On the order of Dedekind Zeta-functions near the line $\sigma = 1$

by

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1. Denote by $K$ an algebraic number field, by $n$ and $d$ the degree and the discriminant of the field $K$, respectively, and by $\zeta_K(s)$, $s = \sigma + it$, the Dedekind Zeta-function (see [4]).

Basing on some estimates of A. V. Sokolovskii connected with the application of I. M. Vinogradov's methods to the theory of Dedekind Zeta-functions (see [7] and compare [10]), refined in [1] with respect to the constants of the field, we shall prove the following

**Theorem.** If $1 - \frac{1}{n+1} \leq \sigma \leq 1$, $t \geq e$, then

$$|\zeta_K(\sigma + it)| \leq e^{e^h + (\log d^h n^{1/2} (\log n - \log e))^{1/2}} \log^{10} t,$$

where $c$ is a positive purely numerical constant.

About possible application of (1.1), see [3]. For the Riemann Zeta-function $\zeta(s)$ the strongest estimate of the form (1.1) is due to H. E. Richert (see [9] and compare [9]).

2. The Dedekind Zeta-function of an algebraic number field $K$ is defined by the series

$$\zeta_K(s) = \sum_a (Na)^{-s}, \quad s = \sigma + it,$$

in the open half-plane $\sigma > 1$, the sum being taken over all ideals of $K$ (see [4]). The function $\zeta_K(s)$ can be continued analytically to a meromorphic function with a simple pole at $s = 1$.

It is known that

$$\zeta_K(s) = \sum_{\sigma} \left( \sum_{a \in \sigma} (Na)^{-s} \right),$$

where the inner sum is taken over all ideals of $K$, belonging to an ideal class $C$ (see [4], p. 57) and the outer sum is taken over all $h$ ideal classes.
It is also known that

\[ f_C(s) = \sum_{a \in \mathcal{O}} (N(a))^{-s} = N(a')^s \sum_{m(\text{mod } a')} |N(a)|^{-s} \]

where the last sum is taken over a complete system of pairwise not associated algebraic integers belonging to any ideal \( a' \in \mathcal{O}^{-1} \) (see [4], p. 58).

If \( a_1, a_2, \ldots, a_n \) form a basis for \( a' \), then every element \( a \) of \( a' \) can be uniquely represented in the form \( a = a_1x_1 + \cdots + a_nx_n \) where \( a_i, i = 1, 2, \ldots, n \) are rational integers.

Every element \( a \in K \) can be considered as an element of the \( n \)-dimensional real space \( \mathbb{R}^n \):

\[ a = (a_1, \ldots, a_n, b_1, \ldots, b_n) \]

where \( a = r_1 + 2r_2 \) (see [2], III, § 3).

Denote by \( \mathbb{M} \) the \( n \)-dimensional lattice in \( \mathbb{R}^n \) formed of images of algebraic integers \( a \in K \) divisible by \( a' \) and denote by \( V \) the fundamental domain of \( K \) (see [2], p. 332). Then the summation in (2.2) reduces to the summation over rational integers \( a_1, \ldots, a_n \) such that \( a(a) \in \mathbb{M} \cap V \).

Denoting

\[ N(a) = N(a(a)) = f(a_1, \ldots, a_n) \]

we can write

\[ f_C(s) = N(a')^s \sum_{a(a) \in \mathbb{M} \cap V} |f(a_1, \ldots, a_n)|^{-s}. \]

We denote by

\[ a^{(0)} = a_1a^{(0)} + \cdots + a_na^{(0)}, \quad i = 1, 2, \ldots, n, \]

the conjugates of \( a \) so that \( a^{(0)} \) are real if \( 1 \leq i \leq r_1 \) and \( a^{(0)} \) are complex conjugates of \( a^{(0-i)} \) if \( r_1 + 1 \leq i \leq r_2 \) hence \( N(a) = a^{(0)} \cdots a^{(0)} \).

Denote further by \( V \) the set which we get multiplying the elements of \( V \) by images of all roots of unity belonging to \( K \). Then the series (2.3) can be written as follows

\[ f_C(s) = \frac{1}{m} N(a')^s \sum_{a(a) \in \mathbb{M} \cap V} \sum_{\sigma(a) \in V} e^{-s \log |f(a_1, \ldots, a_n)|} \]

where \( m \) denotes the number of roots of unity contained in \( K \) (see [7], p. 323).

In the following we shall always assume that

\[ N(a') \leq |a'|^{12}, \]

since in each ideal class \( C \) there exists at least one ideal satisfying (2.5) (see [4], p. 42).

3. The proof of (1.1) will rest on the following lemmas:

**Lemma 1** (see [1], Lemma 4, and compare [7], Lemma 1). If \( a_1, \ldots, a_n \) form a basis for a given ideal \( a \) with \( N(a) \leq |a'|^{10} \), then \( K_{\mathbb{R}}^X \) denotes the set of all systems of real numbers \( (u_1, \ldots, u_n) \) with

\[ \max |u_i| \leq X \]

where \( u_1a_1 + \cdots + u_na_n \) are elements of \( R^a \) which belong to \( V \), then for any system of real numbers \( (u_1, \ldots, u_n) \in K_{\mathbb{R}}^X \), we have the inequality

\[ A_1X < |u_1a_1| + \cdots + |u_na_n| < A_2X, \quad i = 1, \ldots, n, \]

where

\[ A_1 = \exp(-4n^6|a'|^2), \quad A_2 = 2|a'|^n+1. \]

**Lemma 2** (see [1], Lemma 12, and compare [7], Lemmas 5 and 8). Denote

\[ F(a_1, \ldots, a_n) = -\frac{t}{2\pi} \log |N(a(a))| = -\frac{t}{2\pi} \log |f(a_1, \ldots, a_n)|. \]

If

\[ m_1 = \left[ \frac{n+2}{n} \log t \right], \quad A_k = \left[ \frac{1}{n+1} \log X \right], \]

then

\[ |S_d| = \left| \sum_{\sigma(a) \in V} e^{-s \log |f(a_1, \ldots, a_n)|} \right| \leq A_k X^{1 - \frac{1}{4n_k}} \]

where

\[ A_k = \exp(4 \cdot 10^6 n_k^6 |d|^2), \quad A_k = 10^6 n_k^6. \]

**Remark.** Lemma 2 is a slightly completed version of [1], Lemma 12, to that effect that in the present version of the lemma under consideration all the numerical constants are counted out explicitly.

**Lemma 3** (see [1], Lemma 13, and compare [7], Lemma 9). In the region \( \sigma \geq 1 - 1/(n+1) \), \( t > 1 \), \( \sigma = a + ti \) of the complex plane, we have the estimate

\[ |\zeta_K(s) - \sum_{1 \leq m \leq n+1} F(m) n^{-s} | \leq \exp(\sigma n^6 |d|^2) \]

where \( c_2 \) is a pure numerical positive constant.
Lemma 4 (see [8], p. 186). In the region $-1 \leq \sigma \leq 2$, $-\infty < t < +\infty$, of the complex plane, we have the estimate

$$|z(s) - z(s')| \leq A_s(|s| + 1)^{\epsilon_2}; \quad s = \sigma + it,$$

where

$$A_s = c_e^2|s|^{\epsilon_2}, \quad A_s = \frac{3}{4}n + 2,$$

and $c_e$ is a pure numerical constant.

4. Proof of the theorem. Denote

$$K_{s,t} = |s|^{-1/2}$$

(see Lemma 1) where $t = \exp(\log^{2^{23}}t)$, $i$ integer, $i \geq 0$. Owing to (2.1), (2.4) and (3.4) we have in the region $\sigma \geq 1 - 1/(n + 1)$, $t > 1$, the estimate (compare [7], p. 330)

$$|z(s)| \leq \exp(c_1n^2|A|^2) + |A|^{1/2} \sum_{j=1}^{h} \sum_{x_1, \ldots, x_n} |N(x)|^{-s} +$$

$$+ |A|^{1/2} \sum_{j=1}^{h} \sum_{x_1, \ldots, x_n} \sum_{x_{n+1}} |N(x)|^{-s},$$

where $c_j$ are ideals belonging to the inverse classes $O_{j}^{-1}$ and are chosen in such a way that $\|N_0\| \leq |A|^{1/2}$ (see (2.5)) and $h$ is the class-number. For $h$ we use the simplest estimate

$$h \leq |A|^{(\sigma-2)/2n}$$

mentioned in [5], p. 160.

We estimate the second term of (4.1) as follows.

Denoting $K_m = |s|^{-1/2}, m = 0, 1, 2, \ldots$ (see Lemma 1) we have

$$|z(s)| \leq \sum_{|s| = \frac{\log^{23}t}{\log 2}, \ldots, x_{n+1}} |N(x)|^{-s} +$$

$$+ \sum_{|s| = \frac{\log^{23}t}{\log 2}, \ldots, x_{n+1}} |N(x)|^{-s},$$

where $m_0 = \left[\log^{23}t / \log 2\right], t \geq e$, since $t_0 = \exp(\log^{23}t)$.

Estimating the first term on the right of (4.3) trivially, and the second term by the use of Lemma 1, we simply get the inequality

$$|A|^{1/2} \sum_{j=1}^{h} \sum_{x_1, \ldots, x_n} |N(x)|^{-s} \leq 2e^{\log^{23}t} \log^{1/2} |A|^{1/2},$$

valid in the region $\sigma \geq 1 - 1/(n + 1), t \geq e$, of the complex plane.
We estimate the second factor of (5.5) as follows. Consider the polynomial
\[ \varphi(x) = n(1 - \sigma)x - A_0 x^2 - 2A_0 x. \]
This polynomial has a maximum at the point
\[ a_0 = -4A_0 a_0 + 16A_0^2 a_0^2 + 12A_0 a_0^2 n(1 - \sigma) \frac{1}{6A_0 a_0^2}. \]
It is easy to realize that the above maximum of \( \varphi(x) \) is absolute for \( x \geq 0 \).

From the obvious inequality
\[ 0 \leq a_0 \leq \frac{1}{3A_0} \sqrt{n(1 - \sigma)}, \]
we get
\[ \varphi(a_0) \leq \frac{1}{3A_0} \left( n(1 - \sigma) \right)^{\frac{3}{2}}. \]

Therefore, owing to (5.4), we have
\[ 2n(1 - \sigma) - 4A_0 x^2 - 2A_0 x \leq \frac{1}{3A_0} \left( n(1 - \sigma) \right)^{\frac{3}{2}}. \]

Owing to (5.4) and the definition of \( t_0 \), we have for the first factor of (5.5)
\[ a_0^{(1 - \sigma) - 4A_0 x^2 - 2A_0 x} \leq \frac{1}{3A_0} \left( n(1 - \sigma) \right)^{\frac{3}{2}}. \]

Therefore, from (5.5)–(5.7) it follows
\[ |S_{i0}| \leq A_0 t_{\log_{n^2} t_{\sqrt{n^2}}} \frac{1}{2} - \frac{1}{2A_0 x^2}. \]

For the remaining \( |S_{ki}|, \ k = 2, \ldots, 2n \) we get similar estimates. Hence from (5.1), (5.3), (5.4) and (5.5) it follows
\[ \left| A^{|1/2} \sum_{k=1}^{n} \sum_{\sigma < k < \pi_{k/2(k+1)}} \left| X(a) \right|^{-\sigma} \right| \leq c_0 t_{\log_{n^2} A_0} \frac{1}{2} - \frac{1}{2A_0 x^2} \log_{n^2} t_{\sqrt{n^2}}. \]

Owing to (4.1), (4.4), (5.9) we get in the region
\[ 1 - \frac{1}{n + 1} \leq \sigma \leq 1, \ t \geq e^{1.05 n^2 A_0^2} \]
the estimate
\[ |\mathcal{L}(\sigma + it)| \leq \exp(c_0 n^2 |A|^2) \log_{n^2} t_{\sqrt{n^2}}. \]

We split the region (5.10) into the two following regions:
\[ D_1: 1 - \frac{1}{n + 1} \leq \sigma \leq 1, \ t \geq \exp(2.105 n^2 |A|^2), \]
\[ D_2: 1 - \frac{1}{n + 1} \leq \sigma \leq 1 - \frac{1}{n log_{n^2} t_{\sqrt{n^2}}}, \ t \geq \exp(2.105 n^2 |A|^2). \]

From (5.11) it follows that in \( D_1 \)
\[ |\mathcal{L}(\sigma + it)| \leq \exp(c_0 n^2 |A|^2) \log_{n^2} t_{\sqrt{n^2}}. \]

Analogously in \( D_2 \) we get
\[ |\mathcal{L}(\sigma + it)| \leq \exp(c_0 n^2 |A|^2) \exp(2.105 n^2 (1 - \sigma)^2) \log_{n^2} t_{\sqrt{n^2}}. \]

Hence from (5.12), (5.13) we get in the region
\[ 1 - \frac{1}{n + 1} \leq \sigma \leq 1, \ t \geq \exp(2.105 n^2 |A|^2) \]
the estimate
\[ |\mathcal{L}(\sigma + it)| \leq \exp(c_0 n^2 |A|^2) \exp(2.105 n^2 (1 - \sigma)^2) \log_{n^2} t_{\sqrt{n^2}}. \]

Owing to Lemma 4, for
\[ 1 - \frac{1}{n + 1} \leq \sigma \leq 1, \ s \leq t \leq \exp(2.105 n^2 |A|^2), \]
we have simply
\[ |\mathcal{L}(\sigma + it)| \leq \exp(c_0 n^2 |A|^2). \]

From (5.12) and (5.13) the theorem follows.

References

On a paper of Baker and Schinzel

by

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1. Introduction. Let $D$ be an integer, positive or negative but not a square. It was shown by Baker and Schinzel [1] that every genus of primitive binary quadratic forms of discriminant $D$ represents a positive integer, prime to $D$, and less than $O(\varepsilon |D|^{3/8+\varepsilon})$, where $\varepsilon > 0$ and $O(\varepsilon)$ depends only on $\varepsilon$; and they conjectured that in fact the bound could be replaced by $O(\varepsilon |D|^{1/4})$. The object of this paper is to prove the following sharpening of their result.

Theorem. Every genus of primitive binary quadratic forms of discriminant $D$ represents a positive integer, prime to $D$, and less than $O(\varepsilon |D|^{3/8+\varepsilon})$.

Our theorem may be used in place of the result of Baker and Schinzel, in the work of Möller [4], thereby improving his results somewhat. In particular it follows from our theorem that the smallest prime which splits in $Q(\sqrt{-d})$, but does not ramify, is less than $O(\varepsilon |D|^{3/8+\varepsilon})$, where $D$ is the discriminant of the field, and so all the ‘numer idonei’ of Euler are less than $[O(\varepsilon)]^{12}$, for any $\varepsilon$ with $0 < \varepsilon < 1/4$. Thus if $O(\varepsilon)$ were effectively computable then all the numeri idonei could in principle be explicitly determined. But unfortunately, as in [1], the constant $O(\varepsilon)$ is ineffective; this is due to the use of Siegel’s lower bound for $L(1, \chi)$ (see [5]).

Our improved bound results from the use of estimates of Burgess [2] in place of those of Burgess [3] as employed by Baker and Schinzel [1]. Apart from this our argument follows that of [1] closely, but there are two further differences; the first involves the employment of a modified path of integration and the second involves the replacement of a finite sum by the corresponding $L$-function. The latter change is not in fact essential but we believe that it leads to a more elegant exposition.

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2. Bounds for $L$-functions. In place of Lemma 2 of [1] we prove the following.