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On the values of p-adic L-functions at positive integers

by

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1. Introduction. Leopoldt and Kubota, [4], have defined p-adic L-functions so that when n is a positive integer,

$$L_p(1-n,\chi) = -B^n_{\chi_n}(1-\chi_n(p)p^{n-1})/n.$$

In this equation, χ is a primitive Dirichlet character, ω is the character defined by $\omega(a) = \lim_{k\to\infty} a^{p^k}$ when $p\neq 2$ and ω is the non-principal character mod 4 if $p = 2$. χ_n is the primitive character induced by the product $\chi(a)\omega^{-n}(a)$ and B^n_φ is the nth generalized Bernoulli number defined for the character φ with modulus f by

$$\sum_{a=1}^{f}\frac{\varphi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B^n_\varphi\frac{t^n}{n!}.$$

In order to consider p-adic and complex L-functions simultaneously we will use an isomorphism, σ, between the algebraic closure of the rational numbers in Ω_p and the algebraic closure of the rational numbers within the complex numbers. A Dirichlet character can then be considered as a mapping from the integers into either Ω_p or the complex numbers. If we have a character φ in Ω_p, the corresponding character in C determined by σ will also be denoted by φ. Thus the function $L(s,\varphi)$ corresponding to $L_p(s,\varphi)$ is determined by our choice of σ.

It is possible to avoid this dependence on σ by stating our results in terms of sums over the positive integers in a residue class instead of L-functions. In fact, several of our theorems are specifically about such sums.

However, our purpose is to discuss the values of L-functions, so we shall consider σ as fixed throughout this paper and use σ to identify p-adic algebraic numbers with complex algebraic numbers. We shall write $x = y$ when $x\in\Omega_p$, $y\in C$ and $y = \sigma(x)$.

When we compare the formula

$$L(1-n,\varphi) = -B^n_\varphi/n$$

with the formula for $L_p(1-n,\chi)$, we see that when n is a positive integer:

$$L_p(1-n,\chi) = \big(1-\chi_n(p)p^{n-1}\big)L(1-n,\chi_n). \tag{0}$$

If we wish to consider (0) for other integral values of n we must take into account the fact that if the values of the L-functions are transcendental then it is not meaningful to ask that (0) be true. On the other hand, from the approximation theorem, we know that for each n there is a sequence of algebraic numbers which converges to the left side of (0) in Ω_p and to the right side of (0) in C.

Leopoldt ([2], [5]) has found an interesting middle ground in this situation:

When $n = 0$ the formula

$$-\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{a=1}^{f}\bar{\chi}(a)\log(1-\zeta^{-a})$$

gives each side of (0), depending whether we use p-adic log in Ω_p or complex log in C. In [2], Iwasawa asks: are there "similar expressions for the values of $L_p(n;\chi)$, $n\geqslant 2$?"

We shall give an affirmative answer to the following question:

(1) Are there natural infinite series of algebraic numbers and their logarithms which converge to $L_p(1-n,\chi)$ in Ω_p and to

$$\big(1-\chi_n(p)p^{n-1}\big)L(1-n,\chi_n)$$

in C when n is a rational integer?

The series to be given are natural in the sense that they arise from the term-by-term computations of standard integral representations of L-functions and the derivatives of the log gamma function.

For the case $n = 0$ we obtain a new formula for $L_p(1,\chi)$.

Question (1) can be generalized to:

(2) For which pairs of integers n, k, with $k\geqslant 0$, are there natural infinite series of algebraic numbers and their logarithms which converge to

$$D_s^{(k)}L_p(1-s,\chi)|_{s=n}$$

in Ω_p and to

$$D_s^{(k)}\big(1-\chi_n(p)p^{s-1}\big)L(1-s,\chi_n)\big|_{s=n}$$

in C?

We shall show a solution to (2) in the case $n=1$, $k=1$.

Two separate proofs of our formulas will be given. The first approach is to express certain values of complex L-functions in terms of the derivatives of the log gamma function and then use the inverse factorial series for the derivatives of log gamma. No integrals are used in this discussion.

Then we use G_p, a p-adic analog of log gamma, [1], to obtain the corresponding results for p-adic L-functions.

The second approach is to express the values of complex L-functions as definite integrals and then compute the integral. On the p-adic side we use Leopoldt's p-adic Γ-transform, [5], [2]. Since $\Gamma_H(0)$ is comparable to $H(1)-H(0)$, we have a computation which is similar to the integral approach for complex numbers.

In order to be able to work with p-adic inverse factorial series, we have a result which equates inverse factorial series with certain power series in $1/x$. The two series need not have precisely the same domain of convergence, so our results provide a technique for (Krasner) analytic continuation.

We will use Q, Q_p, Z, Z_p, C and Ω_p for, respectively, the field of rational numbers, the p-adic completion of Q, the ring of rational integers, the p-adic completion of Z in Q_p, the field of complex numbers and the completion of the algebraic closure of Q_p. B_n will be the nth Bernoulli number defined by $te^t/(e^t-1)$. ν will be the p-adic valuation on Ω_p with $\nu(p)=1$ and $|\ |_p$ will be the absolute value on Ω_p with $|p|_p = p^{-1}$.

2. Inverse factorial series. An inverse factorial series is a series of the form

$$\sum_{n=0}^{\infty} b_n\begin{bmatrix} n\\ x\end{bmatrix}, \tag{3}$$

where

$$\begin{bmatrix} n\\ x\end{bmatrix} = \frac{n!}{x(x+1)\ldots(x+n)}.$$

Elementary estimates show that if $x\notin Z_p$ and $\nu(x+m)\leqslant a$, for some $a>0$ and for all $m\in Z$, then

$$\nu\begin{bmatrix} n\\ x\end{bmatrix} \geqslant \frac{n}{p^{a+1}(p-1)} - O(\log n).$$

As a consequence, (3) is Krasner analytic on $\Omega_p - Z_p$ if $\nu(b_n) = o(n)$.

In the early eighteenth century, Stirling used a pair of transformations which relate an inverse factorial series to a power series in $1/x$. These transformations correspond to writing the Laplace transform of an analytic function as either an integral from 0 to ∞ or from 0 to 1. In the complex field, the corresponding series are not always both convergent. However, in Ω_p we obtain equal series. The following two theorems show how Stirling's transformations behave in Ω_p.

THEOREM 1. *If*

$$F(x) = \sum_{n=0}^{\infty} b_n\begin{bmatrix} n\\ x\end{bmatrix}\quad \textit{for}\quad |x|_p\geqslant R>1,$$

$$f(t) = \sum_{n=0}^{\infty} b_n(1-t)^n \quad \textit{and} \quad f(e^{-u}) = \sum_{n=0}^{\infty} a_n u^n/n!,$$

then when $|x|_p \geqslant R$,

$$F(x) = \sum_{n=0}^{\infty} a_n/x^{n+1}.$$

Proof.

Case 1: $F(x) = \sum_{n=0}^{m} b_n \begin{bmatrix} n \\ x \end{bmatrix}$.

It is sufficient to consider $F(x) = \begin{bmatrix} n \\ x \end{bmatrix}$. Direct computation, together with

$$\begin{bmatrix} n \\ x \end{bmatrix} = \sum_{r=0}^{n} \frac{\binom{n}{r}(-1)^r}{x+r}$$

establishes the result.

Case 2: $F(x) = \sum_{n=0}^{\infty} b_n \begin{bmatrix} n \\ x \end{bmatrix}$.

We observe that $F(1/x)$ is a uniform limit of holomorphic function on $|x|_p \leqslant 1/R$, so $F(1/x)$ is holomorphic at zero and we have constants A_n so

$$F(x) = \sum_{n=0}^{\infty} A_n/x^{n+1} \quad \text{when} \quad |x|_p \geqslant R.$$

The following simple result will be useful:

LEMMA. *If* $g(x) = \sum_{n=0}^{\infty} c_n x^n$, $g_m(x) = \sum_{n=0}^{\infty} c_{m,n} x^n$ *and* $\lim_{m\to\infty} g_m(x) = g(x)$ *uniformly on a disc of positive radius about* 0, *then*

$$\lim_{m\to\infty} c_{m,n} = c_n \quad \textit{for} \quad n = 0, 1, \ldots$$

We define

$$F_m(x) = \sum_{n=0}^{m} b_n \begin{bmatrix} n \\ x \end{bmatrix},$$

$$f_m(t) = \sum_{n=0}^{m} b_n(1-t)^n \quad \text{and} \quad f_m(e^{-u}) = \sum_{n=0}^{\infty} a_{m,n} u^n/n!.$$

From case 1 we have

$$F_m(x) = \sum_{n=0}^{\infty} a_{m,n}/x^{n+1}.$$

We can apply the lemma to $F_m(1/x)$ and $F(1/x)$ to obtain

$$A_n = \lim_{m\to\infty} a_{m,n}.$$

We can also apply the lemma to $f_m(e^{-u})$ and $f(e^{-u})$ to obtain

$$a_n = \lim_{m\to\infty} a_{m,n}.$$

Thus $A_n = a_n$ and Theorem 1 is proved.

We make the observation that Theorem 1 remains true if $|x|_p \geqslant R > 1$ is replaced by $|x|_p > R \geqslant 1$ and $|x|_p \geqslant R$ is replaced by $|x|_p > R$.

THEOREM 2. *If*

$$F(x) = \sum_{n=0}^{\infty} a_n/x^{n+1} \quad \textit{for} \quad |x|_p > R \geqslant 1,$$

$$g(u) = \sum_{n=0}^{\infty} a_n u^n/n! \quad \textit{and} \quad g(-\log t) = \sum_{n=0}^{\infty} b_n(1-t)^n,$$

then,

$$F(x) = \sum_{n=0}^{\infty} b_n \begin{bmatrix} n \\ x \end{bmatrix} \quad \textit{when} \quad |x|_p > R.$$

Proof. It is easy to show $\sum_{n=0}^{\infty} b_n \begin{bmatrix} n \\ x \end{bmatrix}$ converges for $|x|_p > R$, so Theorem 2 follows from the remark after the proof of Theorem 1.

As a consequence of Theorem 2 we have immediate proofs in Ω_p of several classical formulas:

(i) $$\frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{1}{n} \begin{bmatrix} n \\ x \end{bmatrix} \quad \text{for} \quad x \in \Omega_p - Z_p.$$

(ii) $$\frac{1}{x-a} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{a(a+1)\ldots(a+n-1)}{x(x+1)\ldots(x+n)}$$

for at least $|x|_p > |a|_p$.

(iii) $$G_p''(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \begin{bmatrix} n \\ x \end{bmatrix} \quad \text{for} \quad x \in \Omega_p - Z_p.$$

G_p is a p-adic analog of the log gamma function and is described in [1] and in the next section.

If we define

$$\zeta_p(r) = \frac{1}{r-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{m=0 \\ m\equiv 1 (\mathrm{mod}\, p)}}^{p^k-1} \frac{1}{m^{r-1}},$$

then from the formula for $G_p''(1/p)$ we obtain

$$\zeta_p(2) = \frac{1}{p^2}\sum_{n=0}^{\infty}\frac{1}{n+1}\begin{bmatrix} n \\ 1/p \end{bmatrix}.$$

Putting $p = 1$ in the inverse factorial series we obtain $\zeta(2)$. There are similar formulas for $\zeta_p(r)$ with $r \in Z$ (also see [1]).

3. $L(r, \chi)$ **and** $L_p(r, \chi)$. We will now derive inverse factorial series for $L(r, \chi)$ and $L_p(r, \chi)$ when r is an integer $\geqslant 2$. First, we work in C.

PROPOSITION 1. *If* $\operatorname{Re} x > 0$,

$$D^{(2)}\log\Gamma(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}\begin{bmatrix} n \\ x \end{bmatrix}.$$

This known result can be deduced without the use of integrals by observing that

$$F(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}\begin{bmatrix} n \\ x \end{bmatrix}$$

satisfies:

(i) F is analytic for $\operatorname{Re} x > 0$ (see [6], p. 279),
(ii) $F(x+1) = F(x) - 1/x^2$,
(iii) $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} D^{(2)}\log\Gamma(x) = 0$ for real x.

For the higher derivatives we have

PROPOSITION 2. *If* $r \geqslant 2$, $\operatorname{Re} x > 0$ *and we define* $c_{r,n}$ *by*

$$\sum_{n=0}^{\infty} c_{r,n}(1-t)^n = \frac{(\log t)^{r-1}}{t-1},$$

then

$$D^{(r)}\log\Gamma(x) = \sum_{n=0}^{\infty} c_{r,n}\begin{bmatrix} n \\ x \end{bmatrix}.$$

Proof. This result is just an application to Proposition 1 of the formula

$$D_x\sum_{n=0}^{\infty} b_n\begin{bmatrix} n \\ x \end{bmatrix} = \sum_{n=0}^{\infty} a_n\begin{bmatrix} n \\ x \end{bmatrix}$$

where

$$a_n = -\left(\frac{b_0}{n} + \frac{b_1}{n-1} + \ldots + \frac{b_{n-1}}{1}\right)$$

([6], p. 297).

We begin to relate $D^{(r)}\log\Gamma(x)$ to $L(r, \chi)$ with

PROPOSITION 3. *If* r, a, f *are integers with* $r \geqslant 2$ *and* $0 < a \leqslant f$, *and we let*

$$A_{r,n} = (-1)^r c_{r,n}/(r-1)!$$

then

$$\sum_{\substack{m=1 \\ m\equiv a (\operatorname{mod} f)}}^{\infty}\frac{1}{m^r} = \frac{1}{f^r}\sum_{n=0}^{\infty} A_{r,n}\begin{bmatrix} n \\ a/f \end{bmatrix}.$$

Proof. We combine Proposition 2 with the well known formula

$$D^{(r)}\log\Gamma(x) = (-1)^r(r-1)!\sum_{n=0}^{\infty}(x+n)^{-r}$$

and evaluate at $x = a/f$.

We can now write an inverse factorial series for $L(r, \chi)$.

PROPOSITION 4. *If* r *is an integer* $\geqslant 2$, χ *is a Dirichlet character* mod f *and* c *is an arbitrary positive integer, then*

$$L(r, \chi) = (cf)^{-r}\sum_{a=1}^{cf}\chi(a)\sum_{n=0}^{\infty} A_{r,n}\begin{bmatrix} n \\ a/cf \end{bmatrix}.$$

We obtain the last part of (1) from a

COROLLARY. *If* χ *is a primitive character and* χ_n *is the primitive character* mod f *induced by* $\chi(a)\omega^{-n}(a)$, *then if* n *is a negative integer*,

$$\left(1-\chi_n(p)p^{n-1}\right)L(1-n, \chi_n) = (pf)^{n-1}\sum_{a=1}^{pf}{}^{*}\chi_n(a)\sum_{m=0}^{\infty} A_{1-n,m}\begin{bmatrix} m \\ a/pf \end{bmatrix}.$$

The * in $\sum^*$ indicates that the index of summation omits values which are multiples of p.

Now we proceed to obtain a series for $L_p(r, \chi)$. The function G_p, [1], plays the role of log gamma in Ω_p. For convenience, we list some of its properties. The proofs and other results are in [1].

(i) DEFINITION. For $x \in \Omega_p - Z_p$,

$$G_p(x) = \frac{1}{2} - x + \lim_{k\to\infty}\frac{1}{p^k}\sum_{n=0}^{p^k-1}(x+n)\log(x+n).$$

The p-adic log is defined by the usual power series when $|x-1|_p < 1$ and by setting $\log p = 0$ and using the functional equations for log when $|x-1|_p \geqslant 1$ and $x \neq 0$. There is a complete discussion of this idea in [2].

(ii) (Stirling's series): If $|x|_p > 1$,

$$G_p(x) = (x-\tfrac{1}{2})\log x - x + \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)x^n}.$$

(iii) G_p is locally holomorphic on $\Omega_p - Z_p$ and $G_p''(x)$ is Krasner analytic on $\Omega_p - Z_p$.

The next three results are similar to Propositions 2, 3 and 4.

PROPOSITION 5. *For $r \geqslant 2$ and $x \in \Omega_p - Z_p$,*

$$D^{(r)}G_p(x) = \sum_{n=0}^{\infty} c_{r,n}\begin{bmatrix} n \\ x \end{bmatrix}.$$

Proof. We apply Theorem 2, with $R = 1$, to the formula

$$D^{(r)}G_p(x) = (-1)^r \sum_{n=0}^{\infty} B_n(n+1)\ldots(n+r-2)/x^{n+r-1},$$

and then continue to $\Omega_p - Z_p$ by analytic extension.

PROPOSITION 6. *If r, a, f are integers, $r \geqslant 2$, $0 < a < f$ and $v(a/f) < 0$, then*

$$\frac{1}{r-1}\lim_{k\to\infty}\frac{1}{fp^k}\sum_{\substack{n=0 \\ n\equiv a (\operatorname{mod} f)}}^{fp^k-1}\frac{1}{n^{r-1}} = \frac{1}{f^r}\sum_{n=0}^{\infty} A_{r,n}\begin{bmatrix} n \\ a/f \end{bmatrix}.$$

Proof. We combine Proposition 5 with the formula

$$D^{(r)}G_p(x) = (-1)^r(r-2)!\lim_{k\to\infty}\frac{1}{p^k}\sum_{n=0}^{p^k-1}(x+n)^{1-r}$$

and evaluate at $x = a/f$.

THEOREM 3. *If r is an integer $\geqslant 2$, χ is a primitive Dirichlet character and φ_r is the primitive character* mod f *induced by the product $\chi(a)\omega^{r-1}(a)$, then*

$$L_p(r, \chi) = (pf)^{-r}\sum_{a=1}^{pf}{}^{*}\varphi_r(a)\sum_{m=0}^{\infty} A_{r,m}\begin{bmatrix} m \\ a/pf \end{bmatrix}.$$

Proof. By definition,

$$L_p(r, \chi) = \frac{1}{r-1}\lim_{k\to 0}\frac{1}{fp^k}\sum_{n=1}^{fp^k}{}^{*}\chi(n)\omega^{r-1}(n)/n^{r-1}.$$

When $(n, p) = 1$, $\varphi_r(n) = \chi(n)\omega^{r-1}(n)$. Application of Proposition 6 completes the proof.

If we write $r = 1-n$ in Theorem 3 and compare the result with the Corollary to Proposition 4 we have a solution to (1) when n is a negative integer. Stated precisely, we have

THEOREM 4. *If n is a negative integer, χ is a primitive character and χ_n has conductor f, then*

$$(pf)^{n-1}\sum_{a=1}^{pf}{}^{*}\chi_n(a)\sum_{m=0}^{\infty} A_{1-n,m}\begin{bmatrix} m \\ a/pf \end{bmatrix}$$

is equal to $L_p(1-n, \chi)$ in Ω_p and to $(1-\chi_n(p)p^{n-1})L(1-n, \chi_n)$ in C.

4. $L_p(1, \chi)$ and $L(1, \chi)$. We will use inverse factorial series to find a new formula for $L_p(1, \chi)$. This formula satisfies (1) with $n = 0$. We will also be able to estimate $L_p(1, \chi) - (-M_\chi^k(\log))$ and show that both the Leopoldt χ-mean and the expression $\sum^{*}\chi(n)/n$ are natural parts of $L_p(1, \chi)$ and $(1-\chi(p)p^{-1})L(1, \chi)$.

In C we have the following formula for $L(1, \chi)$.

PROPOSITION 7. *If χ is a non-principal character* mod f *and c is a positive integer, then*

$$L(1, \chi) = \frac{-1}{cf}\sum_{a=1}^{cf}\chi(a)\log a + \frac{1}{cf}\sum_{a=1}^{cf}\chi(a)\sum_{n=0}^{\infty} A_n\begin{bmatrix} n \\ a/cf \end{bmatrix},$$

where A_n is defined by

$$\frac{1}{1-t} + \frac{1}{\log t} = \sum_{n=0}^{\infty} A_n(1-t)^n.$$

Proof. Let $\psi(x) = D^{(1)}\log\Gamma(x)$. From the relation

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right)$$

we easily obtain

$$cfL(1, \chi) = -\sum_{a=1}^{cf}\chi(a)\psi(a/cf).$$

We combine this formula with ([7], p. 286)

$$\psi(x) = \log x - \sum_{n=0}^{\infty} A_n\begin{bmatrix} n \\ x \end{bmatrix}$$

to establish Proposition 7.

The following corollary will be useful.

COROLLARY.

$$(1-\chi(p)p^{-1})L(1,\chi) = -\frac{1}{pf}\sum_{a=1}^{pf}{}^{*}\chi(a)\log a+\frac{1}{pf}\sum_{a=1}^{pf}{}^{*}\chi(a)\sum_{n=0}^{\infty}A_n\begin{bmatrix}n\\ a/pf\end{bmatrix}.$$

In Ω_p there are similar results:

PROPOSITION 8. *For* $|x|_p>1$ *and* $\psi_p(x) = D^{(1)}G_p(x)$

$$\psi_p(x) = \log x-\sum_{n=0}^{\infty}A_n\begin{bmatrix}n\\ x\end{bmatrix}.$$

Proof. This is an application of Theorem 2 to

$$\psi_p(x) = \log x-\sum_{n=1}^{\infty}B_n/nx^n \quad \text{for} \quad |x|_p>1.$$

PROPOSITION 9. *If a, f are positive integers, $a<f$ and $\nu(a/f)<0$, then*

$$-\lim_{k\to\infty}\frac{1}{fp^k}\sum_{\substack{n=1\\ n\equiv a\,(\mathrm{mod}\,f)}}^{fp^k-1}\log n = -\frac{\log a}{f}+\frac{1}{f}\sum_{n=0}^{\infty}A_n\begin{bmatrix}n\\ a/f\end{bmatrix}.$$

Proof. The result follows immediately from the definition of ψ_p. Now we can proceed to $L_p(1,\chi)$.

THEOREM 5. *If χ is a non-principal character* mod f, *then*

$$L_p(1,\chi) = -\frac{1}{pf}\sum_{a=1}^{pf}{}^{*}\chi(a)\log a+\frac{1}{pf}\sum_{a=1}^{pf}{}^{*}\chi(a)\sum_{n=0}^{\infty}A_n\begin{bmatrix}n\\ a/pf\end{bmatrix}.$$

Proof. From [4], we have

$$L_p(1,\chi) = -\lim_{k\to\infty}\frac{1}{fp^k}\sum_{n=1}^{fp^k}{}^{*}\chi(n)\log n.$$

We combine this formula with Proposition 9 to obtain our result.

Now we have a solution of (2) with $k=n=0$.

THEOREM 6. *If χ is a primitive, non-principal character with conductor f, then*

$$L_p(1,\chi) \quad \text{and} \quad (1-\chi_0(p)p^{-1})L(1,\chi_0)$$

are given by the series in Theorem 5.

Proof. Since χ is primitive we have $\chi=\chi_0$. We need only observe that we have the same formula in Theorem 5 and the Corollary to Proposition 7. ■

In the proof of Theorem 5 we can replace pf by $\bar{f}p^k$ where $\bar{f}=[f,q]$ and $q=p$ if $p>2$ and $q=4$ if $p=2$. We then obtain

THEOREM 7. *If χ is a non-principal character* mod f,

$$L_p(1,\chi)-\left(\frac{-1}{\bar{f}p^k}\sum_{a=1}^{\bar{f}p^k}{}^{*}\chi(a)\log a\right) = \frac{1}{\bar{f}p^k}\sum_{a=1}^{\bar{f}p^k}{}^{*}\chi(a)\sum_{n=0}^{\infty}A_n\begin{bmatrix}n\\ a/\bar{f}p^k\end{bmatrix}.$$

The expression on the left is $L_p(1,\chi)-(-M_\chi^k(\log))$ and the expression on the right is a well-behaved series which converges rapidly. Our next result allows us to use Theorem 7 to calculate "how fast" the partial χ-means approach $L_p(1,\chi)$.

PROPOSITION 10. *If A_n is defined by*

$$\frac{1}{1-t}+\frac{1}{\log t} = \sum_{n=0}^{\infty}A_n(1-t)^n,$$

then

$$A_0 = 1/2 \quad \text{and} \quad \nu(A_n)\geqslant\frac{-n-1}{p-1}.$$

Proof. Use induction on n.

If we extract the term for $n=0$ in Theorem 7 we have

$$L_p(1,\chi) = -M_\chi^k(\log)+\frac{1}{2}\sum_{a=1}^{\bar{f}p^k}{}^{*}\frac{\chi(a)}{a}+\frac{1}{\bar{f}p^k}\sum_{a=1}^{\bar{f}p^k}{}^{*}\chi(a)\sum_{n=1}^{\infty}A_n\begin{bmatrix}n\\ a/\bar{f}p^k\end{bmatrix}. \tag{4}$$

When we look at Proposition 7 we see that with $c=\bar{f}p^k/f$, (4) is $(1-\chi(p)p^{-1})L(1,\chi)$. Thus both the partial (Leopoldt) χ-means and the partial Dirichlet sums are natural parts of $L_p(1,\chi)$ and $(1-\chi(p)p^{-1})L(1,\chi)$.

5. $L_p'(0,\chi)$ and $L'(0,\chi)$. The technique we have used can also demonstrate (2) when $n=k=1$. In this case, we want to show

$$L_p'(0,\chi) \quad \text{and} \quad (1-\chi_1(p))L'(0,\chi_1)-\chi_1(p)(\log p)B_{\chi_1}^1$$

have the same series.

A formula by Lerch ([8], p. 271), for the derivative of the Hurwitz zeta function can be used to obtain the following formula for $L'(0,\varphi)$.

PROPOSITION 11. *If φ is a Dirichlet character* mod f *and c is a positive integer, then*

$$L'(0,\varphi) = B_\varphi^1\log(cf)+\sum_{a=1}^{cf}\varphi(a)\big(-(1/2)\log(2\pi)+\log\Gamma(a/cf)\big).$$

The useful corollary is

COROLLARY. *If φ is a Dirichlet character* mod f *then*

$$(1-\varphi(p))L'(0,\varphi)-\varphi(p)(\log p)B_\varphi^1$$
$$= B_\varphi^1(1-\varphi(p))\log(pf)+\sum_{a=1}^{pf}{}^{*}\varphi(a)\big(-(1/2)\log(2\pi)+\log\Gamma(a/pf)\big).$$

In Ω_p we have a formula for $L_p'(0,\chi)$.

THEOREM 8. *If χ is a primitive character and χ_1 is the primitive character* mod f *induced by* $\chi(a)\omega^{-1}(a)$, *then*

$$L_p'(0,\chi) = B^1_{\chi_1}(1-\chi_1(p))\log(pf) + \sum_{a=1}^{pf}{}^{*} \chi_1(a) G_p(a/pf).$$

Proof. This is a straightforward calculation using the definitions of $G_p(x)$ and $L_p(s,\chi)$.

We are now prepared to show (2) is satisfied when $n = k = 1$.

THEOREM 9. *If χ is a primitive character and χ_1 is the primitive character* mod f *induced by* $\chi(a)\omega^{-1}(a)$, *then*

$$L_p'(0,\chi) \quad \textit{and} \quad (1-\chi_1(p))L'(0,\chi_1) - \chi_1(p)(\log p)B^1_{\chi_1}$$

can be given by the same natural series.

Proof. First we observe that

$$G_p(x) \quad \text{and} \quad -(1/2)\log(2\pi) + \log\Gamma(x)$$

are both given by

$$(x-\tfrac{1}{2})\log x - x + \sum_{n=0}^{\infty} c_n \begin{bmatrix} n \\ x \end{bmatrix}$$

where c_n is defined by

$$\sum_{n=0}^{\infty} c_n(1-t)^n = \frac{1}{\log t}\left(\frac{1}{2} - \frac{1}{1-t} - \frac{1}{\log t}\right).$$

The formula for $G_p(x)$ is derived from the Stirling series and Theorem 2; the formula for $\log\Gamma(x)$ is in [7], p. 285.

When we substitute the series for G_p into Theorem 8 and the Corollary to Proposition 11 we obtain the formula which establishes Theorem 9.

6. Proofs with Leopoldt's Γ-transform. We demonstrated our results in C without using definite integrals so that we could show a certain parallel structure in Ω_p. However, if we use definite integrals in C we can still parallel the approach in Ω_p by using antiderivatives and Leopoldt's Γ-transform ([5], [2]).

We will indicate the computations for $L(r,\chi)$ and $L_p(r,\chi)$ with r an integer $\geqslant 2$. As before, φ will be a character mod f. Using the well known identity:

$$n^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-nx}\,dx, \qquad \operatorname{Re}s > 0,$$

we can deduce

$$(1-\varphi(r)p^{-r})L(r,\varphi) = \sum_{a=1}^{pf}{}^{*}\varphi(a)\frac{1}{(r-1)!}\int_0^{\infty}\frac{x^{r-1}e^{-ax}}{1-e^{-pfx}}\,dx.$$

Let $t = e^{-x}$ and

$$A_r(u) = \frac{1}{(r-1)!}\cdot\frac{(-\log u)^{r-1}}{1-u} = \sum_{n=0}^{\infty} A_{r,n}(1-u)^n.$$

Then,

$$(5) \qquad (1-\varphi(r)p^{-r})L(r,\varphi) = (pf)^{1-r}\int_0^1 A_r(t^{pf})\sum_{a=1}^{pf}{}^{*}\varphi(a)t^{a-1}\,dt.$$

We will see the analog of equation (5) in the calculation of $L_p(r,\chi)$.

In order to complete this proof of the Corollary to Proposition 4 we can substitute $z = t^{pf}$. The new integrand does not converge uniformly for z in $[0,1]$ because $0 < a/pf < 1$. However, if we let $x = a/pf$ the integrand converges uniformly in z when $\operatorname{Re}x > 2$. Thus we can integrate term by term when $\operatorname{Re}x > 2$ and then use an analytic extension argument to show that term by term integration is valid in (5). We finally obtain

$$(1-\varphi(r)p^{-r})L(r,\varphi) = (pf)^{-r}\sum_{a=1}^{pf}{}^{*}\varphi(a)\sum_{n=0}^{\infty}A_{r,n}\begin{bmatrix} n \\ a/pf \end{bmatrix}.$$

In Ω_p, we will again prove Theorem 3. Let χ be a primitive character and φ_r be the primitive character mod f induced by $\chi(a)\omega^{r-1}(a)$.

For r fixed, $r \geqslant 2$, we define

$$F(t) = \sum_{a=1}^{pf}{}^{*}\varphi_r(a)\sum_{n=0}^{\infty}A_{r,n}\sum_{j=0}^{n}\frac{\binom{n}{j}(-1)^j t^{pjf+a}}{pjf+a}.$$

F has the following properties:

(i) The series for $F(t)$ converges uniformly for t with

$$\nu(t-1) \geqslant \frac{1}{p(p-1)} + \varepsilon \quad \text{for any} \quad \varepsilon > 0.$$

(ii) If $\nu(t-1) > \dfrac{1}{p(p-1)}$,

$$F'(t) = A_r(t^{pf})\sum_{a=1}^{pf}{}^{*}\varphi_r(a)t^{a-1}.$$

(iii) F has a Γ-transform.

We will use Leopoldt's Γ-transform as it is defined in [5]: The Γ-transform is a continuous linear mapping from a Banach space of functions which are holomorphic at 1 and whose series converges sufficiently rapidly into the space of continuous p-adic functions on $2Z_p$. The Γ-transform is determined by the values

$$\Gamma(u^n)(s) = \begin{cases} \langle n\rangle^s & \text{if} \quad (n,p) = 1, \\ 0 & \text{if} \quad p|n \end{cases}$$

where $\langle n\rangle = n\omega^{-1}(n)$.

The Γ-transform is also discussed in [2], but its domain is translated to functions holomorphic at zero.

Since for certain functions g, $\Gamma_g(0)$ behaves like $g(1)-g(0)$ (note that $g(0)$ may not be defined), equation (5) indicates that we should calculate $\Gamma_F(0)$.

From the formula $\Gamma(u^n)(s) = \langle n\rangle^s$ when $(n,p) = 1$, we obtain

$$\Gamma_F(0) = \sum_{a=1}^{pf}{}^{*} \varphi_r(a) \sum_{n=0}^{\infty} A_{r,n} \sum_{j=0}^{\infty} \frac{\binom{n}{j}(-1)^j}{jpf+a} \tag{6}$$

$$= \frac{1}{pf} \sum_{a=1}^{pf}{}^{*} \varphi_r(a) \sum_{n=0}^{\infty} A_{r,n} \begin{bmatrix} n \\ a/pf \end{bmatrix}.$$

For a second way to calculate $\Gamma_F(0)$ we use the formula

$$\Gamma_{Dg}(s) = s\Gamma_g(s)$$

where $Dg(t) = t(\log t)g'(t)$.

We have

$$DF(t) = t(\log t)A_r(t^{pf}) \sum_{a=1}^{pf}{}^{*} \varphi_r(a)t^{a-1}.$$

In order to calculate $\Gamma_{DF}(s)$ we use

$$\Gamma_g(s) = \lim_{m\uparrow s} \frac{d^m}{dx^m} g(e^x)_{x=0}\Big|.$$

The symbol $\uparrow$ indicates $m\to\infty$ in an archimedean sense, $m\to s$ in Ω_p and $(p-1)|m$ $(2|m$ if $p = 2)$.

$$\Gamma_{DF}(s) = \lim_{m\uparrow s} \frac{d^m}{dx^m} \frac{(pf)^{r-1}(-1)^r}{(r-1)!} \sum_{a=1}^{pf}{}^{*} \frac{\varphi_r(a)xe^{ax}}{e^{pfx}-1}.$$

Using the definition of Bernoulli numbers and the formula, [4],

$$(m-r+1)L_p(r-m,\chi) = -B_{\varphi_r}^{m-r+1}(1-\varphi_r(p)p^{m-r})$$

we have

$$\Gamma_{DF}(s) = (pf)^{r-1}(-1)^{r-1}(s-r+1)\ldots(s)L_p(r-s,\chi)/(r-1)!.$$

Then,

$$\Gamma_F(s) = (pf)^{r-1}(r-1-s)\ldots(1-s)L_p(r-s,\chi)/(r-1)!$$

and $\Gamma_F(0) = (pf)^{r-1}L_p(r,\chi)$.

Comparing this last result with (6), we have Theorem 3.

There are similar p-adic arguments for $L_p(1,\chi)$ and $L'_p(0,\chi)$ in which we also use the generating function for the coefficients of the inverse factorial series.

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