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On sets characterizing additive arithmetical functions, I

by

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To the memory of Professor Paul Turán

1. Throughout this paper f denotes an additive arithmetical function, A and B are subsequences of the natural numbers, formed of the elements $a_1 < a_2 < a_3 < \ldots$ and $b_1 < b_2 < b_3 < \ldots$, respectively.

A is called *a set of uniqueness* (from now on: U-set), if $f(a_k) = 0$, $k = 1, 2, \ldots$, implies $f = 0$. This notion was introduced by I. Kátai.

The problem of the U-sets is a special case of the following general type of characterization of additive functions:

If $f(a_k)$ satisfy certain conditions, then f belongs to a given class of additive functions.

Thus the U-sets characterize the class consisting of the single function $f = 0$ (see [1], [5], [7], [8]). Many characterizations were given for the class of the $c\log n$ functions (see e.g. [2], [6], [9], [10]).

In [3] I showed that

(i) even sets of "arbitrary rarity" can characterize the function $f = 0$, and

(ii) every "sufficiently dense" set has this property.

More precisely:

(i) Let $g(n)$ be an arbitrary real-valued arithmetical function. Then there exists an A, for which

$$a_k > g(k) \tag{1}$$

holds, and A is a U-set; moreover if

$$f(a_k) \text{ is convergent,} \tag{2}$$

then $f = 0$.

(ii) If a set A has upper density 1, then (2) implies $f = 0$, and the assumption of upper density 1 cannot be weakened, even under strengthening the other conditions.

In connection with (i), Professor Paul Erdös recently raised the following problem:

What can be asserted if in (i) we require even stronger conditions of rarity than (1)?

The aim of this paper is to answer this question.

The refinements of condition (1) are the following:

$$a_{k+1}-a_k > g(k), \tag{1a}$$

$$\frac{a_{k+1}}{a_k} > g(k), \tag{1b}$$

$$a_{k+1} > g(a_k). \tag{1c}$$

We shall show that we can find a suitable A which satisfies (1b) and thus (1a) too, and still characterizes the function $f = 0$ (Theorem 1). On the other hand, we prove that (1c) cannot be valid for U-sets either, and we determine the exact rate of the possible "growth" of the elements of a U-set (Theorems 2 and 3).

More precisely, we have the following results:

THEOREM 1. *Let $g(n)$ be an arbitrary real-valued arithmetical function.*

We can construct an A, for which (1b) *holds, and A has the following characterizing property:*

If

$$f(a_{k+1})-f(a_k) \text{ is convergent,} \tag{3}$$

then $f = 0$.

THEOREM 2. I. *Let A be a U-set. Then*

$$\liminf \frac{a_{k+1}}{a_k^2} \leqslant 1;$$

moreover, if we put $\frac{a_{k+1}}{a_k^2} = e_k$, *then*

$$\liminf (e_1 \cdot \ldots \cdot e_k) = 0. \tag{4}$$

II. *Let α_k be an arbitrary sequence of positive numbers, satisfying*

$$\liminf (\alpha_1 \cdot \ldots \cdot \alpha_k) = 0, \tag{5}$$

and

$$\alpha_k \geqslant 2^{-k}. \tag{6}$$

Then there exists an A, for which

$$\frac{a_{k+1}}{a_k^2} \geqslant \alpha_k \tag{7}$$

holds, and A is a U-set; moreover, if

$$\sum_{k=1}^{\infty} f(a_k) \text{ is convergent,} \tag{8}$$

then $f = 0$.

THEOREM 3. I. *Let A be a U-set. Then*

$$\liminf \frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} = 0. \tag{9}$$

II. *Let β_k be an arbitrary sequence of positive numbers, satisfying* $\liminf \beta_k = 0$.

Then there exists an A, for which

$$\frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} \geqslant \beta_k \tag{10}$$

holds, and (8) *implies $f = 0$.*

Remark. In [4] we consider the problem when the characterizing condition (8) is replaced with (3) and (2) respectively.

2. Proof of Theorem 1. First we prove a

LEMMA. *If $f(n)$ is convergent, then $f = 0$.*

Proof. Let j be a natural number, and $\varepsilon > 0$. We show that $|f(j)| < \varepsilon$. $\{f(n)\}_1^\infty$ is a Cauchy-sequence, consequently we can find an N, such that for all $s, m \geqslant N$ we have $|f(s)-f(m)| < \varepsilon$. If we take $t \geqslant N$, $(t, j) = 1$, then

$$|f(j)| = |f(t\cdot j)-f(t)| < \varepsilon. \blacksquare$$

We turn now to the proof of Theorem 1. We may assume that $g(n)$ is an increasing function, and $g(n) > 1$ for all n.

We define first a set B:

$$d_2, 2d_2, d_3, 3d_3, \ldots, d_i, i\cdot d_i, \ldots,$$

where

$$(d_i, i) = 1, \quad d_i > g(4i) \quad \text{and} \quad d_i > (i-1)\cdot d_{i-1}.$$

Then we have

$$b_k > g(2k) \quad \text{and} \quad b_k > b_{k-1}, \quad k = 1, 2, \ldots$$

We form now the required set A:

$$s_1, b_1\cdot s_1, s_2, b_2\cdot s_2, \ldots, s_i, b_i\cdot s_i, \ldots,$$

where

$$(s_i, b_i) = 1 \quad \text{and} \quad \frac{s_{i+1}}{b_i\cdot s_i} > g(2i). \tag{11}$$

(11) assures that

$$\frac{a_{k+1}}{a_k} > g(k) \qquad (\text{and } a_{k+1} > a_k)$$

hold for all k.

Let now f be additive, satisfying (3). For $k = 2i-1$, $i = 1, 2, \ldots$

$$f(a_{2i}) - f(a_{2i-1}) = f(b_i \cdot s_i) - f(s_i) = f(b_i),$$

i.e., f is convergent on B.

Hence

$$f(b_{2n-2}) - f(b_{2n-3}) = f(n \cdot d_n) - f(d_n) = f(n) \to 0,$$

and so by the lemma $f = 0$. ∎

Remarks. 1. By the same method we can easily prove that instead of the convergence of $\Delta f(a_k) = f(a_{k+1}) - f(a_k)$ it is enough to assume that $\Delta^r f(a_k)$ is convergent, where r is an arbitrary but fixed natural number and Δ^r denotes the rth difference. In this case we construct A so that we repeat r times the "doubling" process used in the preceding proof.

2. The set A constructed in the proof of Theorem 1, obviously also has the following property:

If $f(a_{k+1}) - f(a_k)$ is bounded, then f is bounded. Moreover: If

$$f(a_{k+1}) - f(a_k) = O(T(k)) \quad \text{or} \quad o(T(k)),$$

then

$$f(n) = O(T(n)) \quad \text{or} \quad o(T(n)), \quad \text{respectively},$$

if $T(n)$ satisfies some fairly general conditions, e.g.: $T(n)$ is monotone, and $T(n)$ has the same order of magnitude as $T(4n)$ (thus e.g. we may take $T(n) = \log n, n^c, \log\log n$, etc.).

3. Proof of Theorems 2/I and 3/I. These two assertions are equivalent, since

$$\frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} = \frac{a_{k+1}}{a_k^2} \frac{a_k}{a_{k-1}^2} \cdot \ldots \cdot \frac{a_2}{a_1^2} \cdot a_1.$$

We shall prove the statement of Theorem 3/I.

Indirectly, we assume that for some $d > 0$ and N, we have

$$a_{k+1} > d \cdot a_1 \cdot \ldots \cdot a_k, \tag{12}$$

for $k \geq N$.

We shall call p^r (p is a prime, $r \geq 1$) a prime power of n if p has exactly the exponent r in the standard form of n.

For any a_k, either all the prime powers of a_k have already appeared in some previous a_j numbers ($j < k$), or a_k contains a "new" prime power, i.e. one occurring in a_k for the first time in the A sequence.

(α) Let us consider the case when there are infinitely many a_k of the first type, i.e. which contain no "new" prime power. Let a_m and a_t be two such numbers, where $m \geq t \geq N$.

Then a_t is formed of some of the prime powers of $a_1, a_2, \ldots, a_{t-2}$ and a_{t-1}, and so a_m is the product of some of the prime powers of $a_1, \ldots$ $\ldots, a_{t-1}, a_{t+1}, \ldots, a_{m-2}$ and a_{m-1}, i.e.,

$$a_m \leq a_1 \cdot \ldots \cdot a_{t-1} \cdot a_{t+1} \cdot \ldots \cdot a_{m-1}. \tag{13}$$

On the other hand, by (12),

$$a_m > d \cdot a_1 \cdot \ldots \cdot a_{m-1},$$

and so, using (13), we infer

$$d \cdot a_t < 1.$$

But this is impossible if t is large enough.

We obtained that apart from finitely many values of k, each a_k must contain a new prime power.

Thus we may assume N to be so large that for $k \geq N$ a_k has a new prime power.

(β) If an a_i contains more than one new prime power, say P and Q, then put $f(P) = 1$, $f(Q) = -1$, $f(R) = 0$, if R is a prime power of $a_1, \ldots$ $\ldots, a_{k-2}$ or a_{k-1}, and finally we define the values of f successively on the new prime powers of $a_{i+1}, a_{i+2}, \ldots$, by the equations $f(a_k) = 0$, $k = i+1$, $i+2, \ldots$

Thus we define an additive f, for which $f(a_k) = 0$ holds for all k, but $f \neq 0$. This is impossible since A is a U-set.

Therefore we may assume that in each a_k exactly one new prime power appears, if $k \geq N$.

Further, every prime power must appear in some a_k, otherwise we could define the value of f arbitrarily on the "missing" prime powers, but this contradicts to the assumption that A is a U-set.

Let us consider now an arbitrary $k \geq N$, and take three prime powers lying between a_k and $2a_k$, let these be q_1, q_2 and q_3. We check, which of the numbers $a_{k+2}, a_{k+3}, \ldots$ contain these q_i as "new" prime powers. Suppose, we "meet" first q_1 in a_s, and then q_2 in a_t. (We needed the third one only for eliminating the problem of the possible occurrence of one of the q_i already in a_{k+1}.)

Clearly, a_s consists of q_1 and of some of the prime powers in $a_1, a_2, \ldots$ $\ldots, a_{s-1}$, and so a_t is the product of q_2, possibly of q_1, and of some prime powers in $a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{t-1}$.

Thus we have

$$a_l \leqslant q_1 \cdot q_2 \cdot a_1 \cdot \ldots \cdot a_{s-1} \cdot a_{s+1} \cdot \ldots \cdot a_{l-1}.$$

This and the lower estimation (12) applied to a_l imply

$$q_1 \cdot q_2 > d \cdot a_s. \tag{14}$$

Using (12) again, we obtain

$$d \cdot a_s > d^2 \cdot a_1 \cdot \ldots \cdot a_{s-1} > d^2 \cdot a_1 \cdot \ldots \cdot a_{k-1} \cdot a_k^2, \tag{15}$$

since $s \geqslant k+2$.

Furthermore, we have

$$q_1 \cdot q_2 < 4a_k^2 \tag{16}$$

by the choice of the q_i.

(14), (15) and (16) together imply

$$4a_k^2 > d^2 \cdot a_1 \cdot \ldots \cdot a_{k-1} \cdot a_k^2,$$

i.e.

$$\frac{4}{d^2} > a_1 \cdot \ldots \cdot a_{k-1},$$

but this is impossible if k is large enough. ■

Using the same ideas we are able to prove an even sharper result:

THEOREM 4. *If A does not satisfy (4) or (9), i.e. if*

$$\liminf \frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} > 0,$$

then there exists an f, which is unbounded, though $f(a_k) = 0$, $k = 1, 2, \ldots$

Moreover, to any $h(n)$ positive-valued function we can construct a suitable f, which satisfies even

$$\limsup \frac{f(n)}{h(n)} = \infty.$$

Proof. In the preceding proof we actually showed, that if

$$\liminf \frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} > 0,$$

then

(*) a_k contains a "new" prime power if k is large enough,

and

(**) either there are infinitely many prime powers which do not appear in any a_k, or there are infinitely many a_k which contain at least two new prime powers.

To verify (**), indirectly, we choose N so large, that for $k \geqslant N$, a_k contains no more than one new prime power, and all prime powers greater than a_N occur in some (later) a_j. This leads to a contradiction, exactly in the same way, as in the previous proof.

We are now ready to prove Theorem 4.

Case I. There are infinitely many prime powers, $P_1, \ldots, P_j, \ldots$, which do not appear in any a_k. Put $f(P_j) = j \cdot h(P_j)$, $j = 1, 2, \ldots$, and $f(P) = 0$ for all other prime powers P.

Case II. There are infinitely many a_k, say $a_{k_1}, \ldots, a_{k_j}, \ldots$, which have at least two new prime powers, one of these should be Q_j.

We may assume, by (*), that for $k \geqslant k_1$ each a_k contains at least one new prime power.

Put $f(Q_j) = j \cdot h(Q)$, $j = 1, 2, \ldots$, and for the other prime powers P we define the $f(P)$ values so, that $f(a_k) = 0$ should hold for all k. This is possible (see the previous observation), f is not necessarily unique.

The f functions constructed according to the two cases clearly meet the requirements of Theorem 4. ■

4. Proof of Theorem 3/II. We take a sequence (of natural numbers) $t_1, t_2, \ldots$, in which each natural numbers occurs infinitely often.

The required $a_1, a_2, \ldots$ sequence will be the union of successive "blocks".

The ith block consists of N_i+1 elements:

$$u_{i1}, u_{i2}, \ldots, u_{iN_i}, t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i},$$

where the u_{ij} are primes, $(u_{ij}, t_i) = 1$, and the u_{ij} and N_i are specified as follows:

Let us suppose, that we have already constructed the $(i-1)$-st block, and let a_s denote its last element ($s = N_1 + \ldots + N_{i-1} + i - 1$). We take:

$$\begin{aligned}
a_{s+1} &= u_{i1} \geqslant \beta_s \cdot a_1 \cdot \ldots \cdot a_s, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{s+j} &= u_{ij} \geqslant \beta_{s+j-1} \cdot a_1 \cdot \ldots \cdot a_{s+j-1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{s+N_i} &= u_{iN_i} \geqslant \beta_{s+N_i-1} \cdot a_1 \cdot \ldots \cdot a_{s+N_i-1}, \\
a_{s+N_i+1} &= t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i}.
\end{aligned}$$

Now we have to choose an N_i satisfying

$$\begin{aligned}
a_{s+N_i+1} = t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i} &\geqslant \beta_{s+N_i} \cdot a_1 \cdot \ldots \cdot a_{s+N_i} \\
&= \beta_{s+N_i} \cdot a_1 \cdot \ldots \cdot a_s \cdot u_{i1} \cdot \ldots \cdot u_{iN_i}.
\end{aligned}$$

This means

$$\beta_{s+N_i} \cdot a_1 \cdot \ldots \cdot a_s \leqslant t_i.$$

Here $s, t_i, a_1, \ldots, a_s$ are fixed numbers. Since $\liminf \beta_k = 0$, we can find an $r > s$, for which

$$\beta_r \cdot a_1 \cdot \ldots \cdot a_s \leqslant t_i$$

holds. Put $N_i = r - s$, and so (10) is valid.

We take now an f satisfying (8), and fix an arbitrary $\varepsilon > 0$. Using the Cauchy-property we have

$$|f(u_{i1}) + \ldots + f(u_{iN_i})| < \varepsilon/2 \quad \text{and} \quad |f(t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_j})| < \varepsilon/2,$$

if i is large enough.

Thus

$$|f(t_i)| = |\{f(t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i})\} - \{f(u_{i1}) + \ldots + f(u_{iN_i})\}| \leqslant |\ \ | + |\ \ | < \varepsilon,$$

if i is large enough.

Since each natural number occurs infinitely often among the t_i, we obtain $|f(n)| < \varepsilon$ for all n and for all ε, i.e. $f = 0$. ∎

5. Proof of Theorem 2/II. Put $\gamma_k = a_k \cdot (1 + 2^{-k})^2 = a_k \vartheta_k^2$. Since $\prod_{k=1}^{\infty} (1 + 2^{-k})$ is convergent, we have $\liminf (\gamma_1 \cdot \ldots \cdot \gamma_k) = 0$.

The sequence $t_1, t_2, \ldots$, and the structure of the blocks should be the same as in the previous proof, the u_{ij} will be primes again, with $(u_{ij}, t_i) = 1$, while the other specifications are modified as follows:

Let us suppose that the $(i-1)$-st block has already been constructed, and a_s is its last element.

We want to define the set A so that the value of a_{k+1}/a_k^2 should be "about" a_k. This means:

$$\begin{aligned}
a_{s+1} &= u_{i1} \sim a_s \cdot a_s^2, \\
a_{s+2} &= u_{i2} \sim a_{s+1} \cdot (a_{s+1})^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{s+N_i} &= u_{iN_i} \sim a_{s+N_i-1} \cdot (a_{s+N_i-1})^2, \\
a_{s+N_i+1} &= t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i}.
\end{aligned}$$

We denote u_{i1} simply by u, and express all the u_{ij} by u and the a_k:

$$\begin{aligned}
& u_{i2} \sim a_{s+1} \cdot u^2, \\
& u_{i3} \sim a_{s+2} \cdot u_{i2}^2 \sim a_{s+2} \cdot (a_{s+1})^2 \cdot u^4, \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& u_{ij} \sim a_{s+j-1} \cdot (a_{s+j-2})^2 \cdot \ldots \cdot (a_{s+1})^{2^{j-2}} \cdot u^{2^{j-1}}.
\end{aligned}$$

More precisely, we define the u_{ij} as follows: Let $u_{i1} = u$ be a prime, $u \geqslant a_s \cdot a_s^2$, $u > t_i$, and

$$u > 2^{10(s+1)}. \tag{17}$$

Furthermore, for $j \geqslant 2$:

$$M_j = u^{2^{j-1}} \cdot (\gamma_{s+1})^{2^{j-2}} \cdot \ldots \cdot (\gamma_{s+j-2})^2 \cdot \gamma_{s+j-1} \leqslant u_{ij} \leqslant M_j \cdot \vartheta_{s+j} \tag{18}$$

(M_j also depends on i, of course).

If we put $M_1 = u$, then (18) holds also for $j = 1$.

Later we show the u_{ij} can be chosen as primes, i.e. there is at least one prime between M_j and $M_j \vartheta_{s+j}$ (for $j \geqslant 2$). Taking this for granted, we have also $(u_{ij}, t_i) = 1$, since $u_{ij} > u > t_i$.

We check now

$$\frac{a_{s+j+1}}{(a_{s+j})^2} = \frac{u_{i,j+1}}{u_{ij}^2} \geqslant a_{s+j}, \quad \text{for} \quad j = 1, 2, \ldots, N_i - 1.$$

Indeed:

$$\begin{aligned}
\frac{u_{i,j+1}}{u_{ij}^2} &\geqslant \frac{M_{j+1}}{M_j^2 \cdot (\vartheta_{s+j})^2} = \frac{u^{2^j} \cdot (\gamma_{s+1})^{2^{j-1}} \cdot \ldots \cdot (\gamma_{s+j-1})^2 \cdot \gamma_{s+j}}{u^{2^j} \cdot (\gamma_{s+1})^{2^{j-1}} \cdot \ldots \cdot (\gamma_{s+j-1})^2 \cdot (\vartheta_{s+j})^2} \\
&= \frac{\gamma_{s+j}}{(\vartheta_{s+j})^2} = a_{s+j}.
\end{aligned}$$

We need now a suitable N_i for which

$$\frac{a_{s+N_i+1}}{(a_{s+N_i})^2} \geqslant a_{s+N_i}$$

also holds.

$$\begin{aligned}
\frac{a_{s+N_i+1}}{(a_{s+N_i})^2} &= \frac{t_i \cdot u_{i1} \cdot \ldots \cdot u_{iN_i}}{(u_{iN_i})^2} \geqslant \frac{t_i \cdot M_1 \cdot M_2 \cdot \ldots \cdot M_{N_i-1}}{M_{N_i} \cdot \vartheta_{s+N_i}} \\
&= \frac{t_i \cdot u^{1+2+4+\ldots+2^{N_i-2}} \cdot (\gamma_{s+1})^{1+2+4+\ldots+2^{N_i-3}} \cdot \ldots \cdot \gamma_{s+N_i-2}}{u^{2^{N_i-1}} \cdot (\gamma_{s+1})^{2^{N_i-2}} \cdot \ldots \cdot \gamma_{s+N_i-1} \cdot \vartheta_{s+N_i}} \\
&= \frac{t_i}{u \cdot \gamma_{s+1} \cdot \ldots \cdot \gamma_{s+N_i-1} \vartheta_{s+N_i}} = R.
\end{aligned}$$

Since $\liminf (\gamma_1 \cdot \ldots \cdot \gamma_k) = 0$, we have $\liminf_r (\gamma_{s+1} \cdot \ldots \cdot \gamma_{s+r}) = 0$ as well, and thus we can find an N_i for which

$$\gamma_{s+1} \cdot \ldots \cdot \gamma_{s+N_i} \leqslant \frac{t_i}{2u} \leqslant \frac{t_i}{u \cdot \vartheta_{s+N_i}}$$

holds.

Hence

$$R \geqslant \gamma_{s+N_i} > a_{s+N_i},$$

which completes the verification of (7) for all k.

We have to show that the u_{ij} may be chosen as primes, i.e. the interval $[M_j, M_j\cdot\vartheta_{s+j}]$ always contains a prime ($j\geqslant 2$). As e.g. $[n, n+n^{0.9}]$ always contains a prime, it suffices to show that

$$M_j\cdot\vartheta_{s+j}-M_j > M_j^{0.9},$$

i.e.

(19) $$M_j > 2^{10(s+j)}.$$

By (17)

$$M_1 = u > 2^{10(s+1)}$$

thus (19) holds for $j=1$.

Assuming the validity of (19) for j, we prove it for $j+1$. Using $\gamma_k > a_k > 2^{-k}$, we obtain:

$$M_{j+1} = M_j^2\cdot\gamma_{s+j} > 2^{20(s+j)}\cdot 2^{-s-j} > 2^{10(s+j+1)}.$$

Finally, if f satisfies (8), we conclude that $f=0$, exactly in the same way as we did in the proof of Theorem 3/II.

Remarks. 1. The most interesting case of Theorem 2/II is when $\liminf a_k = 1$, and this shows that (6) is not a serious restriction. We also observe, that (6) can also be replaced by considerably weaker conditions.

2. Theorems 3/II and 2/II do not imply each other, though Theorem 3/II is "almost" a corollary of Theorem 2/II (condition (6) is the only obstacle in this direction).

3. Analysing the proof of Theorem 2/II we may observe that the condition (7) for the set A can also be replaced by stronger prescriptions. E.g. if $\liminf a_k = d$ ($d\leqslant 1$ by Theorem 2/I), and $L\geqslant 1$, $M\geqslant L/d$ are arbitrary finite or infinite values, then we can find an A satisfying

$$\liminf\frac{a_{k+1}}{a_k\cdot a_k^2} = L \quad\text{and}\quad \limsup\frac{a_{k+1}}{a_k\cdot a_k^2} = M.$$

Similar assertions hold also for Theorem 3/II.

4. The sets A constructed in the proofs of Theorems 2/II and 3/II have also the following property:

If the $\sum_{r=1}^{k} f(a_r)$ sums are bounded, then f is bounded.

The proof is straightforward.

6. Finally we mention the following generalization of Theorem 2/II, which shows that the a_k elements of the characterizing set A may be chosen arbitrarily large, even with respect to a_{k-1}, for almost all k.

THEOREM 5. *By $K=\{k_n\}_{n=1}^{\infty}$ we shall denote a suitable subsequence of the natural numbers.*

Let $h(n)$ be an arbitrary function tending to infinity, and $g(n)$ an arbitrary function. Furthermore, let a_k satisfy (5) *and* (6).

Then we can find an A and K, for which the following hold:

(i) $$a_{k+1} > g(a_k) \quad \text{if} \quad k\in K,$$

(ii) $$\frac{a_{k+1}}{a_k^2} > a_k \quad \text{if} \quad k\notin K,$$

(iii) *for infinitely many values of n $\sum_{k\in K, k\leqslant n} 1 > n-h(n)$ (i.e. the upper density of K is "very strongly" 1),*

(iv) *if f satisfies* (8) *or the $\sum_{r=1}^{k} f(a_r)$ sums are bounded, then $f=0$, or f is bounded, resp.*

Proof. We have to modify the proof of Theorem 2/II only slightly: after each block we insert "very many" "stuffing" elements of "arbitrarily rapid growth".

I am indebted to Professor Paul Erdös for his valuable remarks.

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