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On B. Segre's construction of an ovaloid

by

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Let $GF(q)$ be the finite field of order $q = 2^h \geqslant 8$ and let C be the set of elements $a \in GF(q)$ for which the roots of the equation $x^2 + x = a$ are in $GF(q)$. Put $D = GF(q) \setminus C$. Obviously C is an additive subgroup of $GF(q)$ with index 2. Denote those elements of $GF(q)$, which are different from 0 and 1 by $a_1, a_2, \ldots, a_{q-2}$ taken in any order with the only restriction $a_2 = a_1 + 1$. The following condition $S(q)$ plays an important rôle in Segre's construction of a non-quadric ovaloid in the Galois space $S_{3,q}$:

$S(q)$: There exist $q-2$ not necessarily distinct elements $b_1, b_2, \ldots, b_{q-2} \in D$, such that $(a_i b_j + a_j b_i) \in D$ whenever $i, j = 1, 2, \ldots, q-2$ and $i \neq j$.

More exactly, Segre's theorem VI in [2] says, that if $S(q)$ holds for $q = 2^h \geqslant 8$, then there exists an ovaloid in $S_{3,q}$, which is not a quadric. By theorem VIII in [2], $S(8)$ holds while $S(16)$ is false. We generalize this result:

THEOREM. *Condition $S(q)$ is false for $q = 2^h > 8$.*

Proof. Suppose $S(q)$ is true for $q = 2^h > 8$. Then for every $k = 3, 4, \ldots \ldots, q-2$ precisely one of the elements $a_1 b_k$ and $a_2 b_k = a_1 b_k + b_k$ is contained in C. By assumption we have $(a_1 b_k + a_k b_1), (a_2 b_k + a_k b_2) \in D$, hence exactly one of the following two elements $a_k b_1$, $a_k b_2$ is in C. We conclude $b_1 \neq b_2$ and $a_k(b_1 + b_2) \in D$. This implies

$$|\{a_k(b_1+b_2);\ k = 3, 4, \ldots, q-2\}| = q-4 \leqslant |D| = q/2,$$

which is a contradiction.

Consequently, B. Segre's method for constructing new ovaloids is applicable only to the case $q = 8$. According to a computer result of G. Fellegara [1], Segre's ovaloid belongs to an infinite class of ovaloids discovered by J. Tits [3] with quite different — group theoretical — methods.

References

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[2] B. Segre, *On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two*, Acta Arith. 5 (1959), pp. 315–332.
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Fast konstante Folgen

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In dieser Arbeit geben wir einen Beweis eines Resultats von Rauzy, [2], über gleichverteilte Folgen mod 1, der sich auf den Fall der Gleichverteilung in kompakten Gruppen übertragen läßt. Wir zeigen ein analoges Resultat für $\boldsymbol{R}^n$, in $\boldsymbol{Z}$ ist die Situation anders. Allgemeines zur Theorie der Gleichverteilung findet man in [1].

Sei

M die Menge aller glv. Folgen mod 1,

C die Menge aller Folgen (c_n) mit $(x_n) \in M \Rightarrow (c_n + x_n) \in M$,

C^0 die Menge aller Folgen (c_n): $\exists a > 1$ mit $c_n = c_m$, wenn für ein $k \in \boldsymbol{N}$ gilt: $a^{k-1} \leqslant n, m < a^k$.

Sei

$$d(u, v) = \min\{|u - v - k|, k \in \boldsymbol{Z}\}, \quad u, v \in \boldsymbol{R},$$
$$g\big((c_n), (d_n)\big) = \overline{\lim}(1/N) \sum_{n \leqslant N} d(c_n, d_n).$$

Theorem 0 (Rauzy). *Folgende Bedingungen sind äquivalent*

(i) $(c_n) \in C$;

(ii) *Für* $\forall \varepsilon > 0$, $\exists \delta > 0$, *sodaß für jede Indexfolge* n_k *mit* $n_{k+1}/n_k \to a$, $1 < a < 1 + \delta$:

$$\overline{\lim_{k\to\infty}}(1/n_k) \sum_{h \leqslant k} \inf_{y \in \boldsymbol{R}} \sum_{n_h \leqslant n < n_{h+1}} d(c_n, y) < \varepsilon;$$

(iii) $\forall \varepsilon > 0$, $\exists (d_n) \in C^0$ *mit* $g\big((c_n), (d_n)\big) < \varepsilon$.

Die Äquivalenz von (ii) und (iii) ist einfach zu zeigen. (iii) $\Rightarrow$ (i) folgt sofort aus Lemma 1. Der nichttriviale Teil des Beweises ist die Implikation (i) $\Rightarrow$ (ii). Wir benötigen dazu Lemmata 2–4.

Lemma 1. $C^0 \subseteq C$.