leq x \) for which \( V(n, \varphi(n)) = k \). Our main result here is the following theorem.

**Theorem.** For each \( k \geq 0 \), we have

\[
A_k(n) = \frac{(1 + o(1)) x e^{-\gamma}(\log x)^k}{k \log^3 x}
\]

The proof will require several lemmas and intermediate results. The first two lemmas are due to Erdős [1].

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Lemma 1. Let $p$ be a fixed prime. Then
\[
\sum_{s \leq x} \frac{1}{s} \ll \frac{1}{p} (\log p + \log_4 x),
\]
where the asterisk indicates that the sum is over primes $s \leq x$, $s \equiv 1 \pmod{p}$.

We remark here that unless otherwise stated, $p$, $q$, and $s$ will denote primes.

Lemma 2. Let $p < (\log_4 x)^{1+\varepsilon}$. Then the number of $pm \leq x$ such that $m$ has no prime divisor $\equiv 1 \pmod{p}$ is $o(\log_4 x)$ uniformly in $p$.

Lemma 3. Let $H_k(x)$ be the number of $n \leq x$ of the form $n = p_1 p_2 \ldots p_k m$, where
(i) $p_i < (\log_4 x)^{1+\varepsilon}$, $i = 1, 2, \ldots, k$;
(ii) all the prime divisors of $m$ are $\not\equiv 1 \pmod{p}$;
(iii) $[m, \varphi(m)] = 1$.
Then for each fixed $k$,
\[
H_k(x) = \frac{1 + o(1)x^{1-\varepsilon}(\log_4 x)^k}{k\log_4 x}.
\]

Proof. By definition,
\[
H_k(x) = \sum_p \sum_{m} \frac{1}{\prod_{i=1}^{k} p_i m}
\]
where the outer sum is over all $p_i < (\log_4 x)^{1+\varepsilon}$ and the inner sum is over $m \leq x/p_1 \ldots p_k$ satisfying (ii) and (iii). Erdős' proof of (1) shows that
\[
\sum_{m} \frac{1}{m} = \frac{1 + o(1)x^{1-\varepsilon}(\log_4 x)^k}{k\log_4 x}
\]
and as the product $p_1 \ldots p_k$ is obtained $k!$ times in the $k$-fold outer sum of (3), we get
\[
H_k(x) = \frac{1 + o(1)x^{1-\varepsilon}(\log_4 x)^k}{k\log_4 x} \left( \sum_{p} \frac{1}{p} \right)^k - \frac{1 + o(1)x^{1-\varepsilon}(\log_4 x)^k}{k\log_4 x}
\]
proving the lemma.

We are now ready to prove our theorem.

Proof of theorem. We shall give the proof for $k = 1$ and then sketch the modifications needed for general $k$. Write
\[
A_1(x) = A'_1(x) + A''_1(x)
\]
where $A'_1(x)$ counts the contribution of squarefree $n$ to $A_1(x)$ and $A''_1(x)$ counts the remaining $n$. First we estimate $A'_1(x)$. If $n$ is not squarefree and $V(n, \varphi(n)) = 1$ then certainly $n = p^m m$ ($n \geq 2$) with $(p, m) = 1$ and
\[
(m, \varphi(m)) = 1. \text{ The number of such } n \leq x \text{ with } p \geq \log_4 x = y \text{ (say) is clearly}
\]
\[
\sum_{p \geq y} \sum_{m \leq x/p} \frac{x}{p^{\varepsilon}} \ll \frac{x}{y},
\]
and the number of remaining non-squarefree $n$ in $A_1(x)$ is
\[
\ll \sum_{p \leq x} \sum_{m \leq x/p} A_0 \left( \frac{x}{p^2} \right) \ll \frac{x}{\log_4 x},
\]
using (1). Thus, we have
\[
A'_1(x) \ll x/\log_4 x.
\]

If $n$ is squarefree, then $V(n, \varphi(n)) = 1$ implies that
\[
A''_1(x) \ll x/\log_4 x.
\]

Let $\varepsilon > 0$ be fixed. Then, we write
\[
A_1(x) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]
where the sums are over those $n \leq x$ in $A'_1(x)$ of the form (7) and $\Sigma_1$, $p > (\log_4 x)^{1+\varepsilon}$;
in $\Sigma_2$, $\log_4 x^{1-\varepsilon} \ll p \ll (\log_4 x)^{1+\varepsilon}$;
in $\Sigma_3$, $p < (\log_4 x)^{1-\varepsilon}$ and at least one prime divisor of $m$ is $< (\log_4 x)^{1-\varepsilon}$;
in $\Sigma_4$, $p < (\log_4 x)^{1-\varepsilon}$ and all the prime divisors of $m$ are $> (\log_4 x)^{1-\varepsilon}$.
Clearly, we have by Lemma 1,
\[
\Sigma_1 \ll \sum_{p \leq x} \sum_{m \leq x/p} \frac{x}{p^{\varepsilon}} \ll \sum_{p \leq x/p} \frac{x}{p^{\varepsilon}} (\log p + \log_4 x)
\]
\[
\ll \frac{x}{(\log_4 x)^{1+\varepsilon}} \log_4 x \log_4 x = o(x \log_4 x \log_4 x).
\]
Also, we get from (1) that
\[
\Sigma_2 \ll \sum_{p \leq x} A_0 \left( \frac{x}{p} \right) \ll \sum_{p \leq x/p} \frac{x}{(p \log_4 (x/p))} \ll x/\log_4 x,
\]
where all the sums are over $p$ in the range indicated for $\Sigma_2$. Now from Lemma 2, the number of $m \leq x$, $[m, \varphi(m)] = 1$ which have a prime divisor $< (\log_4 x)^{1-\varepsilon}$ is
\[
oo{m/\log_4 x} \log_4 x \log_4 x = o(x/\log_4 x).
\]
Hence,
\[
\Sigma_3 = o \left( \frac{x}{\log_4 x} \sum_{p \leq x} \frac{1}{p} \right) = o \left( \frac{x \log_4 x}{\log_4 x} \right)
\]
where the sum is over \( p < (\log x)^{1-\varepsilon} \). For \( \Sigma_4 \), we write

\[
\Sigma_4 = \Sigma_4' + \Sigma_4''
\]

where in the first sum, all the prime divisors of \( m \) are \( > (\log x)^{1+\varepsilon} \), and the second sum contains the remaining \( n \) of \( \Sigma_4 \). Thus, for the \( n \) in \( \Sigma_4' \), there is a prime divisor \( q \) (say) with

\[
(\log x)^{1-\varepsilon} < q < (\log x)^{1+\varepsilon}
\]

so

\[
\Sigma_4' \leq \sum_p \sum_{a|n} A_a(pq) \leq \frac{\pi}{\log a} \sum_{n=1}^{\infty} \frac{1}{pq}
\]

where the sum over \( q \) is in the range (12) and the sum over \( p \) is in the range specified for \( \Sigma_4 \). The sum over \( q \) is clearly \( < \varepsilon \) so we get

\[
\Sigma_4' < \varepsilon \log \log x.
\]

Finally, recalling the definition of \( H_k(x) \) from Lemma 3, noting that our \( n \) are now squarefree, and that in the range of \( \Sigma_4 \) every number in (7) satisfies \( \psi(n, \phi(n)) = 1 \), we get

\[
H_1(x) \geq \Sigma_4' \geq H_1(x) - T(a)
\]

where \( T(a) \) is the number of \( psn = n < x \) such that \( m \) has no prime divisor \( = 1 \pmod{p} \) and \( p \) is in the range specified for \( \Sigma_4 \). Lemmas 2 and 3 imply that

\[
\Sigma_4' = \frac{1 + o(1)x e^{-\gamma} \log x}{\log x}
\]

so that combining (5), (6), (8)-(11), (13) and (14), and noting that \( \varepsilon > 0 \) was arbitrary, the proof for \( k = 1 \) is completed.

Now we sketch the modifications needed in the above proof, for general \( k \). As before, we write \( \Delta_k(x) = \Delta_k'(x) + \Delta_k''(x) \) using the notation as in (5). Recalling \( y = \log x \), we get

\[
\Delta_k'(x) \leq \sum_{p < y} \sum_{a > 1} A_{k-1} \left( \frac{p^a}{x} \right) + \frac{x}{y} < \frac{\alpha(\log x)^{k-1}}{\log x}
\]

by induction. To estimate \( \Delta_k'(x) \), we write the \( n < x \) that are counted, in the form

\[
n = p_1 \ldots p_k m, \quad (m, \phi(m)) = 1 \quad \text{and} \quad (n, \phi(n)) = p_1 \ldots p_k.
\]

Then as before,

\[
\Delta_k'(x) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]

where now, the sums are over those \( n < x \) of the form (16) and

in \( \Sigma_1 \), some \( p_i > (\log x)^{1+\varepsilon} \),

in \( \Sigma_2 \), some \( p_i \) satisfies \( (\log x)^{1-\varepsilon} < p_i < (\log x)^{1+\varepsilon} \),

in \( \Sigma_3 \), all \( p_i < (\log x)^{1-\varepsilon} \) and at least one prime divisor of \( m \) is \( < (\log x)^{1-\varepsilon} \),

in \( \Sigma_4 \), all \( p_i < (\log x)^{1-\varepsilon} \) and all the prime divisors of \( m \) are \( > (\log x)^{1-\varepsilon} \).

For \( \Sigma_1 \), the estimate (8) holds as before, and also

\[
\Sigma_4 < \pi \log x, \quad \Sigma_5 = o(x(\log x)^k/\log x),
\]

by simple modifications in (9) and (10). Finally, writing \( \Sigma_4 = \Sigma_4' + \Sigma_4'' \) in the same notation as in (11), we find again by a simple modification that

\[
\Sigma_4' = o(\alpha(\log x)^{k}/\log x)
\]

and

\[
H_k(x) \geq \Sigma_4 \geq H_k(x) - \sum_{j=0}^{k-1} A_j(x)
\]

as clearly all \( n \) counted by \( H_k(x) \) satisfy \( \psi(n, \phi(n)) = k \). By induction,

\[
\sum_{j=0}^{k-1} A_j(x) \leq x(\log x)^{k-1}/\log x,
\]

so that by Lemma 3, we get from (19) that

\[
\Sigma_4' = \frac{1 + o(1)x e^{-\gamma}(\log x)^{k}}{k! \log x}
\]

and combining (15), (8), (17), (18) and (20), the proof of the theorem is complete.

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