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On a paper of Baker and Schinzel

by

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1. Introduction. Let D be an integer, positive or negative but not a square. It was shown by Baker and Schinzel [1] that every genus of primitive binary quadratic forms of discriminant D represents a positive integer, prime to D, and less than $C(\varepsilon)|D|^{3/8+\varepsilon}$, where $\varepsilon > 0$ and $C(\varepsilon)$ depends only on ε; and they conjectured that in fact the bound could be replaced by $C(\varepsilon)|D|^{\varepsilon}$. The object of this paper is to prove the following sharpening of their result.

THEOREM. *Every genus of primitive binary quadratic forms of discriminant D represents a positive integer, prime to D, and less than $C(\varepsilon)|D|^{1/4+\varepsilon}$.*

Our theorem may be used in place of the result of Baker and Schinzel, in the work of Möller [4], thereby improving his results somewhat. In particular it follows from our theorem that the smallest prime which splits in $Q(\sqrt{-d})$, but does not ramify, is less than $C(\varepsilon)|D|^{1/4+\varepsilon}$, where D is the discriminant of the field, and so all the 'numeri idonei' of Euler are less than $(2C(\varepsilon))^{1/\varepsilon}$, for any ε with $0 < \varepsilon < 1/4$. Thus if $C(\varepsilon)$ were effectively computable then all the numeri idonei could in principle be explicitly determined. But unfortunately, as in [1], the constant $C(\varepsilon)$ is ineffective; this is due to the use of Siegel's lower bound for $L(1, \chi)$ (see [5]).

Our improved bound results from the use of estimates of Burgess [2] in place of those of Burgess [3] as employed by Baker and Schinzel [1]. Apart from this our argument follows that of [1] closely, but there are two further differences; the first involves the employment of a modified path of integration and the second involves the replacement of a finite sum by the corresponding L-function. The latter change is not in fact essential but we believe that it leads to a more elegant exposition.

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2. Bounds for L-functions. In place of Lemma 2 of [1] we prove the following.

LEMMA. *Let χ be a real character with modulus k and conductor f. Let r be an integer $\geqslant 2$ and let $\varepsilon > 0$. Then we have*

$$L(s,\chi) \ll |s| f^{1/(4r)} k^{\varepsilon}$$

where $\mathrm{Re}(s) = 1-1/(r+1)$, *and the implied constant depends on r and ε only.*

The bound in this lemma could be improved by an appeal to Burgess [3]. However the improvement would not result in a sharpening of our main theorem.

We prove the lemma first in the case when χ is primitive, with modulus k. If k is 1, $L(s,\chi) = \zeta(s)$ and the lemma follows from a basic estimate in the theory of the Riemann zeta-function (Titchmarsh [6] (2.12.2)). We now assume χ is non-principal. We define

$$S(u) = \sum_{n \leqslant u} \chi(n).$$

By Burgess [2], Corollary to Theorem 1, we have

$$(1) \qquad S(u) \ll u^{1-1/(r+1)} k^{1/(4r)+\varepsilon}.$$

Also, by the Polyá–Vinogradov inequality we have

$$(2) \qquad S(u) \ll k^{1/2+\varepsilon}.$$

In the identity

$$\sum_{n=1}^{\infty} \chi(n) n^{-s} = s \int_{1}^{\infty} S(u) u^{-s-1} du$$

we use the estimate (1) for $1 \leqslant u \leqslant k$, and (2) otherwise. Then

$$L(s,\chi) \ll |s| \Big\{ \int_{1}^{k} u^{-1/(r+1)-\sigma} k^{1/(4r)+\varepsilon} du + \int_{k}^{\infty} u^{-1-\sigma} k^{1/2+\varepsilon} du \Big\}.$$

Since $\mathrm{Re}(s) = \sigma = 1-1/(r+1)$, and r is fixed, we deduce

$$L(s,\chi) \ll |s| \{ k^{1/(4r)+\varepsilon} \log k + k^{1/(r+1)-1/2+\varepsilon} \} \ll |s| k^{1/(4r)+2\varepsilon}.$$

This proves the lemma for primitive characters.

Now let χ have conductor f, and be induced by χ_f, a primitive character with modulus f. Then χ_f is also real, so that

$$L(s,\chi_f) \ll |s| f^{1/(4r)+\varepsilon}$$

for $\mathrm{Re}(s) = 1-1/(r+1)$. However

$$L(s,\chi) = L(s,\chi_f) \prod (1-\chi_f(p) p^{-s}),$$

where the product is over prime factors p of k which do not divide f. Then

$$\prod (1-\chi_f(p) p^{-s}) \ll \prod_{p|k} 2 \ll k^{\varepsilon},$$

whence

$$L(s,\chi) \ll |s| f^{1/(4r)+\varepsilon} k^{\varepsilon}.$$

This completes the proof of the lemma.

3. Proof of the theorem. Following Baker and Schinzel [1], § 3, we define $D = e^2 D_0$, where D_0 is a fundamental discriminant, and let χ_0 be the principal character mod D. We denote by U the set of generic characters for D, and by T the set of generic characters for D_0. T is a non-empty subset of U. If S is any subset of U we write

$$\chi_S = \chi_0 \prod_{\chi \in S} \chi.$$

Now consider a genus determined by the values $\varepsilon_\chi = \pm 1$ to be taken by χ in U. The values ε_χ satisfy

$$(3) \qquad \prod_{\chi \in T} \varepsilon_\chi = 1.$$

On the assumption that no positive integer less than or equal to x and prime to D is represented by the genus, we have, as in [1],

$$(4) \qquad \sum_{S \subseteq U} E(x,S) \prod_{\chi \in S} \varepsilon_\chi = 0,$$

where

$$E(x,S) = \sum_{n \leqslant x} (1-n/x)^3 \sum_{n=uv} \chi_S(u) \chi_{T \dot{-} S}(v).$$

We now apply Lemma 1 of [1] which results in

$$E(x,S) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{6x^s}{s(s+1)(s+2)(s+3)} L(s,\chi_S) L(s,\chi_{T \dot{-} S}) ds.$$

We denote the integrand by $F(x,S,s)$ for brevity. $F(x,S,s)$ has a pole at $s=1$ when $S = \emptyset$ or $T \dot{-} S = \emptyset$, in which cases χ_S or $\chi_{T \dot{-} S}$ are principal, but not for other S. Hence if we move the line of integration to $\mathrm{Re}(s) = \sigma = 1-1/(r+1)$, where $r \geqslant 2$ is an integer, we obtain

$$E(x,S) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(x,S,s) ds + R(S).$$

$R(S)$ is zero unless $S = \emptyset$ or $T \dot{-} S = \emptyset$ in which cases

$$R(S) = \frac{x}{4} \frac{\varphi(|D|)}{|D|} L(1,\chi_T).$$

By the lemma of § 2 we have, for $\mathrm{Re}(s) = 1-1/(r+1)$

$$F(x, S, s) \ll x^{1-1/(r+1)}(f_S f_{T\dot{-}S})^{1/(4r)}|D|^{\varepsilon}|s|^{-2},$$

where f_S denotes the conductor of χ_S. However $f_S f_{T\dot{-}S}$ divides D and we conclude that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(x, S, s)\,ds \ll x^{1-1/(r+1)}|D|^{1/(4r)+\varepsilon}.$$

When we substitute the resulting estimate for $E(x, S)$ in (4) we obtain

$$R(\emptyset)+R(T)\prod_{\chi\in T}\varepsilon_\chi \ll x^{1-1/(r+1)}|D|^{1/(4r)+\varepsilon}\sum_{S\subset U}1.$$

Now the number of generic characters is at most two more than the number $\nu(|D|)$ of distinct prime factors of D. Hence,

$$\sum_{S\subset U}1 \ll 2^{\nu(|D|)} \ll |D|^{\varepsilon}.$$

We now apply (3) together with the fact that

$$R(\emptyset) = R(T) = \frac{x}{4}\frac{\varphi(|D|)}{|D|}L(1, \chi_T).$$

These yield

$$\frac{x}{2}\frac{\varphi(|D|)}{|D|}L(1, \chi_T) \ll x^{1-1/(r+1)}|D|^{1/(4r)+2\varepsilon}.$$

We also have

$$|D|/\varphi(|D|) \ll |D|^{\varepsilon},$$

and by Siegel's estimate [5]

$$L(1, \chi_T) \gg |D|^{-\varepsilon}.$$

A simple rearrangement now yields

$$x^{1/(r+1)} \ll |D|^{1/(4r)+4\varepsilon},$$

or

$$x \ll |D|^{1/4+1/(4r)+4(r+1)\varepsilon}.$$

Finally let ε' be given, take $r = 1+[1/(2\varepsilon')]$ and $\varepsilon = \varepsilon'/(8r+8)$. Then

$$x \ll |D|^{1/4+\varepsilon'}$$

and the theorem is proved.

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