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On the order of Dedekind Zeta-functions near the line $\sigma = 1$

by

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1. Denote by K an algebraic number field, by n and Δ the degree and the discriminant of the field K, respectively, and by $\zeta_K(s)$, $s = \sigma + it$, the Dedekind Zeta-function (see [4]).

Basing on some estimates of A. V. Sokolovskiĭ connected with the application of I. M. Vinogradov's methods to the theory of Dedekind Zeta-functions (see [7] and compare [10]), refined in [1] with respect to the constants of the field, we shall prove the following

THEOREM. *If* $1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1$, $t \geqslant e$, *then*

$$|\zeta_K(\sigma+it)| \leqslant e^{cn^8|\Delta|^2} t^{6\cdot 10^2 n^2 (n(1-\sigma))^{3/2}} \log^{2/3} t \tag{1.1}$$

where c is a positive pure numerical constant.

About possible application of (1.1), see [3]. For the Riemann Zeta-function $\zeta(s)$ the strongest estimate of the form (1.1) is due to H. E. Richert (see [6] and compare [9]).

2. The Dedekind Zeta-function of an algebraic number field K is defined by the series

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}, \quad s = \sigma + it,$$

in the open half-plane $\sigma > 1$, the sum being taken over all ideals of K (see [4]). The function $\zeta_K(s)$ can be continued analytically to a meromorphic function with a simple pole at $s = 1$.

It is known that

$$\zeta_K(s) = \sum_{C} \Big(\sum_{\mathfrak{a} \in C} (N\mathfrak{a})^{-s} \Big), \tag{2.1}$$

where the inner sum is taken over all ideals of K, belonging to an ideal class C (see [4], p. 57) and the outer sum is taken over all h ideal classes.

It is also known that

$$f_C(s) = \sum_{\mathfrak{a}\in C} (N\mathfrak{a})^{-s} = N(\mathfrak{a}')^s \sum_{a\equiv 0 \pmod{\mathfrak{a}'}} |N(a)|^{-s} \tag{2.2}$$

where the last sum is taken over a complete system of pairwise not associated algebraic integers belonging to any ideal $\mathfrak{a}' \in C^{-1}$ (see [4], p. 58).

If $a_1, a_2, \ldots, a_n$ form a basis for $\mathfrak{a}'$, then every element a of $\mathfrak{a}'$ can be uniquely represented in the form $a = a_1\alpha_1 + \ldots + a_n\alpha_n$ where a_i, $i = 1, 2, \ldots, n$ are rational integers.

Every element $a \in K$ can be considered as an element of the n-dimensional real space $\boldsymbol{R}^n$:

$$x(a) = (x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}, z_1, \ldots, z_{r_2})$$

where $n = r_1 + 2r_2$ (see [2], II, § 3).

Denote by $\mathfrak{M}$ the n-dimensional lattice in $\boldsymbol{R}^n$ formed of images of algebraic integers $a \in K$ divisible by $\mathfrak{a}'$ and denote by V the fundamental domain of K (see [2], p. 352). Then the summation in (2.2) reduces to the summation over rational integers $a_1, \ldots, a_n$ such that $x(a) \in \mathfrak{M} \cap V$.

Denoting

$$N(a) = N(x(a)) = f(a_1, \ldots, a_n)$$

we can write

$$f_C(s) = N(\mathfrak{a}')^s \sum_{a_1} \ldots \sum_{\substack{a_n \\ x(a)\in\mathfrak{M}\cap V}} |f(a_1, \ldots, a_n)|^{-s}. \tag{2.3}$$

We denote by

$$a^{(i)} = a_1\alpha_1^{(i)} + \ldots + a_n\alpha_n^{(i)}, \quad i = 1, 2, \ldots, n,$$

the conjugates of a so that $a^{(i)}$ are real if $1 \leqslant i \leqslant r_1$ and $a^{(i)}$ are complex conjugates of $a^{(i+r_2)}$ if $r_1+1 \leqslant i \leqslant r_2$ hence $Na = a^{(1)} \ldots a^{(n)}$.

Denote further by $\overline{V}$ the set which we get multiplying the elements of V by images of all roots of unity belonging to K. Then the series (2.3) can be writen as follows

$$f_C(s) = \frac{1}{m} N(\mathfrak{a}')^s \sum_{a_1} \ldots \sum_{\substack{a_n \\ x(a)\in\mathfrak{M}\cap\overline{V}}} \frac{e^{-it\log|f(a_1,\ldots,a_n)|}}{|f(a_1, \ldots, a_n)|^\sigma} \tag{2.4}$$

where m denotes the number of roots of unity contained in K (see [7], p. 323).

In the following we shall always assume that

$$N\mathfrak{a}' \leqslant |\Delta|^{1/2}, \tag{2.5}$$

since in each ideal class C there exists at least one ideal satisfying (2.5) (see [4], p. 42).

3. The proof of (1.1) will rest on the following lemmas:

LEMMA 1 (see [1], Lemma 4, and compare [7], Lemma 1). *If $\alpha_1, \ldots, \alpha_n$ form a basis for a given ideal $\mathfrak{a}$ with $N\mathfrak{a} \leqslant |\Delta|^{1/2}$ and $K_{\mathfrak{a}}^X$ denotes the set of all systems of real numbers $(u_1, \ldots, u_n)$ with*

$$\max_{1\leqslant i\leqslant n} |u_i| \leqslant X$$

where $u_1x(\alpha_1) + \ldots + u_nx(\alpha_n)$ are elements of $\boldsymbol{R}^n$ which belong to $\overline{V}$, then for any system of real numbers $(u_1, \ldots, u_n) \in K_{\mathfrak{a}}^{2X} \setminus K_{\mathfrak{a}}^X$ we have the inequality

$$A_1X < |u_1\alpha_1^{(i)} + \ldots + u_n\alpha_n^{(i)}| < A_2X, \quad i = 1, \ldots, n, \tag{3.1}$$

where

$$A_1 = \exp(-4n^3|\Delta|^2), \quad A_2 = 2|\Delta|n^{n+1}.$$

LEMMA 2 (see [1], Lemma 12, and compare [7], Lemmas 5 and 8). *Denote*

$$F(a_1, \ldots, a_n) = -\frac{t}{2\pi}\log|N(x(a))| = -\frac{t}{2\pi}\log|f(a_1, \ldots, a_n)|.$$

If

$$m_1 = \left[11\frac{n+2}{n}\frac{\log t}{\log X}\right],$$

$$1 < X < A_1^{-1}t^{(n+1)/n}, \quad A_1 = \exp(-4n^3|\Delta|^2), \tag{3.2}$$

$$t > \exp(2\cdot 10^5 n^7|\Delta|^2)$$

then

$$|S_i| = \Bigg|\sum_{\substack{a<a_i\leqslant a' \\ (a_1,\ldots,a_n)\in K_{\mathfrak{a}}^{2X}\setminus K_{\mathfrak{a}}^X}} e^{2\pi iF(a_1,\ldots,a_n)}\Bigg| \leqslant A_3X^{1-\frac{1}{A_4m_1^2}} \tag{3.3}$$

where

$$A_3 = \exp(4\cdot 10^3 n^2|\Delta|^2), \quad A_4 = 10^6 n^4.$$

Remark. Lemma 2 is a slightly completed version of [1], Lemma 12, to that effect that in the present version of the lemma under consideration all the numerical constants are counted out explicitly.

LEMMA 3 (see [1], Lemma 13, and compare [7], Lemma 9). *In the region $\sigma \geqslant 1 - 1/(n+1)$, $t > 1$, $s = \sigma + it$, of the complex plane, we have the estimate*

$$\Big|\zeta_K(s) - \sum_{1\leqslant m<t^{n+1}} F(m)m^{-s}\Big| \leqslant \exp(c_1n^4|\Delta|^2) \tag{3.4}$$

where c_1 is a pure numerical positive constant.

LEMMA 4 (see [8], p. 185). *In the region* $-1 \leqslant \sigma \leqslant 2$, $-\infty < t < +\infty$, *of the complex plane, we have the estimate*

$$|(s-1)\zeta_K(s)| \leqslant A_5(|t|+1)^{A_6}; \quad s = \sigma + it, \tag{3.5}$$

where

$$A_5 = c_2^n |\Delta|^{3/2}, \quad A_6 = \tfrac{3}{2}n+2,$$

and c_2 *is a pure numerical constant.*

4. Proof of the theorem. Denote

$$K_{i,t_0} = K_{\mathfrak{a}_j'}^{2^i t_0}$$

(see Lemma 1) where $t_0 = \exp(\log^{2/3} t)$, i integer, $i \geqslant 0$. Owing to (2.1), (2.4) and (3.4) we have in the region $\sigma \geqslant 1-1/(n+1)$, $t > 1$, the estimate (compare [7], p. 330)

$$|\zeta_K(s)| \leqslant \exp(c_1 n^4 |\Delta|^2) + |\Delta|^{1/2} \sum_{j=1}^{h} \Big| \sum_{(a_1,\ldots,a_n)\in K_{0,t_0}} |Nx(a)|^{-s} \Big| + \tag{4.1}$$
$$+ |\Delta|^{1/2} \sum_{j=1}^{h} \sum_{i\geqslant 1} \Big| \sum_{\substack{(a_1,\ldots,a_n)\in K_{i,t_0}\setminus K_{i-1,t_0} \\ 0<|Nx(a)|<N\mathfrak{a}_j' t^{n+1}}} |Nx(a)|^{-s} \Big|$$

where $\mathfrak{a}_j'$ are ideals belonging to the inverse classes C_j^{-1} and are chosen in such a way that $N\mathfrak{a}_j' \leqslant |\Delta|^{1/2}$ (see (2.5)) and h is the class-number. For h we use the simplest estimate

$$h \leqslant |\Delta|^{(n^2+1)/2} \tag{4.2}$$

mentioned in [5], p. 160.

We estimate the second term of (4.1) as follows.

Denoting $K_m = K_{\mathfrak{a}_j'}^{2^m}$, $m = 0, 1, 2, \ldots$ (see Lemma 1) we have

$$\Big| \sum_{(a_1,\ldots,a_n)\in K_{0,t_0}} |N(x(a))|^{-s} \Big| \leqslant \sum_{(a_1,\ldots,a_n)\in K_0} |N(x(a))|^{-\sigma} + \tag{4.3}$$
$$+ \sum_{m=1}^{m_0} \sum_{(a_1,\ldots,a_n)\in K_m\setminus K_{m-1}} |N(x(a))|^{-\sigma}$$

where $m_0 = \left[\dfrac{\log^{2/3} t}{\log 2}\right]$, $t \geqslant e$, since $t_0 = \exp(\log^{2/3} t)$.

Estimating the first term on the right of (4.3) trivially, and the second term by the use of Lemma 1, we simply get the inequality

$$|\Delta|^{1/2} \sum_{j=1}^{h} \Big| \sum_{(a_1,\ldots,a_n)\in K_{0,t_0}} |N(x(a))|^{-s} \Big| \leqslant 2e^{6n^4|\Delta|^2} t^{\frac{n(1-\sigma)}{\log 2 \log^{1/3} t}} \log^{2/3} t \tag{4.4}$$

valid in the region $\sigma \geqslant 1-1/(n+1)$, $t \geqslant e$, of the complex plane.

5. We shall now estimate the last term on the right of the inequality (4.1). It is easy to notice that

$$\sum_{\substack{(a_1,\ldots,a_n)\in K_{i,t_0}\setminus K_{i-1,t} \\ 0<|N(x(a))|\leqslant N\mathfrak{a}_j' t^{n+1}}} |N(x(a))|^{-s} = \sum_{k=1}^{2n} \sum_{\substack{(a_1,\ldots,a_n)\in B_{k,i} \\ 0<|N(x(a))|\leqslant N\mathfrak{a}_j' t^{n+1}}} |N(x(a))|^{-s} = \sum_{k=1}^{2n} S_{k,i} \tag{5.1}$$

where $B_{k,i}$ is the set of points $(a_1, \ldots, a_n) \in K_{i,t_0} \setminus K_{i-1,t_0}$ such that

$$\begin{array}{ll} 2^{i-1}t_0 < a_k \leqslant 2^i t_0, & -2^i t_0 \leqslant a_{k-n} < -2^{i-1} t_0 \\ |a_1| \leqslant a_k & |a_1| \leqslant -a_{k-n} \\ \ldots\ldots & \ldots\ldots\ldots \\ |a_n| \leqslant a_k & |a_n| \leqslant -a_{k-n} \\ (k = 1, 2, \ldots, n) & (k = n+1, \ldots, 2n). \end{array}$$

Hence

$$|S_{1i}| = \Big| \sum_{\substack{(a_1,\ldots,a_n)\in B_{1,i} \\ 0<|N(x(a))|<N\mathfrak{a}_j' t^{n+1}}} |N(x(a))|^{-s}$$
$$= \Big| \sum_{2^{i-1}t_0 \leqslant a_1 < 2^i t_0} \sum_{|a_2|\leqslant a_1} \cdots \sum_{|a_n|\leqslant a_1} |N(x(a))|^{-s} \Big|.$$

Following [7], p. 331, we get by partial summation and by the use of Lemma 1 and Lemma 2, with

$$X_0 = t_0 = \exp(\log^{2/3} t) < X = 2^i t_0 < |\Delta|^{1/2n} t^{(n+1)/n},$$

in the region

$$1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1, \quad t \geqslant \exp(2\cdot 10^5 n^7 |\Delta|^2),$$

the estimate

$$|S_{1i}| \leqslant A_7 (2^i t_0)^{n(1-\sigma) - A_8 \frac{\log^2(2^i t_0)}{\log^2 t}} = A_7 (2^i t_0)^{n(1-\sigma) - A_8 \left(i \frac{\log 2}{\log t} + \frac{1}{\log^{1/3} t}\right)^2} \tag{5.2}$$

where

$$A_7 = \exp(c_3 n^4 |\Delta|^2), \quad A_8 = (10^6 n^4)^{-1}. \tag{5.3}$$

Denoting for the sake of brevity

$$a = a(t) = \frac{\log 2}{\log t}, \quad \beta = \beta(t) = \frac{1}{\log^{1/3} t}, \tag{5.4}$$

we get

$$|S_{1i}| \leqslant A_7 (2^i t_0)^{n(1-\sigma) - A_8(ai+\beta)^2} \tag{5.5}$$
$$\leqslant A_7 t_0^{n(1-\sigma) - A_8\beta^2} 2^{n(1-\sigma)i - A_8 a^2 i^3 - 2A_8 a\beta i^2} 2^{-A_8\beta^2 i}.$$

We estimate the second factor of (5.5) as follows. Consider the polynomial

$$\varphi(x) = n(1-\sigma)x - A_8\alpha^2x^3 - 2A_8\alpha\beta x^2.$$

This polynomial has a maximum at the point

$$x_0 = \frac{-4A_8\alpha\beta + \sqrt{16A_8^2\alpha^2\beta^2 + 12A_8\alpha^2 n(1-\sigma)}}{6A_8\alpha^2}.$$

It is easy to realize that the above maximum of $\varphi(x)$ is absolute for $x \geqslant 0$.

From the obvious inequality

$$0 \leqslant x_0 \leqslant \frac{1}{\sqrt{3A_8}} \frac{\sqrt{n(1-\sigma)}}{\alpha},$$

we get

$$\varphi(x_0) \leqslant \frac{1}{\sqrt{3A_8}} \frac{\big(n(1-\sigma)\big)^{3/2}}{\alpha}.$$

Therefore, owing to (5.4), we have

$$(5.6) \qquad 2^{n(1-\sigma)i - A_8\alpha^2 i^3 - 2A_8\alpha\beta i^2} \leqslant t^{\frac{1}{\sqrt{3A_8}}(n(1-\sigma))^{3/2}}.$$

Owing to (5.4) and the definition of t_0, we have for the first factor of (5.5)

$$(5.7) \qquad t_0^{n(1-\sigma) - A_8\beta^2} \leqslant t^{\frac{n(1-\sigma)}{\log^{1/3} t}}.$$

Therefore, from (5.5)–(5.7) it follows

$$(5.8) \qquad |S_{1i}| \leqslant A_7 t^{\frac{n(1-\sigma)}{\log^{1/3} t}} t^{\frac{1}{\sqrt{3A_8}}(n(1-\sigma))^{3/2}} 2^{-A_8\beta^2 i}.$$

For the remaining $|S_{k,i}|$, $k = 2, \ldots, 2n$ we get similar estimates. Hence from (5.1), (5.3), (5.4) and (5.8) it follows

$$(5.9) \qquad |\Delta|^{1/2} \sum_{j=1}^{h} \sum_{i \geqslant 1} \Bigg| \sum_{\substack{(a_1, \ldots, a_n) \in K_{i,t_0} \setminus K_{i-1,t_0} \\ 0 < |N(x(a))| < N\alpha_j' t^{n+1}}} \big|N\big(x(\alpha)\big)\big|^{-s} \Bigg|$$
$$\leqslant c_4 e^{c_5 n^4 |\Delta|^2} t^{\frac{n(1-\sigma)}{\log^{1/3} t}} t^{\frac{10^3}{\sqrt{3}} n^2 (n(1-\sigma))^{3/2}} \log^{2/3} t.$$

Owing to (4.1), (4.4), (5.9) we get in the region

$$(5.10) \qquad 1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1, \quad t \geqslant e^{2\cdot 10^5 n^7 |\Delta|^2}$$

the estimate

$$(5.11) \qquad |\zeta_K(\sigma+it)| \leqslant \exp(c_1 n^4 |\Delta|^2) +$$
$$+ 2\exp(6n^4|\Delta|^2) t^{\frac{n(1-\sigma)}{\log 2 \log^{1/3} t}} \log^{2/3} t +$$
$$+ c_4 \exp(c_5 n^4 |\Delta|^2) t^{\frac{n(1-\sigma)}{\log^{1/3} t}} t^{\frac{10^3}{\sqrt{3}} n^2 (n(1-\sigma))^{3/2}} \log^{2/3} t.$$

We split the region (5.10) into the two following regions:

$$D_1\colon\ 1 - \frac{1}{n\log^{2/3} t} \leqslant \sigma \leqslant 1, \ t \geqslant \exp(2\cdot 10^5 n^7 |\Delta|^2),$$

$$D_2\colon\ 1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1 - \frac{1}{n\log^{2/3} t}, \ t \geqslant \exp(2\cdot 10^5 n^7 |\Delta|^2).$$

From (5.11) it follows that in D_1

$$(5.12) \qquad |\zeta_K(\sigma+it)| \leqslant \exp(c_6 n^4 |\Delta|^2) \log^{2/3} t.$$

Analogously in D_2 we get

$$(5.13) \qquad |\zeta_K(\sigma+it)| \leqslant \exp(c_7 n^4 |\Delta|^2) t^{6\cdot 10^2 n^2 (n(1-\sigma))^{3/2}} \log^{2/3} t.$$

Hence from (5.12), (5.13) we get in the region

$$1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1, \quad t \geqslant \exp(2\cdot 10^5 n^7 |\Delta|^2)$$

the estimate

$$(5.14) \qquad |\zeta_K(\sigma+it)| \leqslant \exp(c_8 n^4 |\Delta|^2) t^{6\cdot 10^2 n^2 (n(1-\sigma))^{3/2}} \log^{2/3} t.$$

Owing to Lemma 4, for

$$1 - \frac{1}{n+1} \leqslant \sigma \leqslant 1, \quad e \leqslant t \leqslant \exp(2\cdot 10^5 n^7 |\Delta|^2),$$

we have simply

$$(5.15) \qquad |\zeta_K(\sigma+it)| \leqslant \exp(c_9 n^8 |\Delta|^2).$$

From (5.12) and (5.13) the theorem follows.

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On a paper of Baker and Schinzel

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1. Introduction. Let D be an integer, positive or negative but not a square. It was shown by Baker and Schinzel [1] that every genus of primitive binary quadratic forms of discriminant D represents a positive integer, prime to D, and less than $C(\varepsilon)|D|^{3/8+\varepsilon}$, where $\varepsilon > 0$ and $C(\varepsilon)$ depends only on ε; and they conjectured that in fact the bound could be replaced by $C(\varepsilon)|D|^{\varepsilon}$. The object of this paper is to prove the following sharpening of their result.

THEOREM. *Every genus of primitive binary quadratic forms of discriminant D represents a positive integer, prime to D, and less than $C(\varepsilon)|D|^{1/4+\varepsilon}$.*

Our theorem may be used in place of the result of Baker and Schinzel, in the work of Möller [4], thereby improving his results somewhat. In particular it follows from our theorem that the smallest prime which splits in $Q(\sqrt{-d})$, but does not ramify, is less than $C(\varepsilon)|D|^{1/4+\varepsilon}$, where D is the discriminant of the field, and so all the 'numeri idonei' of Euler are less than $(2C(\varepsilon))^{1/\varepsilon}$, for any ε with $0 < \varepsilon < 1/4$. Thus if $C(\varepsilon)$ were effectively computable then all the numeri idonei could in principle be explicitly determined. But unfortunately, as in [1], the constant $C(\varepsilon)$ is ineffective; this is due to the use of Siegel's lower bound for $L(1, \chi)$ (see [5]).

Our improved bound results from the use of estimates of Burgess [2] in place of those of Burgess [3] as employed by Baker and Schinzel [1]. Apart from this our argument follows that of [1] closely, but there are two further differences; the first involves the employment of a modified path of integration and the second involves the replacement of a finite sum by the corresponding L-function. The latter change is not in fact essential but we believe that it leads to a more elegant exposition.

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2. Bounds for L-functions. In place of Lemma 2 of [1] we prove the following.