On the existence of a density

by

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We shall give the details which demonstrate a formula for a number theoretical density which played a vital role in our paper [2], but doubts about existence and correctness of the formula have been expressed by A. Garcia, H. Moeller, and the editors. In the meantime Everett [1] has used our encoding idea to derive a new proof for one of our assertions.

We shall recall some of the conventions and symbolisms in our paper. We considered a function $T$, mapping the positive integers into themselves, given by

$$ T_n = \frac{3^{X(n)} n + X(n)}{2}, $$

where $X(n) = 1$ when $n$ is odd and $X(n) = 0$ when $n$ is even.

Given an integer $n$ we considered iterated partities $n, T_n, T^2n, \ldots, T^kn$ and we agreed to stop the iteration at the very first instance when $T^kn < n$. This stopping time was denoted by $\chi(n) = k$. Infinite values for the stopping time were permitted. We also introduced a second stopping time $\tau(n)$ which had a periodicity property. The quantity $P[\tau = k]$ was defined to be the proportion of integers in $[1, 2^k]$ which satisfy the relation $\tau(n) = k$. The quantities $P[\tau < k]$ and $P[\tau \geq k]$ were defined similarly in the same block of integers.

If $A$ is a set of positive integers then the density of $A$ is defined in terms of the counting function $\mu$ to be

$$ \delta(A) = \lim_{n \to \infty} \frac{1}{m} \mu \{ n \leq m \mid n \in A \} $$

provided this limit exists. We now set $[\chi = k] = \{ n \geq 0 \mid \chi(n) = k \}$, and we define $[\tau < k]$ and $[\tau \geq k]$ in a similar manner.

Theorem. The density of the set $[\chi \geq k]$ exists and is given by

$$ \delta[\chi \geq k] = P[\tau \geq k]. $$

Proof. The trick involved is to get this formula without forming any infinite sums. In [2] we established the formula $\delta[\chi = k] = P[\tau = k]$. Finite additivity of density gives $\delta[\chi < k] = P[\tau < k]$. Since the sets
[x < k] and [x ≥ k] are complementary sets of positive integers one has that \( \delta[x < k] \) exists and that

\[
\delta[x < k] + \delta[x ≥ k] = 1.
\]

One also has the relation

\[
P[x < k] + P[x ≥ k] = 1,
\]

which holds because we have defined the quantities involved in terms of a finite block of integers \([1, 2^k]\). The assertion of the theorem follows.

The relation \( P[x ≥ k] = \delta[x ≥ k] \) enables one to compute explicit values of \( \delta[x ≥ k] \) for quite large values of \( k \). In [3] we consider two quite distinct general algorithms in a probabilistic context which enable one to perform such a computation. A table of these density values already appeared in [2].

References


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