On a paper of Baker and Schinzel

by

D. R. HEATH-BROWN (Cambridge)

1. Introduction. Let \( D \) be an integer, positive or negative but not a square. It was shown by Baker and Schinzel [1] that every genus of primitive binary quadratic forms of discriminant \( D \) represents a positive integer, prime to \( D \), and less than \( O(\varepsilon |D|^{3/4 + \varepsilon}) \), where \( \varepsilon > 0 \) and \( O(\varepsilon) \) depends only on \( \varepsilon \); and they conjectured that in fact the bound could be replaced by \( O(\varepsilon |D|^\varepsilon) \). The object of this paper is to prove the following sharpening of their result.

**Theorem.** Every genus of primitive binary quadratic forms of discriminant \( D \) represents a positive integer, prime to \( D \), and less than \( O(\varepsilon |D|^{3/4 + \varepsilon}) \).

Our theorem may be used in place of the result of Baker and Schinzel, in the work of Möller [4], thereby improving his results somewhat. In particular it follows from our theorem that the smallest prime which splits in \( \mathbb{Q}(\sqrt{-d}) \), but does not ramify, is less than \( O(\varepsilon |D|^{3/4 + \varepsilon}) \), where \( D \) is the discriminant of the field, and so all the “numer idonei” of Euler are less than \( [2O(\varepsilon)]^{2\varepsilon} \), for any \( \varepsilon \) with \( 0 < \varepsilon < 1/4 \). Thus if \( O(\varepsilon) \) were effectively computable then all the numeri idonei could in principle be explicitly determined. But unfortunately, as in [1], the constant \( C(\varepsilon) \) is ineffective; this is due to the use of Siegel’s lower bound for \( L(1, \varepsilon) \) (see [5]).

Our improved bound results from the use of estimates of Burgess [2] in place of those of Burgess [3] as employed by Baker and Schinzel [1]. Apart from this our argument follows that of [1] closely, but there are two further differences; the first involves the employment of a modified path of integration and the second involves the replacement of a finite sum by the corresponding \( L \)-function. The latter change is not in fact essential but we believe that it leads to a more elegant exposition.

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2. **Bounds for \( L \)-functions.** In place of Lemma 2 of [1] we prove the following.
Lemma. Let \( \chi \) be a real character with modulus \( k \) and conductor \( f \). Let \( r \) be an integer \( \geq 2 \) and let \( s > 0 \). Then we have
\[
L(s, \chi) \asymp |s|^\frac{f(4r)}{2} k^s
\]
where \( \Re(s) = 1 - \frac{1}{r+1} \), and the implied constant depends on \( r \) and \( s \) only.

The bound in this lemma could be improved by an appeal to Burgess [3]. However the improvement would not result in a sharpening of our main theorem.

We prove the lemma first in the case when \( \chi \) is primitive, with modulus \( k \). If \( k \) is 1, \( L(s, \chi) = \zeta(s) \) and the lemma follows from a basic estimate in the theory of the Riemann zeta-function (Titchmarsh [6] (2.12.3)). We now assume \( \chi \) is non-principal. We define
\[
S(u) = \sum_{n \leq u} \chi(n).
\]

By Burgess [2], Corollary to Theorem 1, we have
\[
S(u) \ll u^{1-(r+1)/2} \log^{(r+1)/2} u.
\]

Also, by the Polya–Vinogradov inequality we have
\[
S(u) \ll k^{1/2+\epsilon}.
\]

In the identity
\[
\sum_{n=1}^\infty \chi(n) u^{-s} = s \int_1^\infty S(u) u^{-s-1} \, du
\]
we use the estimate (1) for \( 1 \leq u \leq k \), and (2) otherwise. Then
\[
L(s, \chi) \ll |s| \left( \frac{1}{2} \int_1^k u^{1-(r+1)/2} \log^{(r+1)/2} u \, du + \int_k^\infty u^{-1-(r+1)/2} \log^{(r+1)/2} u \, du \right).
\]

Since \( \Re(s) = \sigma = 1 - \frac{1}{r+1} \), and \( r \) is fixed, we deduce
\[
L(s, \chi) \ll |s|^{(r+1)/2} \log k + k^{1/2+1/r+1} \ll |s|^{f(4r)/2+\epsilon}.
\]

This proves the lemma for primitive characters.

Now let \( \chi \) have conductor \( f \), and be induced by \( \chi_f \), a primitive character with modulus \( f \). Then \( \chi_f \) is also real, so that
\[
L(s, \chi) \ll |s|^{f(4r)/2+\epsilon}
\]
for \( \Re(s) = 1 - \frac{1}{r+1} \). However
\[
L(s, \chi) = L(s, \chi_f) \prod_p (1 - \chi_f(p) p^{-s}),
\]
where the product is over prime factors \( p \) of \( k \) which do not divide \( f \). Then
\[
\prod_p (1 - \chi_f(p) p^{-s}) \ll \prod_{p \leq k} 2 \ll k^\epsilon,
\]
whence
\[
L(s, \chi) \ll |s|^{f(4r)/2+\epsilon} k^s.
\]

This completes the proof of the lemma.

3. Proof of the theorem. Following Baker and Schinzel [1], \( \S \), we define \( D = e^r D_0 \), where \( D_0 \) is a fundamental discriminant, and let \( \chi_0 \) be the principal character mod \( D \). We denote by \( U \) the set of generic characters for \( D \), and by \( T \) the set of generic characters for \( D_0 \). \( T \) is a non-empty subset of \( U \). If \( S \) is any subset of \( U \) we write
\[
\chi_S = \chi_0 \prod_{\chi \in S} \chi.
\]

Now consider a genus determined by the values \( \epsilon \) to be \( \pm 1 \) to be taken
by \( \chi \) in \( U \). The values \( \epsilon \) satisfy
\[
\prod_{\chi \in S} \epsilon = 1.
\]

On the assumption that no positive integer less than or equal to \( x \) and prime to \( D \) is represented by the genus, we have, as in [1],
\[
\sum_{S \subseteq U} B(x, S) \prod_{\chi \in S} \epsilon = 0,
\]
where
\[
B(x, S) = \sum_{\chi \in S} (1-n|\varepsilon|)^2 \sum_{n \leq x} \chi_S(n) \chi_{T^{x=6}}(n).
\]

We now apply Lemma 1 of [1] which results in
\[
B(x, S) = \frac{1}{2 \pi i} \int_{\sigma=1}^{ \sigma = \text{Re}(s) + \epsilon} \frac{\theta(s) \chi(S) L(s, \chi)}{s(s+1)(s+2)(s+3)} \, ds.
\]

We denote the integrand by \( F(s, S, \varepsilon) \) for brevity. \( F(s, S, \varepsilon) \) has a pole at \( s = 1 \) when \( S = \emptyset \) or \( T = S = \emptyset \), in which cases \( \chi_S \) or \( \chi_{T^{x=6}} \) are principal, but not for other \( S \). Hence if we move the line of integration to \( \Re(s) = 1 - \frac{1}{r+1} \), where \( r \geq 2 \) is an integer, we obtain
\[
B(x, S) = \frac{1}{2 \pi i} \int_{\sigma=1}^{ \sigma = \text{Re}(s) + \epsilon} \frac{F(s, S, \varepsilon) \, ds + R(S)}{s(s+1)(s+2)(s+3)}.
\]

\( R(S) \) is zero unless \( S = \emptyset \) or \( T = S = \emptyset \) in which cases
\[
R(S) = \frac{\sigma^2}{4} \frac{\varphi(|D|)}{|D|} L(1, \chi_D).
\]
By the lemma of §2 we have, for \( \Re(s) = 1 - \frac{1}{(r+1)} \)

\[
F(x, S, s) \ll x^{1-\frac{1}{(r+1)}} (f_{S}f_{S^{-1}})^{1/4(s-r)}|D|^{s/2} |s|^{-2},
\]

where \( f_{S} \) denotes the conductor of \( \chi_{S} \). However \( f_{S}f_{S^{-2}} \) divides \( D \) and we conclude that

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(x, S, s) \, ds \ll x^{1-\frac{1}{(r+1)}} |D|^{1/4(s-r)+2}. \]

When we substitute the resulting estimate for \( F(x, S) \) in (4) we obtain

\[
E(\Theta) + E(T) \prod_{2 \leq r \leq T} \zeta(s) \ll \frac{x^{-\frac{1}{2(r-1)}} |D|^{1/4(r)+1}}{s} \sum_{S \in \mathcal{U}} 1.
\]

Now the number of generic characters is at most two more than the number \( \nu(|D|) \) of distinct prime factors of \( D \). Hence,

\[
\sum_{S \in \mathcal{U}} 1 \ll 2^{\nu(|D|)} \ll |D|^{1/2}.
\]

We now apply (3) together with the fact that

\[
E(\Theta) = E(T) = \frac{\varphi(|D|)}{|D|} L(1, \chi_{D}).
\]

These yield

\[
\frac{\varphi(|D|)}{2 |D|} L(1, \chi_{D}) \ll \frac{x^{-\frac{1}{2(r-1)} |D|^{1/4(r)+2}}}{s}.
\]

We also have

\[
|D|/\varphi(|D|) \ll |D|^{1/2},
\]

and by Siegel's estimate [5]

\[
L(1, \chi_{D}) \gg |D|^{-\varepsilon},
\]

A simple rearrangement now yields

\[
x^{\frac{1}{2(r+1)}} \ll |D|^{1/4(r)+1/2},
\]

or

\[
x \ll |D|^{1/4+1/4(r)+1/2}.
\]

Finally let \( \alpha' \) be given, take \( r = 1 + [1/(2\alpha')] \) and \( \alpha = \alpha'/(8r+8) \). Then

\[
x \ll |D|^{1/4+\alpha'}
\]

and the theorem is proved.