On sets characterizing additive arithmetical functions, I

by

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To the memory of Professor Paul Turán

1. Throughout this paper $f$ denotes an additive arithmetical function, $A$ and $B$ are subsequences of the natural numbers, formed of the elements $a_1 < a_2 < a_3 < \ldots$ and $b_1 < b_2 < b_3 < \ldots$, respectively.

$A$ is called a set of uniqueness (from now on: $U$-set), if $f(a_k) = 0$, $k = 1, 2, \ldots$, implies $f = 0$. This notion was introduced by L. Kátai.

The problem of the $U$-sets is a special case of the following general type of characterization of additive functions:

If $f(a_k)$ satisfy certain conditions, then $f$ belongs to a given class of additive functions.

Thus the $U$-sets characterize the class consisting of the single function $f = 0$ (see [1], [5], [7], [8]). Many characterizations were given for the class of the $cloga$ functions (see e.g. [2], [9], [10], [10]).

In [3] I showed that

(i) even sets of "arbitrary rarity" can characterize the function $f = 0$, and

(ii) every "sufficiently dense" set has this property.

More precisely:

(i) Let $g(n)$ be an arbitrary real-valued arithmetical function. Then there exists an $A$, for which

\[ a_k > g(k) \]

holds, and $A$ is a $U$-set; moreover if

\[ f(a_k) \text{ is convergent,} \]

then $f = 0$.

(ii) If a set $A$ has upper density 1, then (2) implies $f = 0$, and the assumption of upper density 1 cannot be weakened, even under strengthening the other conditions.
In connection with (i), Professor Paul Erdős recently raised the following problem:

What can be asserted if in (i) we require even stronger conditions of rarity than (1)?

The aim of this paper is to answer this question.

The refinements of condition (1) are the following:

(1a) \[ a_{k+1} - a_k > g(k) \]

(1b) \[ \frac{a_{k+1}}{a_k} > g(k) \]

(1c) \[ a_{k+1} > g(a_k) \]

We shall show that we can find a suitable \( A \) which satisfies (1b) and thus (1c) too, and still characterizes the function \( f = 0 \) (Theorem 1). On the other hand, we prove that (1c) cannot be valid for \( U \)-sets either, and we determine the exact rate of the possible “growth” of the elements of a \( U \)-set (Theorems 2 and 3).

More precisely, we have the following results:

Theorem 1. Let \( g(n) \) be an arbitrary real-valued arithmetical function.

We can construct an \( A \), for which (1b) holds, and \( A \) has the following characterizing property:

If

(3) \[ f(a_{k+1}) - f(a_k) \text{ is convergent}, \]

then \( f = 0 \).

Theorem 2. Let \( A \) be a \( U \)-set. Then

\[ \liminf \frac{a_{k+1}}{a_k} < 1; \]

moreover, if we put \( \frac{a_{k+1}}{a_k} = \epsilon_k \), then

(4) \[ \liminf (\epsilon_k) = 0. \]

Theorem 3. Let \( \beta \) be an arbitrary sequence of positive numbers, satisfying

\[ \liminf \frac{a_{k+1}}{a_k} \geq \beta_k. \]

holds, and \( \beta = 0 \), then \( f = 0 \).

Theorem 3. Let \( A \) be a \( U \)-set. Then

(9) \[ \liminf \frac{a_{k+1}}{a_k} = 0. \]

II. Let \( \beta \) be an arbitrary sequence of positive numbers, satisfying

\[ \liminf \beta_k = 0. \]

Then there exists an \( A \), for which

(10) \[ \frac{a_{k+1}}{a_k} \geq \beta_k \]

holds, and (8) implies \( f = 0 \).

Remark. In [4] we consider the problem when the characterizing condition (8) is replaced with (3) and (3) respectively.

2. Proof of Theorem 1. First we prove a lemma. If \( f(n) \) is convergent, then \( f = 0 \).

Proof. Let \( j \) be a natural number, and \( \varepsilon > 0 \). We show that \( |f(j)| < \varepsilon \).

For any \( \varepsilon > 0 \), there exists a \( N \), such that for all \( n \geq N \), we have \( |f(n)| < \varepsilon \). If we take \( t > N \), then

\[ |f(j)| = |f(t+j)-f(t)| < \varepsilon. \]

We turn now to the proof of Theorem 1. We may assume that \( g(n) \) is an increasing function, and \( g(n) > 1 \) for all \( n \).

We define first a set \( B \):

\[ \bar{d}_1, 2\bar{d}_1, \bar{d}_2, 3\bar{d}_2, \ldots, \bar{d}_i, i\cdot \bar{d}_i, \ldots, \]

where

\[ (\bar{d}_i, i) = 1, \quad \bar{d}_i > g(\bar{d}_i) \quad \text{and} \quad \bar{d}_i > (i-1) \cdot \bar{d}_{i-1}. \]

Then we have

\[ b_k > g(2k) \quad \text{and} \quad b_k > b_{k-1}, \quad k = 1, 2, \ldots \]

We form now the required set \( A \):

\[ s_1, b_1 \cdot s_1, s_2, b_2 \cdot s_2, \ldots, s_i, b_i \cdot s_i, \ldots, \]

where

\[ s_1, b_1 \cdot s_1, s_2, b_2 \cdot s_2, \ldots, s_i, b_i \cdot s_i, \ldots, \]

and

\[ b_{i} \cdot s_{i} > g(2i). \]
(11) assures that
\[
\frac{a_{k+1}}{a_k} > g(k) \quad \text{(and } a_{k+1} > a_k) \]
hold for all \( k \).

Let now \( f \) be additive, satisfying (3). For \( k = 2i - 1, i = 1, 2, \ldots \)
\[
f(a_{2i}) - f(a_{2i-1}) = f(b_i) - f(a_i) = f(b_i),
\]
i.e., \( f \) is convergent on \( B \).

Hence
\[
f(b_{2n-1}) - f(b_{2n-3}) = f(n \cdot d_n) - f(d_n) = f(n) \to 0,
\]
and so by the lemma \( f = 0 \). \( \blacksquare \)

Remarks. 1. By the same method we can easily prove that instead of the convergence of \( \Delta f(a_k) = f(a_{k+1}) - f(a_k) \) it is enough to assume that \( \Delta^r f(a_k) \) is convergent, where \( r \) is an arbitrary but fixed natural number and \( \Delta^r \) denotes the \( r \)th difference. In this case we construct \( A \) so that we repeat \( r \) times the "doubling" process used in the preceding proof.

2. The set \( A \) constructed in the proof of Theorem 1, obviously also has the following property:
If \( f(a_{k+1}) - f(a_k) \) is bounded, then \( f \) is bounded. Moreover: If
\[
f(a_{k+1}) - f(a_k) = O(T(k)) \quad \text{or} \quad O(T(k)),
\]
then
\[
f(a) = O(T(a)) \quad \text{or} \quad O(T(a)), \quad \text{respectively},
\]
it \( T(a) \) satisfies some fairly general conditions, e.g.: \( T(a) \) is monotone, and \( T(a) \) has the same order of magnitude as \( T(4a) \) (thus e.g. we may take \( T(a) = \log a, \log \log a, \log \log \log a, \) etc.).

3. Proof of Theorems 2/1 and 3/1. These two assertions are equivalent, since
\[
\frac{a_{k+1}}{a_k} \frac{a_k}{a_{k-1}} \cdots \frac{a_2}{a_1} = \frac{a_{k+1}}{a_k} \frac{a_k}{a_{k-1}} \cdots \frac{a_2}{a_1}. \frac{a_2}{a_1}. \frac{a_1}{a_0}.
\]
We shall prove the statement of Theorem 3/1.

Indirectly, we assume that for some \( \varepsilon > 0 \) and \( N \), we have
\[
a_{k+1} > \varepsilon \cdot a_1 \cdot \cdots \cdot a_k,
\]
for \( k \geq N \).

We shall call \( p^r \) (\( p \) is a prime, \( r \geq 1 \)) a prime power of \( n \) if \( p \) has exactly the exponent \( r \) in the standard form of \( n \).

For any \( a_k \), either all the prime powers of \( a_k \) have already appeared in some previous \( a_l \) numbers (\( j < k \)), or \( a_k \) contains a "new" prime power, i.e. one occurring in \( a_k \) for the first time in the \( A \) sequence.

(a) Let us consider the case when there are infinitely many \( a_k \) of the first type, i.e. which contain no "new" prime power. Let \( a_m \) and \( a_t \) be two such numbers, where \( m \geq t \geq N \).

Then \( a_t \) is formed of some of the prime powers of \( a_1, a_2, \ldots, a_{t-2} \) and \( a_{t-1} \), and so \( a_t \) is the product of some of the prime powers of \( a_1, \ldots, a_{t-2}, a_{t+1}, \ldots, a_{m-2} \) and \( a_m \), i.e.,
\[
a_t \leq a_1 \cdot \cdots \cdot a_{t-2} \cdot a_{t+1} \cdot \cdots \cdot a_{m-1}.
\]

On the other hand, by (12),
\[
a_m > d \cdot a_1 \cdot \cdots \cdot a_{m-1},
\]
and so, using (13), we infer
\[
d \cdot a_t < 1.
\]
But this is impossible if \( t \) is large enough.

We obtained that apart from finitely many values of \( k \), each \( a_k \) must contain a new prime power.

Thus we may assume \( N \) to be so large that for \( k \geq N \) \( a_k \) has a new prime power.

(b) If an \( a_k \) contains more than one new prime power, say \( P \) and \( Q \), then put \( f(P) = 1, f(Q) = -1, f(R) = 0 \), if \( R \) is a prime power of \( a_1, \ldots, a_{t-2}, a_{t-1} \), and finally we define the values of \( f \) successively on the new prime powers of \( a_{t+1}, a_{t+2}, \ldots \), by the equations \( f(a_k) = 0, k = t + 1, t + 2, \ldots \).

Thus we define an additive \( f \), for which \( f(a_k) = 0 \) holds for all \( k \), but \( f \neq 0 \). This is impossible since \( A \) is a \( U \)-set.

Therefore we may assume that in each \( a_k \) exactly one new prime power appears, if \( k \geq N \).

Further, every prime power must appear in some \( a_k \), otherwise we could define the value of \( f \) arbitrarily on the "missing" prime powers, but this contradicts the assumption that \( A \) is a \( U \)-set.

Let us consider now an arbitrary \( k \geq N \), and take three prime powers lying between \( a_k \) and \( 2a_k \), let these be \( q_1, q_2 \) and \( q_3 \). We check, which of the numbers \( a_{k+1}, a_{k+2}, \ldots \) contain these \( q_1 \) as "new" prime powers. Suppose, we "meet" first \( q_1 \) in \( a_s \), and then \( q_3 \) in \( a_t \). (We needed the third one only for eliminating the problem of the possible occurrence of one of the \( q_3 \) already in \( a_{k+1} \).)

Clearly, \( a_t \) consists of \( q_3 \) and of some of the prime powers in \( a_1, a_2, \ldots, a_{t-1} \), and so \( a_t \) is the product of \( q_3 \), possibly of \( q_1 \), and of some prime powers in \( a_2, \ldots, a_{t-1}, a_{k+1}, \cdots, a_{t-1} \).
Thus we have
\[ a_k \leq g_1 \cdot g_2 \cdot a_1 \cdot \ldots \cdot a_{k-1} \cdot a_{k+1} \ldots \cdot a_{k-1}. \]
This and the lower estimation (12) applied to \( a_k \) imply
\[(14) \quad g_1 \cdot g_2 > d \cdot a_k. \]
Using (12) again, we obtain
\[(15) \quad d \cdot a_s > d^2 \cdot a_1 \cdot \ldots \cdot a_{k-1} > d^2 \cdot a_1 \ldots \cdot a_{k-1} \cdot a_1^s, \]
where \( s \geq 2 \).
Furthermore, we have
\[(16) \quad g_1 \cdot g_2 < 4a_k, \]
by the choice of the \( g_i \).
(14), (15) and (16) together imply
\[4a_k^2 > d^2 \cdot a_1 \cdot \ldots \cdot a_{k-1}, \]
i.e.
\[\frac{4}{d^2} > a_1 \cdot \ldots \cdot a_{k-1}; \]
but this is impossible if \( k \) is large enough. \( \square \)

Using the same ideas we are able to prove an even sharper result:

**Theorem 4.** If \( A \) does not satisfy (4) or (9), i.e. if
\[ \liminf \frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} > 0, \]
then there exists an \( f \), which is unbounded, though \( f(a_k) = 0 \), \( k = 1, 2, \ldots \)
Moreover, to every \( \mathbb{N}^+ \) positive-valued function we can construct a suitable \( f \), which satisfies even
\[ \limsup \frac{f(n)}{\mathbb{N}^+(n)} = \infty. \]

**Proof.** In the preceding proof we actually showed, that if
\[ \liminf \frac{a_{k+1}}{a_1 \cdot \ldots \cdot a_k} > 0, \]
then
\[ (\ast) \quad a_k \text{ contains a "new" prime power if } k \text{ is large enough, and} \]
\[ (*\ast) \quad \text{either there are infinitely many prime powers which do not appear in any } a_k, \text{ or there are infinitely many } a_k \text{ which contain at least two new prime powers.} \]

To verify (\( \ast \)), indirectly, we choose \( N \) so large, that for \( k \geq N, a_k \) contains no more than one new prime power, and all prime powers greater than \( a_N \) occur in some (later) \( a_k \). This leads to a contradiction, exactly in the same way, as in the previous proof.

We are now ready to prove Theorem 4.

**Case I.** There are infinitely many prime powers, \( P_1, \ldots, P_j, \ldots \), which do not appear in any \( a_k \). Put \( f(P_j) = j \cdot h(P_j), j = 1, 2, \ldots \), and \( f(P_j) = 0 \) for all other prime powers \( P_j \).

**Case II.** There are infinitely many \( a_k \), say \( a_{k_1}, \ldots, a_{k_2}, \ldots \), which have at least two new prime powers, one of these should be \( Q_j \).

We may assume, by (\( \ast \)), that for \( k \geq k_1 \) each \( a_k \) contains at least one new prime power.

Put \( f(Q_j) = j \cdot h(Q_j), j = 1, 2, \ldots \), and for the other prime powers \( P \) we define the \( f(P) \) values so, that \( f(a_k) = 0 \) should hold for all \( k \). This is possible (see the previous observation), \( f \) is not necessarily unique.

The \( f \) functions constructed according to the two cases clearly meet the requirements of Theorem 4. \( \square \)

**Proof of Theorem 3.** We take a sequence (of natural numbers) \( i_1, i_2, \ldots \), in which each natural number occurs infinitely often.

The required \( a_{i_1}, a_{i_2}, \ldots \) sequence will be the union of successive "blocks".

The \( i^\text{th} \) block consists of \( N_i + 1 \) elements:
\[ u_{i_1}, u_{i_2}, \ldots, u_{i_{N_i}}, i_{i_1}, u_{i_2}, \ldots, u_{i_{N_i}}, \]
where the \( u_{i_j} \) are primes, \( (u_{i_j}, d) = 1 \), and the \( u_{i_j} \) and \( N_i \) are specified as follows:

Let us suppose, that we have already constructed the \((i-1)^\text{st}\) block, and let \( a_k \) denote its last element \((i = N_i + \ldots + N_i + i - 1)\). We take:
\[ a_{k+1} = u_{i_1} \geq u_{i_2} \geq \beta_{i_1} \cdot a_1 \cdot \ldots \cdot a_{k-1}, \]
\[ \ldots \ldots \ldots \]
\[ a_{k+i} = u_{i_1} \geq \beta_{i_2} \cdot a_{i_1+1} \cdot \ldots \cdot a_{k+i-1}, \]
\[ \ldots \ldots \ldots \]
\[ a_{k+N_i} = u_{i_1} \geq \beta_{i_2} \cdot a_{i_1+1} \cdot \ldots \cdot a_{k+N_i-1} \]
\[ a_{k+N_i+1} = i_{i_1}, u_{i_2}, \ldots, u_{i_{N_i}} \cdot u_{i_1}. \]

Now we have to choose an \( N_i \) satisfying
\[ a_{k+N_i+1} = i_{i_1}, u_{i_2}, \ldots, u_{i_{N_i}}, \beta_{i_1} \cdot a_1 \cdot \ldots \cdot a_{k+N_i} \]
\[ = \beta_{i_1} \cdot a_1 \cdot \ldots \cdot a_{k+N_i} \cdot u_{i_1} \cdot \ldots \cdot u_{i_{N_i}}. \]
This means
\[ \beta_{i+1} \cdot a_{i_1} \cdot \ldots \cdot a_{k+N_i} \leq i_{i_1}. \]
Here \( s, t, a_1, \ldots, a_n \) are fixed numbers. Since \( \liminf \beta_n = 0 \), we can find an \( r > s \), for which

\[
\beta_r \cdot a_1 \cdots a_n \leq t_i
\]

holds. Put \( N_\ell = r - s \), and so (10) is valid.

We take now an \( f \) satisfying (8), and fix an arbitrary \( e > 0 \). Using the Cauchy-property we have

\[
|f(u_1) + \cdots + f(u_{N_\ell})| < \varepsilon/2 \quad \text{and} \quad |f(t_i \cdot u_1 \cdots u_{N_\ell})| < \varepsilon/2,
\]

if \( i \) is large enough.

Thus

\[
|f(t_i)| = |f(t_i \cdot u_1 \cdots u_{N_\ell}) - (f(u_1) + \cdots + f(u_{N_\ell}))| \leq \varepsilon,
\]

if \( i \) is large enough.

Since each natural number occurs infinitely often among the \( t_i \), we obtain \( |f(n)| < \varepsilon \) for all \( n \) and for all \( \varepsilon \), i.e. \( f = 0 \).

5. Proof of Theorem 2/II. Put \( \gamma_n = a_n \cdot (1 + 2^{-b}^k)^2 = a_n \cdot b^2 \). Since

\[
\prod_{k=1}^{\infty} (1 + 2^{-b}^k)
\]

is convergent, we have liminf of \( \gamma_1, \ldots, \gamma_n \) is zero.

The sequence \( t_1, t_2, \ldots \), and the structure of the blocks should be the same as in the previous proof, and the \( u_{ij} \) will be primes again, with \( \langle u_{ij}, t_i \rangle = 1 \), while the other specifications are modified as follows:

Let us suppose that the \( (i-1) \)-st block has already been constructed, and \( a_i \) is its last element.

We want to define the set \( A \) so that the value of \( a_{n+1}/a_n^2 \) should be "about" \( a_n \). This means:

\[
a_{n+1} = u_{11} \sim a_n \cdot a_n^2,
\]

\[
a_{n+1} = u_{12} \sim a_n \cdot (a_n + 2^k)^2,
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
a_{n+N_\ell} = u_{1N_\ell} \sim a_n \cdot (a_{n-1} + 2^k)^2
\]

\[
a_{n+N_\ell+1} = u_{21} \sim a_n \cdot (a_{n-1} + 2^k)^2
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
a_{n+N_\ell+1} = u_{2N_\ell} \sim a_n \cdot (a_{n-1} + 2^k)^2
\]

We denote \( a_0 \) by 1, and express all the \( u_{ij} \) by \( u \) and the \( a_i \):

\[
\begin{align*}
u_0 & \sim a_{n+1} \cdot u^2, \\
u_1 & \sim a_{n+1} \cdot a_0^2 \sim a_{n+1} \cdot (a_{n+1})^2 \cdot u^2, \\
u_2 & \sim a_{n+1} \cdot (a_{n+1} + 1)^2 \cdot u^2, \\
\vdots & \quad \vdots \quad \vdots \\
u_{N_\ell} & \sim a_{n+1} \cdot (a_{n+1} - 1)^2 \cdot u^2, \\
u_{N_\ell+1} & \sim a_{n+1} \cdot (a_{n+1} + 2^k)^2 \cdot u^2, \\
u_{N_\ell+2} & \sim a_{n+1} \cdot (a_{n+1} + 3^k)^2 \cdot u^2, \\
\vdots & \quad \vdots \quad \vdots \\
u_{N_\ell+N_\ell+1} & \sim a_{n+1} \cdot (a_{n+1} + N_\ell)^2 \cdot u^2.
\end{align*}
\]

More precisely, we define the \( u_{ij} \) as follows: Let \( u_{ij} = u \) be a prime, \( u > a_n \cdot a_n^2 \), \( u > t_i \), and

\[
u > 2^{(n+1)}.
\]

Furthermore, for \( j \geq 2 \):

\[
M_j = u^{(j-1)} \cdot (\gamma_{n+1})^{(j-1)} \cdots (\gamma_{n+N_\ell})^{(j-1)} = u^{(j-1)} \leq M_j \cdot b^{(j-1)}
\]

(\( M_j \) also depends on \( i \), of course).

If we put \( M_j = u_j \), then (18) holds also for \( j = 1 \).

Later we show the \( u_{ij} \) can be chosen as primes, i.e. there is at least one prime between \( M_j \) and \( M_j \cdot b^{(j-1)} \) (for \( j \geq 2 \)). Taking this for granted, we have also \( (u_{ij}, t_i) = 1 \), since \( u_{ij} > u > t_i \).

We check now

\[
\frac{a_{n+i+1}}{a_{n+N_\ell}} \leq \frac{u_{n+i+1}}{u_{n+N_\ell}} \leq a_{n+i+1}, \quad \text{for} \quad j = 1, 2, \ldots, N_\ell - 1.
\]

Indeed:

\[
\frac{u_{n+i+1}}{u_{n+N_\ell}} \geq \frac{M_{n+1}}{M_{n+N_\ell}} = \frac{u^{(j-1)} \cdot (\gamma_{n+1})^{(j-1)} \cdots (\gamma_{n+N_\ell})^{(j-1)} \cdot (\theta_{n+1})^{(j-1)}}{\theta_{n+N_\ell}^{(j-1)}} = \frac{\gamma_{n+1}}{\theta_{n+1}} \geq a_{n+i+1}.
\]

We need now a suitable \( N_\ell \) for which

\[
\frac{a_{n+i+1} \cdot N_\ell}{(a_{n+N_\ell})^2} \geq \gamma_{n+i+1}
\]

also holds.

\[
\frac{a_{n+i+1} \cdot N_\ell}{(a_{n+N_\ell})^2} = \frac{t_i \cdot u_{n+i} \cdots u_{n+N_\ell}}{(u_{n+N_\ell})^2} = \frac{t_i \cdot M_{n+1} \cdot M_{n+2} \cdots M_{n+N_\ell}}{M_{n+N_\ell} \cdot \theta_{n+N_\ell}} = \frac{t_i \cdot u_{n+i} \cdots u_{n+N_\ell}}{(u_{n+N_\ell})^2} \geq \frac{t_i \cdot u_{n+i} \cdots u_{n+N_\ell}}{\gamma_{n+i+1} \cdots \gamma_{n+N_\ell}} \geq \frac{t_i \cdot \gamma_{n+i+1}}{\theta_{n+N_\ell}} = R.
\]

Since \( \liminf (\gamma_1, \ldots, \gamma_n) = 0 \), we have \( \liminf (\gamma_{n+i+1}, \ldots, \gamma_{n+N_\ell}) = 0 \) as well, and thus we can find an \( N_\ell \) for which

\[
\gamma_{n+i+1} \cdots \gamma_{n+N_\ell} \leq \frac{t_i}{2^{(n+1)}} \leq \frac{t_i}{a_n \cdot a_n^2} = \gamma_{n+i+1}
\]

holds.

Hence

\[
R \geq \gamma_{n+i+1} \geq a_{n+i+1}.
\]

which completes the verification of (7) for all \( k \).
We have to show that the $a_q$ may be chosen as primes, i.e. the interval $[M_j, M_{j+1}]$ always contains a prime $(j \geq 2)$. As e.g. $[n, n + n^\gamma]$ always contains a prime, it suffices to show that

$$M_{j+1} - M_j > M_j^\gamma,$$

i.e.

$$M_j > 2^{a_{j+1}}.$$ 

(19)

By (17)

$$M_j = u > 2^{a_{j+1}}$$

thus (19) holds for $j = 1$.

Assuming the validity of (19) for $j$, we prove it for $j+1$. Using $\gamma_k > a_k > 2^{-s}$, we obtain:

$$M_{j+1} = M_j - 2^{a_{j+1} - 2} > 2^{a_{j+1}} > 2^{a_{j+1} + 1}.$$

Finally, if $f$ satisfies (8), we conclude that $f = 0$, exactly in the same way as we did in the proof of Theorem 3/II.

Remarks. 1. The most interesting case of Theorem 2/II is when $\liminf a_q = 1$, and this shows that (6) is not a serious restriction. We also observe, that (6) can also be replaced by considerably weaker conditions.

2. Theorems 3/II and 2/II do not imply each other, though Theorem 3/II is "almost" a corollary of Theorem 2/II (condition (6) is the only obstacle in this direction).

3. Analysing the proof of Theorem 2/II we may observe that the condition (7) for the set $A$ can also be replaced by stronger prescriptions. E.g. if $\liminf a_q = d$ ($d < 1$ by Theorem 2/I), and $L > 1$, $M > L/d$ are arbitrary finite or infinite values, then we can find an $A$ satisfying

$$\liminf \frac{a_{k+1}}{a_k} = L \quad \text{and} \quad \limsup \frac{a_{k+1}}{a_k} = M.$$

Similar assertions hold also for Theorem 3/II.

4. The sets $A$ constructed in the proofs of Theorems 2/II and 3/II have also the following property:

If the $\sum_{i=1}^n f(a_i)$ sums are bounded, then $f$ is bounded.

The proof is straightforward.

6. Finally we mention the following generalization of Theorem 2/II, which shows that the $q_k$ elements of the characterizing set $A$ may be chosen arbitrarily large, even with respect to $a_{k-1}$, for almost all $k$.

**Theorem 5.** By $K = \{ k, \ldots , \}$ we shall denote a suitable subsequence of the natural numbers.