2. Proofs. Let \( \chi(F) \), \( \chi(K) \) be the residues of \( \zeta_F(s) \), \( \zeta_K(s) \) respectively at \( s = 1 \). Then since \( \zeta_F(s) = \zeta_K(s)L(s, \chi) \) we have

\[
L(1, \chi) = \frac{\chi(F)}{\chi(K)}.
\]

Under the assumptions of the Theorem, \( L(s, \chi) \) is an entire function, it follows that if \( \zeta_K(\tfrac{1}{2}) = 0 \) then \( \zeta_K(\tfrac{1}{2}) = 0 \). We use this fact to obtain a lower bound for \( \chi(F) \), and since an upper bound for \( \chi(K) \) is easily got we can prove Theorem 4.

**Lemma 1.** If \( K \) is an algebraic number field of degree \( n \geq 2 \), then

\[
\chi(K) \leq 2^{2n} \pi^n e(1.3)^{n+1} (\log |d_K|)^{n-1}.
\]

And if \( K \) is a totally real field, then

\[
\chi(K) \leq 2^n e(1.3)^{n+1} (\log |d_K|)^{n-1}.
\]

**Proof.** This is Lemma 2.1 of [4].

**Lemma 2.** If \( \zeta_K(\tfrac{1}{2}) = 0 \), then

\[
\chi(F) \geq 2^{-4n+4} e^{-\log |d_F|} |d_F|^{-1/4}.
\]

**Proof.** Take \( s_0 = \tfrac{1}{2}, N = [F : Q] = 2n \) in Lemma 3, p. 323 of [3]. Thus together Lemmas 1 and 2 give

\[
L(1, \chi) \gg |d_F|^{-1/4},
\]

and under the further assumptions of the Corollary we have from the first part of the proof of Theorem 4.1 of [4] (see (7)) that

\[
L(1, \chi) \leq (2\pi)^n \frac{h(F)}{h(K)} |d_F|^{1/2} |d_K|^{1/2},
\]

and so

\[
h(F) \gg L(1, \chi) |d_F|^{1/2} \gg |d_F|^{1/4}.
\]

Thus together Lemmas 1 and 2 give

\[
L(1, \chi) \gg |d_F|^{-1/4},
\]

and under the further assumptions of the Corollary we have from the first part of the proof of Theorem 4.1 of [4] (see (7)) that

\[
L(1, \chi) \leq (2\pi)^n \frac{h(F)}{h(K)} |d_F|^{1/2} |d_K|^{1/2},
\]

and so

\[
h(F) \gg L(1, \chi) |d_F|^{1/2} \gg |d_F|^{1/4}.
\]

**References**


Existence of an indecomposable positive quadratic form in a given genus of rank at least 14

by

G. L. Watson (London)

**0. Introduction.** We shall prove the following

**Theorem.** Let \( f \) be a positive-definite quadratic form with integer coefficients in \( n \geq 14 \) variables. Then in the genus of \( f \) there is at least one class that contains no disjoint form.

There is also (for \( n \geq 12 \)) at least one class that does contain a disjoint form; see [4], pp. 77, 78, Theorem 47.

The constant 14 is best possible; to see this, we define genera each of which consists entirely of classes that contain disjoint forms. Twelve suitable genera may be defined by

\[
(0.1) \quad f = x_1^2 + x_2^2 + \ldots + x_n^2, \quad 2 \leq n \leq 11 \text{ or } n = 13,
\]

\[
(0.2) \quad f = x_1^2 + x_2^2 + \ldots + x_n^2 + 2x_1x_2, \quad n = 12.
\]

In a number of papers, references to which may be found in [1], it has been shown that

\[
(0.3) \quad \text{each of (0.1), (0.2) implies } f \sim x_1^2 + h, \text{ for some } (n-1)-\text{ary form } h = h(x_2, \ldots, x_n).
\]

Denote by \( c(f) \) the class-number of \( f \), that is, the number of classes in the genus of \( f \). In the counter-examples (0.1), (0.2) we have \( c(f) = 1 \) for \( n = 8 \); 2 for \( n = 9, 10, 11 \); 3 for \( n = 13 \); 4 for \( n = 12 \). Many other counter-examples, with \( n \leq 10 \) and \( c(f) = 1 \), may be found in [5]. For the smaller values of \( n \) many examples with \( c(f) > 1 \) could be given. For example, with \( n = 2 \) and \( f = x_1^2 + 14x_2^2 \sim 2x_1^2 + 7x_2^2 \), we have \( c(f) = 2 \).

We shall use the classical formula, see [2], [3] for the weight of a positive genus. The weight, \( w(f) \), of the genus of \( f \) is the sum of the weights of its constituent classes. Temporarily, let \( w(f) \) be the sum of the weights of the classes that contain disjoint forms; and define \( W(f) \) as \( w(f)/w(f) \).

Then trivially \( W(f) \leq 1 \); and the theorem may be expressed as:

\[
(0.4) \quad W(f) < 1 \text{ for every positive-definite } f \text{ in } n \geq 14 \text{ variables}.
\]
I have shown in [6], p. 182, Theorem 8, that the theorem follows in all cases if proved for \( f \) with certain conveniently simple arithmetical properties, without which the estimation of \( W(f) \) would be impossibly complicated. See (6.1), (7.1), below.

1. Formulae for \( W(f) \). It suffices (by permuting the variables) to consider disjoint forms of the shape

\[
g(a_1, \ldots, a_n) + h(a_{k+1}, \ldots, a_n), \quad 0 < k < n.
\]

For brevity we call this form \( g + h \), and for symmetry we write \( l \) for \( n - k \), the rank of \( h \). We define \( W_k(f) \) in the same way as \( W(f) \), except that we count only those classes in the genus of \( f \), that contain at least one form \( g + h \) with given \( k \) but do not (if \( k \geq 2 \)) contain any form \( g' + h' \) with \( g' \) of rank \( k' \), \( 0 < k' < k \). Clearly this makes \( k \leq l \), or \( k \leq \frac{1}{2} n \), which lessens the symmetry but gives

\[
W(f) = W_1(f) + W_2(f) + \ldots + W_{[n]}(f).
\]

We now show that, for \( 1 \leq k \leq \frac{1}{2} n \),

\[
W_k(f) \leq \sum \omega(g) \omega(h) / \omega(f), \quad \text{rank } g = k, \ g + h \sim f,
\]

where the accent means that if \( 2 | a \) the term with \( k = \frac{1}{2} n \) is to be halved, and the summation is over ordered pairs of genera. To see this, let \( a, b \) be the number of integral automorphs (with determinant \( \pm 1 \)) of \( g, h \) respectively. Then by definition the weights of \( g, h \) (or of their classes) are \( 1/a, 1/b \); and that of \( g + h \) is at most \( 1/ab \), since it has trivially at least \( ab \) automorphs. Now let \( g \) range over a set of representatives of the classes in some fixed \( k \)-ary genus, and similarly for \( h \), with the two genera such that \( g + h \sim f \). The sum of the weights of the \( g + h \) is at most

\[
\sum (1/ab) = \sum (1/a) \sum (1/b) = \omega(g) \omega(h);
\]

the sum of their contributions to \( W_k(f) \) is at most \( \omega(g) \omega(h) / \omega(f) \), whence we have (1.3) except for the accent. Now if \( k = l = \frac{1}{2} n \) we need not count both \( g + h, h + g \) unless \( g \sim h \), in which case we may suppose \( g = h \). Then \( g + h \) is unaffected by interchanging \( a_i \) and \( a_{i+1} \) for \( i = 1, \ldots, \frac{1}{2} n \); so it has at least \( 2ab \) automorphs, and the above estimate for its weight may be halved, which completes the proof of (1.3).

We conclude this section by showing that (1.3) remains valid if we impose the additional summation conditions

\[
W(g) < 1 \text{ if } k \geq 2, \quad W_r(h) < 1 \text{ if } 1 \leq r \leq k - 1.
\]

To see this, note that if \( W(g) = 1 \) then every class in the genus of \( g \) contains a disjoint form and so is to be disregarded in the above definition of \( W_k(f) \). Similarly for \( h \) if the second part of (1.4) fails.

2. The weight formula. For the classical formula for the weight of a positive genus, see [2], p. 96, (2), with a factor \( \frac{1}{2} \) since Pauli there excludes automorphs with determinant \( -1 \), and so doubles the weight. He also omits to mention that another factor \( \frac{1}{2} \) is needed in the trivial case \( n = 1 \).

I also replace the \( d \) in Pauli’s formula by \( \det A \), where

\[
A = A(f) = (\partial f / \partial x_i \partial x_j)_{i=1, \ldots, n}
\]

is the matrix of \( f \) in the notation I prefer. In Gaussian notation this \( A \) is the matrix of \( 2f \); but clearly \( w(f) = w(2f) \), so we have

\[
w(f) = \{1 + \text{sgn}(n - 1)\} \pi^{-1 + n + 1} \prod_{i=1}^{n} \Gamma(\frac{1}{2} i) \prod_{p} \left( \alpha_p(A) \right)^{-1} \left( \det A \right)^{n+1}.
\]

Here the second product is taken over all primes \( p \), and the \( p \)-adic density \( \alpha_p(A) \) of the matrix \( A = A(f) \) is defined as in [2].

Now let \( \det A \) be the highest power of \( p \) dividing \( \det A \), and define

\[
\beta_p(f) = \beta_p(A) = \left( \det A \right)^{-n+1 - \alpha_p(A)}.
\]

It is easily seen that this gives \( \beta_p(a) = \beta_p(f) \) for every positive integer \( a \), and we may replace (2.2) by

\[
w(f) = \{1 + \text{sgn}(n - 1)\} \pi^{-1 + n + 1} \prod_{i=1}^{n} \Gamma(\frac{1}{2} i) \prod_{p} \left( \beta_p(f) \right)^{-1}.
\]

We define also

\[
\theta(k, l) = \theta(l, k) = \pi^{1/2} \prod_{i=1}^{k} \left( \Gamma(\frac{1}{2} i) / \Gamma(\frac{1}{2} i + \frac{1}{2} l) \right);
\]

\[
\theta_p(g, h) = \beta_p(g + h) / \beta_p(g) \beta_p(h),
\]

\[
S(f, k) = \sum_{p} \left( \theta_p(g, h) \right) \text{ rank } g = k, \ g + h \sim f, (1.4) \}
\]

The sum in (2.7) is over ordered pairs of genera as in (1.3). With these definitions, (1.3) gives, for \( 1 \leq k \leq \frac{1}{2} n, n > 3 \),

\[
W_k(f) \leq \{1 + \text{sgn}(k - 1)\} \{1 + \text{sgn}(n - 2k)\} \theta(k, n - k) S(f, k).
\]

3. Further definitions. Let \( \delta \) denote a positive square-free integer. When \( g, h \) satisfy the summation conditions of (2.7) we must clearly have, for some \( d \),

(3.1) \( \text{rank} g = k, \ g + h \sim f, \) and \( d^{-1} \det \{ \frac{1}{2} A(g) \} \) is a rational square.
So if we define
\[ S_d(f, k) = \sum \left\{ \prod_{p} \theta_p(g, h): (1.4), (3.1) \right\} \]
we have
\[ S(f, k) = \sum_{d} S_d(f, k). \]
This last sum may be taken over \( d = 1, 2, 3, 5, \ldots \); but clearly only finitely many terms are non-zero.

Now for each prime \( p \) (3.1) implies, for some integer \( s \), that
\[ \text{rank } g = k, g + k \sim_{p} f, \quad \text{and } d^{-1} p^{-2s} \det \left( \frac{1}{2} A(g) \right) \]
is the square of a \( p \)-adic unit;
\[ \sim_{p} \]
denotes equivalence over the ring of \( p \)-adic integers.

We define
\[ T_d(f, k, p, s) = \sum \left\{ \theta_p(g, h): (3.4) \right\}. \]
Here the summation is over ordered pairs of classes under \( \sim \); and the accent means that terms for which (3.4) is inconsistent with (1.4) are to be omitted. Next, we define
\[ U_d(f, k, p) = \sum_{s} T_d(f, k, p, s); \]
the sum may be taken over a finite set of integers \( s \) for which (3.4) is possible; and we have
\[ S_d(f, k) \leq \prod_{p} U_d(f, k, p). \]
Here strict inequality is to be expected, because there may not exist positive forms \( g, h \) satisfying a given set of conditions (3.4), with \( p = 2, 3, 5, \ldots \).

Now define
\[ M_p(f, k) = \max_{\sim_{p}} U_d(f, k, p) + \max_{\sim_{p}} U_d(f, k, p). \]
Then clearly (3.3) implies
\[ S(f, k) \leq \prod_{p} M_p(f, k), \]
which will be used for \( k = 1 \) and for \( 3 \leq k \leq \frac{1}{2} n; \) but the product diverges for \( k = 2 \).

4. Inequalities for \( \theta(k, n - k) \). We notice that (2.5) gives
\[ \theta(1, n - 1) = \pi^{kn}/\Gamma(\frac{1}{2} n), \]
\[ \theta(2, n - 2)/\theta(1, n - 1) = \pi^{(n-2)}/\Gamma(\frac{1}{2}n-\frac{1}{2}). \]
Each of these expressions is a decreasing function of \( n \) for \( n \geq 14 \). For the gamma function is logarithmically convex and so
\[ \Gamma(a \frac{1}{2}) > \{ \Gamma(a - 1) \}^{-\frac{1}{2}} \{ \Gamma(a) \}^{\frac{1}{2}} = (a - 1)^{\frac{1}{2}} \Gamma(a) > a^{\frac{1}{2}} \Gamma(a) \]
if \( a > \frac{1}{2} \). Using the duplication formula \( \Gamma(a) \Gamma(a + \frac{1}{2}) = 2^{1-2a} \Gamma(2a) \), (6.1) and (6.2) give
\[ \theta(2, n - 2)/(2n - 2)! \]
and (2.5) gives
\[ \theta(k, n - k)/(2n - k)! \]
for \( n > 19 \) we shall later use (4.1), (4.2) and
\[ \theta(k, n - k)/(2n - k)! \leq \left( \begin{array}{l} 1/4 \end{array} \right) \text{ for } n \geq \min(19, 2k + 4), \]
for \( k \geq 2 \).
To prove (4.3) for fixed \( k \geq 8 \), note that the right member of (4.4) is a decreasing function of \( n \) for \( n \geq 2k + 4 \), and reduces for \( n = 2k + 1 \) to \( 2^{k+1}(k+1)(k+2) < 40/90 \). For fixed \( k \leq 7 \), the right member of (4.4) is decreasing for \( n \geq 19 \), so greatest for \( n = 19 \). So (4.5) is easily verified.

For \( 14 \leq n \leq 18 \), we use (4.1)-(4.4) to calculate the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 14 )</th>
<th>( 15 )</th>
<th>( 16 )</th>
<th>( 17 )</th>
<th>( 18 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.20</td>
<td>2.87</td>
<td>1.89</td>
<td>1.199</td>
<td>0.740</td>
</tr>
<tr>
<td>2</td>
<td>7.91</td>
<td>3.83</td>
<td>1.72</td>
<td>0.719</td>
<td>0.382</td>
</tr>
<tr>
<td>3</td>
<td>10.08</td>
<td>4.60</td>
<td>1.15</td>
<td>0.327</td>
<td>0.085</td>
</tr>
<tr>
<td>4</td>
<td>10.59</td>
<td>5.09</td>
<td>0.69</td>
<td>0.159</td>
<td>0.025</td>
</tr>
<tr>
<td>5</td>
<td>10.26</td>
<td>5.19</td>
<td>0.42</td>
<td>0.066</td>
<td>0.008</td>
</tr>
<tr>
<td>6</td>
<td>9.52</td>
<td>4.99</td>
<td>0.28</td>
<td>0.036</td>
<td>0.003</td>
</tr>
<tr>
<td>7</td>
<td>9.04</td>
<td>4.70</td>
<td>0.22</td>
<td>0.021</td>
<td>0.002</td>
</tr>
<tr>
<td>8</td>
<td>--</td>
<td>--</td>
<td>0.20</td>
<td>0.016</td>
<td>--</td>
</tr>
<tr>
<td>9</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0.001</td>
</tr>
</tbody>
</table>

5. Estimation of certain products over primes. For brevity, write
\[ P(k, p) = \prod_{j=0}^{\lfloor k/2 \rfloor} \{ 1 - p^{-2j}; j = 0, 1, \ldots, \lfloor k/2 \rfloor \} \]
for prime \( p \) and \( k \geq 0 \), and
\[ R(k, p) = \min \{ P(k, p) P(l, p); k_2 \geq 0, k_2 \geq 0, k_1 + k_2 = k \}. \]
Clearly we have
\[ P(k, p) = (1 - p^{-2})^{\lfloor k/2 \rfloor} \text{ for } k \leq 3, \]
\[ R(k, p) = (1 - p^{-2})^{\lfloor k/2 \rfloor} \text{ for } k \leq 5. \]
For larger $k$, the product over $j \geq 2$ can be estimated numerically by straightforward calculations; and for $j = 1$ we use the well known $\prod (1 - p^{-2}) = 6/\pi^2$, giving

$$\prod_p (P(k, p))^{-1} \leq 1.66, 1.79, 1.82, 1.83,$$

for $k \leq 3, \leq 5, \leq 7, \geq 8$ respectively.

Now, for square-free positive $d$, define

$$L_d(p) = 1 - p^{-1}(-d/p), \quad L_d = \prod_p L_d(p),$$

where the Legendre symbol $(-d/p)$ may be interpreted as 0 if $p|2d$.

We shall prove that

$$L_d < 0.36 d \quad \text{for} \quad d \geq 3.$$

There exists a Dirichlet character $\chi$, modulo $4d$, such that $\chi(p) = (-d/p)$ for every $p$, whence (5.6) gives

$$L_d = \sum (N^{-1} \chi(N); \ N = 1, 2, \ldots).$$

When $d = 3$ we have $\chi(N) = 1$ for $N = 1$ or 5, $-1$ for $N = 7$ or 11, modulo 12; so we have the case $d = 3$ of (5.7) by a straightforward calculation. For $d \geq 5$, define

$$u_m = \sum N^{-1} |\chi(N)|,$$

with summation over $2d < N \leq 2d(m+1)$.

It is easily seen that $L_d \leq u_0 - u_1 + u_2 - \ldots$, and that $u_m$ is monotone decreasing. So $L_d < u_2$. Subtract $\frac{1}{2} \log d$ from each side of this inequality, and note that $2|N$ implies $\chi(N) = 0$. We have

$$L_d - \frac{1}{2} \log d < 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2d-1} - \frac{1}{2} \log d;$$

and the term $1/5$ on the right may be omitted if $d = 5$, giving (5.7).

Allowing $d$ to take non-square-free values, and differenting, the right member of (5.9) is a decreasing function of $d$, for $d \geq 1$, so it is less than 1 for $d \geq 6$, and we have

$$d^{-1} L_d < d^{-1} + \frac{1}{2} d^{-1} \log d.$$

By differentiating the right member of (5.10) we find that it is a decreasing function for real $d \geq 6$, and so is at most $6^{-1} (1 + \frac{1}{2} \log 6) < 3^{-1}$, which completes the proof of (5.7).

6. Odd primes. For each odd prime $p$ we shall assume that there exist forms $f_1, f_2$, with integer coefficients, depending on $f_1, p$, such that

$$f = f_1 + pf_2, \quad p \det A(f_1) \det A(f_2),$$

where the right member is disjoint as in (1.1) unless $f_2$ is identically 0. If so, $\det A(f_2)$ is to be interpreted as 1. Defining

$$n_i = n_i(f, p) = \text{rank} f_i \ (i = 1, 2), \quad \text{whence} \quad n_1 + n_2 = n,$$

we shall also assume that

$$n_i(f, p) \geq \frac{1}{2} n \quad \text{for every} \ p.$$ 

As far as odd primes are concerned these are just the assumptions justified by the theorem quoted at the end of § 0.

Now for $i = 1, 2$ we define

$$e_i = \begin{cases} 0 & \text{if} \ 2|m_i, \\ (-1)^{n_i} \det A(f_i)(p) & \text{if} \ 2|m_i; \end{cases}$$

$$m_i = [\frac{1}{2} n_i], \quad P_i = \prod (1 - p^{-2j}; j = 0, 1, \ldots, m_i);$$

$$X_i = 1 + e_i p^{-m_i}.$$ 

Later, if there is any risk of confusion, we may write more explicitly $e_i = e_i(f_1, p), \ldots$. We note that $P_i = P(n_i, p)$, see (5.1).

It now becomes convenient to write $\exp_{p^e}$ for $p^e$, if the exponent $e$ is complicated. With this notation, we show that

$$a_p(A(f)) = 2 \exp_p(\frac{1}{2} \log 1 + \frac{1}{2} \log 2) P_x X_1 X_2 X_3^{-1}.$$ 

To prove (8.7) for $n_2 > 0$ we put $s = 3$ in (23), 101, (21)–(23) and correct an error; the last sign in (23) should be $+$. When $n_2 = 0$, the convention $\det A(f_1) = 1$ gives $P_1 = e_1 = 1, X_2 = 2$, and the right member of (6.7) reduces to $P_1 X_1^{-1}$, agreeing with the case $s = 1$ of the formulae just quoted.

Now (2.3), with $(\det A)_p = \exp_p(n_3)$ by (6.1), (6.2), gives

$$b_p(f) = 2 \exp_p(-\frac{1}{2} n_1 n_2) P_1 P_2 X_1 X_2^{-1}.$$ 

We now suppose that

$$f \sim g + h, \quad \text{rank} g = k, \quad \text{rank} h = l = n - k.$$ 

It is clear that each of $g, h$ must be of the shape (6.1). For brevity we write, for $i = 1, 2$,

$$k_i = n_i(g), \quad l_i = n_i(h), \quad e_i = e_i(g), \quad X_i = X_i(g), \quad Z_i = X_i(h).$$

By differentiating the right member of (5.9) we find that it is a decreasing function for real $d \geq 6$, and so is at most $3^{-1} (1 + \frac{1}{2} \log 6) < 3^{-1}$, which completes the proof of (5.7).
From (6.9) and (6.10) it follows easily that

\[(6.11) \quad h_1 + h_2 = k, \quad l_1 + l_2 = l, \quad \text{and} \quad h_i + l_i = n_i \quad \text{for} \quad i = 1, 2;\]

also that (with \(g, h\) as in (6.1) with \(g, h\) for \(f\))

\[(6.12) \quad (\det A \langle f, i \rangle | p) = (\det A \langle g, i \rangle | p) (\det A \langle h, i \rangle | p) \quad (i = 1, 2).\]

Now from (2.6), (6.8), and (6.10) we have

\[(6.13) \quad \theta_p (g, h) = p^{-k} (2X_1X_2)^{-1} \prod_i P_i (\langle f, i \rangle | p);\]

where the product is over \(i = 1, 2\) and, see (6.11),

\[(6.14) \quad t = t(g, h, p) = k_1 l_2 + k_2 l_1 - n_1 n_2 - k_1 k_2 - l_1 l_2.\]

7. The prime 2. By the theorem quoted at the end of § 0 we may assume that

\[(7.1) \quad f \sim q_0 + 2q_1 + 2q_2 + 4q_3,\]

where each \(q_i\) is a diagonal form with odd coefficients and each \(w_i\) is a form of even rank of one of the shapes

\[(7.2) \quad a_4 a_2 + a_3 a_3 + \ldots + a_{2m-1} a_{2m} \quad (m \geq 0),\]

\[(7.3) \quad x_1^2 + a_1 x_1 x_2 + a_2 x_2 + \ldots + a_{m-1} a_m \quad (m \geq 1).\]

One or more of the summands in (7.1) may be identically 0, and no two of them have a variable in common. We define

\[(7.4) \quad \nu_i = \text{rank } q_i \quad (i = 0, 1), \quad 2m_i = \text{rank } w_i \quad (i = 1, 2), \quad \text{and} \quad n_i = 2n_i + \nu_i \quad (i = 1, 2).\]

This gives \(n_1 + n_2 = n\) and we assume \(n_1 \geq \frac{1}{2} n\) as in (8.3).

We need to normalize (7.1) so as to make the \(m_i\) and \(\nu_i\) into invariants under \(\sim\) and to define invariants \(s_i = s_i(f, 2)\). To do so, we appeal to [3], p. 97, Theorem 3: the normalization is as follows.

(i) A 3-year \(\varphi\) can be put into the shape (7.1) with summands of ranks 1, 2, 0, 0, 0. So we can have

\[(7.5) \quad \nu_i = 0, 1, \text{ or } 2 \quad (\text{for } i = 0, 1);\]

and this makes the \(m_i\) and \(\nu_i\) invariant.

(ii) For \(i = 1, 2\), if \(n_{i-1} \neq 0\), we can take \(w_i\) to be of the shape (7.2), and we define \(s_i = 0\).

(iii) For \(i = 1, 2\), if \(\nu_{i-1} = 2\) and the diagonal coefficients of \(q_i\) have the same residue \((\pm 1)\) modulo \(4\), we can again take \(w_i\) to be of the shape (7.2), and we again define \(s_i = 0\).

(iv) For \(i = 1, 2\), if neither of (ii), (iii) above implies \(s_i = 0\), we define \(s_i\) to be 1 or \(-1\) according as \(w_i\) is of the shape (7.2) or (7.3). And now the \(s_i\) and \(v_i\) are invariant.

(v) For \(i = 0, 1\), we can take each term of \(v_i\) to be \(\pm Q\) or \(\pm 3Q\), where \(Q\) denotes a unary \(\varphi\) with coefficient 1. For \(r_1 = 2\) we can further by using the obvious

\[(7.6) \quad aQ + bQ \not\sim bQ + aQ \not\sim (a + 4)Q + (b + 4)Q \quad \text{if } 2 \mid ab.\]

(vi) We can further normalize \(q_0 + 2p_1\) by using

\[(7.7) \quad aQ + bQ \not\sim (a + 2b)Q + (b + 2a)Q \quad \text{if } 2 \mid ab.\]

(vii) Now let \(f, f'\) be two forms of the shape (7.1), each normalized as above, of the same rank \(n\), and with \(\det A \langle f \rangle \det A \langle f' \rangle\) the square of a 2-adic unit, whence \(n_i (f') = n_i (f)\) for \(i = 1, 2\). Suppose also that \(m_i (f') = m_i (f)\) for \(i = 1, 2\), but \(f' \not\sim f\) is false. Then there is at most one possibility for \(f'\) up to \(\sim\) when \(f\) is given, and we must have

\[(7.8) \quad e_i (f') = -e_i (f) \quad (i = 1, 2).\]

I leave it to the reader to verify this assertion; and note that it obviously holds good with an odd \(p\) instead of 2, and without the hypothesis about the \(m_i\), which by (6.5) is redundant for odd \(p\).

Assuming that \(f\) is of the shape (7.1), normalized as above, we define \(P_i = P_i (f, 2)\) and \(X_i = X_i (f, 2)\) as in (6.5), (6.6), noticing that the \(= \text{sign}\) in (6.5) has to be replaced by \(<\), see (7.5). Now we further define

\[(7.9) \quad v = v (f) = n_1 (n_1 + 1) + n_2 (n_2 + 1),\]

\[(7.10) \quad q = q (f) = (\text{sgn } n_1) n_1 + (\text{sgn } n_2) n_2 + (\text{sgn } n_3),\]

\[(7.11) \quad E = E (f) = \exp \left( -2 - \text{sgn } n_1 - \text{sgn } n_2 \right) X_1 X_2;\]

and we shall show that, with \(A = A (f)\) as in (2.1), we have

\[(7.12) \quad \sigma_2 (A) = 2^{v + q - 1} PE^{-1}, \quad \text{where} \quad P = P_1 P_2.\]

Pall's formula for \(\alpha_2\), see [2], p. 105, (47), is very complicated and contains obvious errors, so I refer instead to [8], p. 96, Theorem 1, and p. 106, (13.4). Now we have to specialize, replacing the formally infinite sum treated in [8] by the right member of (7.1). This specialization is straightforward except for the factor \(E = \prod F_j\), taken over \(-\infty < j < \infty\).
The $E_j$ of [8], p. 96, (2.9) is $\frac{1}{2}X_j$ for $j = 1, 2$ in the present notation, and 1, in case (7.1), for other $j$ except 0, 3. $E_0 = 1$ if $n_0 = 0, \frac{1}{2}$ if not; and $E_3 = 1$ if $n_3 = 0, \frac{1}{2}$ if not. So the $E$ of [8] reduces in case (7.1) to that of (7.11), and we have (7.12).

From (2.1) (7.1), det $A$ is exactly divisible by $\exp_2(n_1 + 2n_3)$, so from (2.3), (7.12) we have

$$\beta_3(f) = P_3P_4E^{-1}\exp_2(n - q - 1 - \frac{1}{2}n_1n_3)$$

after a little simplification.

8. The prime 2, continued. Now suppose, cf. (6.9), that

$$f \tau g + h, \quad \text{rank } g = k, \quad \text{rank } h = l = n - k.$$  

Using the notation of (6.10), (6.11) follows. Obviously $g, h$ have to be of the shape (7.1). By (i) of § 7 we have

$$m_i(f) - m_i(g) - m_i(h) = \begin{cases} 0 & \text{if } n_{i-1}(g) + n_{i-1}(h) \leq 2, \\ 1 & \text{otherwise}; \quad \text{for } i = 1, 2. \end{cases}$$

For brevity we define, for $i = 1, 2$,

$$\eta_i = \sgn n_{i-1}(f), \quad \sigma_i = \sgn n_{i-1}(g), \quad \tau_i = \sgn n_{i-1}(h).$$

We note that, by (7.5), the $\eta_i$ and $\sigma_i$ determine the $m_i(f)$, and that

$$2 \eta_i, \eta_i, \eta_i, \eta_i$$ imply respectively $\eta_i, \eta_i, \tau_i = 1,$

$$n_i, \eta_i, \eta_i, \eta_i$$ imply respectively $\eta_i, \eta_i, \tau_i = 0,$

$$\eta_i = 0 = \sigma_i = \tau_i = 0;$$

each for $i = 1, 2$. For (8.6) see (8.2), (7.5).

We prove next:

**Lemma 1.** If $f$ and $g$, of the shape (7.1), are given then for given $m_i(h), m_i(k)$ there is at most one possibility, up to $f\tau g$, for $h$ satisfying (8.1).

**Proof.** We refer to [8], p. 96, Theorem 2. Part (i) of that theorem shows that the result is true if $g$ is of the shape (7.1) with each $q_i$ null; and part (ii), with the hypothesis about the $m_i(h)$, shows that it is true for $g = aQ (= a\eta_1)$, $a$ odd. By considering the forms adjoint to $f, g, h$ the result follows for $g = 2aQ$. Repeating these arguments, we can split off the summands in the normalized form of $g$ one at a time, and the lemma follows.

A similar result for odd $p$, with (6.9) for (8.1), is trivial; in it, the $m_i(h)$ need not be supposed given, since they are determined by (6.5) and (6.11).

From (7.10) and (2.6) we see that $\theta_2(g, h)$ is the product of the following four factors:

$$2^{-i}P_1(f)P_2(f)/P_1(g)P_2(g)P_1(h)P_2(h),$$

with $t = t(g, h, 2)$ as in (6.14);

$$\exp_2(1 - q(f) + q(g) + q(h));$$

$$\exp_2[-1 + \sum_{i=1}^2 (q_i - \sigma_i - \tau_i)];$$

$$(2X_1X_3)^{-1}Y_1Y_2Z_1Z_2.$$  

For (8.9), we use (7.11) and (8.3).

We define $J$ by

$$\log J = -\eta_1 \eta_2 + \sigma_1 \sigma_2 + \tau_1 \tau_2 +$$

$$+ \sum_{i=1}^2 (q_i(k_i - 1) + \tau_i(k_i - 1) - \sigma_i(n_i - 1)).$$

It is easily verified that $J$ is the product of the expressions (8.8), (8.9) (eliminate $q$ by using (7.10)). So we have

$$\theta_2(g, h) = 2^{-i}J(2X_1X_3)^{-1}Y_1Y_2Z_1Z_2 \prod_{i=1}^2 (P_i(f)/P_i(g)P_i(h)).$$

9. Estimation of $T_4(f, n, p, s)$. It is convenient to begin by replacing (6.13), (8.12) by more convenient inequalities. We notice first that (6.5) and (6.11) give $m_i(f) \geq m_i(h)$ for $p = 3$, and this holds also for $p = 2$ by (8.2). $P_i(f) \leq P_i(h)$ follows, and we have

$$P_1(f)P_2(f)/P_1(g)P_2(h) \leq 1.$$  

In one special case we shall need to notice that if for some $i, m$ we have $m_i(f) = m > m_i(h)$ then the one on the right of (9.1) may be replaced by $1 - p^{-m} - p^{-2m}$.

Again referring to (6.5), with the first $-$ sign replaced by $\leq$ if $p = 2$, as noted above (before (7.9)), we have $P(f) \geq P(h, p)$ and so by (5.2), (6.11) we have

$$P_1(g)P_2(g) \geq R(k, p).$$

Now (6.11) gives $k_2 \leq n_2$, so if $n_2 = n_2(f, p) = 1$ we have $k_2 \leq 1$ and $P_2(g) = 1$ by (6.5), (6.10). Using $k_1 \leq k$ to estimate $P_1(g)$, we have, see (5.1),

$$P_1(g)P_2(g) \geq [P(k, p)]^2 \quad \text{if } n_2 \leq 1,$$

$$P_1(g)P_2(g) \geq [P(k, p)(1 - p^{-2})] \quad \text{if } n_2 \leq 3.$$  

With $g$ of the shape (6.1) or (7.1), with $k_0$ for $n_2$, we see that

$$k_2 = 2s \text{ if } p \nmid d, \quad 2s + 1 \text{ if } p \mid d;$$

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from which

\[(9.5) \quad T_d(f, 2, p, 0) \leq p^{-1/2}T_d(p)X_2^{-1}Z_1 \quad \text{if} \quad p \geq 2.\]

(because the hypotheses with (9.4) give \(\max(k_1, k_3) = 1\)). We now have, for \(p \geq 2\),

\[(9.6) \quad \theta_0(g, h) \leq p^{-1/2}J[R(k, p)]^{-1}(2X_1X_2)X_4X_5Z_1Z_2,
\]

with the convention that \(J = 1\) if \(p > 2\), and subject to obvious improvement if we can use (9.9) or the remark following (9.1). We define \(V = V(g, h)\) by

\[(9.7) \quad V = (2X_1X_2)^{-1}(X_4X_5Z_1Z_2 + (2 - \eta_1)(2 - \eta_2)(2 - Z_1)(2 - Z_2)).\]

We now show that, with \(\varepsilon = [\frac{1}{2}k_2]\) by (9.4), we have

\[(9.8) \quad T_d(f, k, p, \varepsilon) \leq (R(k, p))^{-1}Vp^{-1/2} \quad \text{if} \quad p > 2.\]

To prove (9.8), we may obviously suppose that there is at least one pair \(g, h\) satisfying (6.9) and (3.4); we consider the possibilities for a different pair satisfying the same conditions; denote such a pair by \(g', h'\). By (9.4) and (6.11), (3.4) determines all the \(k_1, l_1\), also the \(P_i(g)\) and \(\tau_i\), see (6.14). By the remark following (7.8) there is at most one possibility for \(g', h'\), up to \(\sim\), and, see (6.10), we have \(g'(g') = -g(g) = -\eta_i\), giving, see (8.6), \(X_i'(g') = 2 - X_i(2 - \eta_i)\).

Now (6.9) and (3.4) imply that \(h\) satisfies a condition of the same shape as (3.4); so what has been proved above for \(g'\) applies to \(h'\) with obvious changes of notation. The pairs \(g, h\) and \(g', h'\) can be excluded since \(g\) determines \(h\) by the remark following Lemma 1. Summing over \(g, h\) and \(g', h'\), we have (9.8); and the possibilities for improving on the \(R\)-factor are just the same as for (9.6).

When \(p = 2\) we can argue exactly as above if we strengthen (3.4) by fixing the \(m_i(g)\) and the \(m_i(h)\), or equivalently the \(\eta_i, \tau_i\) which fixes \(J\). (See (8.3)-(8.6) and the preceding remark, also (8.11).) Then summing over the possibilities for the \(\eta_i, \tau_i\) we have, with \(\varepsilon = [\frac{1}{2}k_2]\),

\[(9.9) \quad T_d(f, k, 2, \varepsilon) \leq 2^{-1/2}[R(k, 2)]^{-1}\sum(\eta_1, \eta_2)X_1X_2^{-1/2} \text{if} \quad k = 2.\]

Now the case \(k = 2\) requires special treatment. Suppose that \(p \geq 2\) and \(s = 0\), whence \(k_1 = 2, k_3 = 0\). From (3.4), (5.6)-(6.6) and (6.10) we find \(\eta_1 = -dD(p), X_1 = 1 + p^{-1}\eta_1, P_1(g) = (1 - p^{-1})\). Similarly, with the convention for \(\eta_3 = 0\) in (6.1), we have \(\eta_3 = 1, X_3 = 2\), and trivially \(X_2 = Z_2\), since \(g_2\) is null and so \(f, h_2\) are equal. The second term on the right of (9.7) is zero since \(2 - Z_1 = 0\) and \(2 - Z_2 = 0\); and it is easy to see that \(g'\) does not exist. So \(T_d(f, 2, p, 0)\) is the same as \(T_d(g, h)\), the sum in (3.5) having just one term. So, see (5.6), we have, from (9.6),

\[(9.10) \quad T_d(f, 2, p, 0) \leq p^{-1/2}T_d(p)X_2^{-1}Z_1 \quad \text{if} \quad p \geq 2d,\]

on noting that (6.14) gives \(t = 2t,\) whence by (6.11) \(t = 2n_2\).

Similarly we find that

\[(9.11) \quad T_d(f_2, 2, p, 1) \leq p^{-1/2}T_d(p)X_2^{-1}Z_1 \quad \text{if} \quad p \geq 2d.\]

For \(2 < p \leq d\), giving \(s = 1\) and \(t = l_1 + l_2 = n - 2\), we find, using (9.5),

\[(9.12) \quad T_d(f, 2, p, 0) \leq p^{-1/2}X_1 \quad \text{(for} \quad 2 < p \leq d,\)

and note that \(Y_1 = Y_2 = 1,\) so that \(V\) simplifies.

For \(k = 2\), we shall use (9.9), with the \(R\)-factor is \(4/3\).

10. Estimation of \(V\), see (9.7), for odd \(p\). We shall prove:

**Lemma 2.** For \(p \geq 3\) and \(n \geq 14\) we have \(V \leq 2\). For \(p \geq 3\) and \(k_3 = 0\) we have

\[(10.1) \quad V \leq \begin{cases} 1 + p^{-1/2} & \text{if} \quad 2k_1 = 2\ell_1, \\
1 + p^{-1/2} & \text{if} \quad 2k_1 = 2\ell_2, \\
(1 - p^{-1/2})^{-1} & \text{if} \quad 2k_1, \\
(1 - p^{-1/2})^{-1} & \text{if} \quad k = l_2 = 0 \pmod{3}.
\end{cases}\]

Proof. In the case \(k_3 = 0\) we have \(X_3 = 0\) and \(X_2 = Z_2\) as in the proof of (9.10), and so \(V = X_2^{-1}Y_1Z_2\). If \(k\) is even and \(l_3\) odd, then \(n\) is odd by (6.11), whence \(\eta_1 = \eta_2 = 0\) by (6.4), (6.10), giving \(X_1 = Z_1 = 1\) and \(V = X_1, 1 + p^{-1}X_2\) by (6.5), (6.6), and the first case of (10.1) follows. The second and third cases are dealt with similarly. In the fourth case \(\eta_1, \eta_2, \eta_3\) are each \(\pm 1\), and we have (10.1) unless they are \(1, 1, -1\). From (6.4), (6.9) this is clearly impossible. So (10.1) is proved; and with (6.11) it clearly gives \(V \leq 2\).

It now suffices to prove \(V \leq 2\) in the case \(k_1, k_2, k_3 \neq 0\) since the case \(k_1, k_2, k_3 = 0\) can be dealt with as above. With this assumption, \(p \geq 3\), the \(\eta_1, \eta_2\) all lie between \(1 \pm 1/3\), and (9.7) gives, crudely, the first part of

\[(10.2) \quad V \leq (2X_1)^{-1}(4/3)^2 \leq (17/10)^2\]

For the second part of (10.2), we use \(n \geq 14\), giving \(n \geq 7\) by (6.3), and so \(X_2 \geq 80/81\) by (6.5), (6.6).

Now (10.2) gives \(V \leq 2\) unless \(X_2 < 17/20 < 1 - p^{-2}\), which is possible only if \(n_2 = n_3 < 2\). That, with the \(k_1, k_2\) non-zero, gives \(n_2 = 2, k_2 = n_1 = 1\), whence \(X_2 = Z_1 = 1, X_2 \geq 2\). We can now replace the fourth powers in (10.2) by squares, and \(V \leq 2\) easily follows.

11. Estimation of \(V\) for \(p = 2\). We shall show that

\[(11.1) \quad V \leq 2 \quad \text{if} \quad p = 2.\]

We begin by noting, see [4], p. 58, Theorem 34, that

\[(11.2) \quad a_2a_2 + a_2a_2 + a_2a_2 \geq \begin{cases} a_2^2a_2 + 2a_2a_2 \quad \text{if} \quad a_2 = 0 \pmod{4}, \\
a_2a_2a_2 + a_2a_2a_2 \quad \text{if} \quad a_2 = 0 \pmod{4}.
\end{cases}\]
We use (11.2) to prove that

\[(11.3) \quad \epsilon_i \eta_i \xi_i \neq -1 \quad (i = 1, 2).\]

Taking \(i = 1\) (the same argument works for \(i = 2\)), we note that \(f\) may be supposed to be of the shape (7.1) with \(\varphi_1 = \varphi_1(f)\) null, else (ii) of § 7 gives \(\epsilon_1 = 0\); and (iii) of § 7 may be supposed not to be applicable to \(f\), else again \(\epsilon_1 = 0\). Now by (iv) of § 7 we have \(\eta_1 = 1\) or \(-1\) according as \(\psi_1 = \psi_1(f)\) is of the shape (7.2) or (7.3). Similarly with \(g, \eta_1, \) or \(k, \xi_1,\) for \(f, \xi_1,\)

In the first case of (8.2) the foregoing gives us

\[(11.4) \quad \psi_1(f) = \psi_1(g) + \psi_1(k),\]

and the right member is easily seen to be \(\psi_1\) (7.3) if both or neither of its summands are so, \(\psi \neq \psi_1\) (7.3) if not. That gives \(\epsilon_1 \eta_1 \xi_1 = 1\). In the second case of (8.2) we have to normalize the 3-ary or 4-ary form \(\varphi_1(g) + \varphi_1(k)\) as in (i) of § 7. If it so doing we can use (11.2), as it stands or with one more unary summand on each side, then (11.4) holds with one more binary summand (7.2) on the right, which does not affect the argument. If we cannot so use (11.2) then one of \(\varphi_1(g), \varphi_1(k)\) is a binary form \(\equiv \pm (x^2 + y^2) \pmod{4}\); but then by (iii) of § 7 we have \(\eta_1 = 0\) or \(\xi_1 = 0\). So (11.3) is proved.

From (6.8), with \(p = 2\), we see that (11.3) gives

\[(11.5) \quad X_i - 1 = (X_i - 1)(Z_i - 1) \quad \text{if} \quad \epsilon_i \eta_i \xi_i \neq 0 \quad \text{and} \quad (8.2)_i \quad \text{holds},
\]

\[(11.6) \quad X_i - 1 = \frac{1}{2}(X_i - 1)(Z_i - 1) \quad \text{if} \quad \epsilon_i \eta_i \xi_i \neq 0 \quad \text{and} \quad (8.2)_i \quad \text{holds},
\]

\[(11.7) \quad X_i, Y_i, \text{or} Z_i = 1 \quad \text{if} \quad \epsilon_i \eta_i \xi_i = 0.\]

Suppose now that \(\epsilon_1 = \epsilon_2 = 0\), that is, that \(f, g, k, h\) are of the shape (7.1) with each \(\varphi_i\) null. Then the \(\epsilon_i, \eta_i, \xi_i\) are all non-zero by (iv) of § 7, and (8.2), holds. We multiply the inequality \(V \leq 2\) by \(2X_1 X_2\) and use (11.5) to express it as a linear inequality in the \(X_i, Z_i\), which obviously all lie between 0 and 2. We need only verify it when each of the \(X_i, Z_i\) is 0 or 2. So \(V \leq 2\) is proved in this case.

Next suppose \(\epsilon_1 = \epsilon_2 = 1\). Then each \(\varphi_1(f)\) is non-null, see (8.3) and so each \(\epsilon_i, \eta_i, \xi_i\) is 0 by (ii) of § 7, giving \(X_i = Z_i = 1\). (8.6) gives \(\sigma_i\) or \(\tau_i = 1\), so one at least of \(\eta_i, \xi_i\) is 0 and one of \(X_i, Z_i\) is 1, for \(i = 1, 2\). \(V \leq 2\) is now trivial.

So to prove \(V \leq 2\) we may suppose \(\epsilon_1 + \epsilon_2 = 1\). By symmetry (provided that we do not use \(n_1, n_2\)) we may suppose \(\epsilon_1 = 1, \epsilon_2 = 0\). As above this gives \(\sigma_1, \tau_1 = 1\) and \(X_1 = 1, Y_1, Z_1 = 0\). By symmetry (provided that we do not use \(n < \frac{1}{2}\)) we may suppose \(Y_2 = 1\); and then \(V \leq 2\) can be expressed as

\[(11.8) \quad Y_1 Z_2 + (2 - Y_1)(2 - Z_2)(2 - Z_2) \leq 4 X_1.\]

If we can use (11.5), with \(i = 1\), we substitute for \(X_1\) on the right of (11.8), and it suffices to verify the resulting inequality for \(Y_1 = 3, Y_1 = 2\) and for \(Y_1 = Z_1 = 0\), each of these being trivial.

If we can use (11.6), with \(i = 1\), we substitute for \(X_1\) in (11.8), giving an inequality which can be verified as above if we further suppose \(1/2 < Z_1 < 3/2\); whence we may suppose \(Z_1 = 0\) or 2. But from (7.11) and (7.12), with \(k, f\) for \(f_i, f\) respectively we see that \(Z_1 = 0\); so \(Z_1 = 2\). Now (11.8) reduces to \(Y_1 + Z_1 \leq 3\). From \(Z_1 = Z_2 = 2\) it would follow that \(k\) is null, which is excluded. So \(Z_1 \leq 3/2\) and we may suppose that \(Y_1 = 2, Z_1 > 1\). From \(Z_1 > 1\) and \(Z_2 > 1\) it follows, see (ii) of § 7, that \(h\) is of the shape (7.1) with each \(\eta_i\) null; then the first case of (8.2) holds.

We may therefore assume (11.7), with \(i = 1\). If any two of \(X_1, Y_1, Z_1\) are equal to 1, then, with \(X_1 \neq 0\) as above and so \(1/2 < Z_1 < 3/2\) (11.8) is easily verified. So suppose that exactly one of \(X_1, Y_1, Z_1\) is equal to 1, that is, exactly one of \(\sigma_1, \tau_1, \xi_1\) is 0, see (6.6). By (ii)–(iv) of § 7, exactly one of \(\varphi_1(f), \varphi_1(g), \varphi_1(h)\) is congruent to \(\pm (Q + Q) \pmod{4}\) as in (iii). This is trivially impossible if either of the others is null; so suppose not, that is \(\sigma_1 = \tau_1 = 1\), giving \(Y_1 = Z_1 = 1\) by (ii). Substituting in (11.8), we have an inequality which is easily deduced from (11.7), and the proof of (11.1) is complete.

12. Investigation of \(J,\) see (8.11). We shall show that

\[(12.1) \quad \sum_{J: (8.4) - (8.6)} \leq 1, \quad n \geq 14,
\]

the summation variables being the \(\sigma_1, \tau_1\) of (8.3). With (11.1) this will enable us to replace the last factor on the right of (8.8) by 2; this simple but crude result will however have to be improved in some important cases.

We begin by noticing that if \(\sigma_1 = \tau_1 = 0\) then by (8.6) the \(\sigma_i, \tau_i\) are all zero, so the sum in (12.1) has just one term, which is 1 by (8.11).

Next suppose \(\sigma_1 = 1, \tau_1 = 0\). We have \(\sigma_2 = \tau_2 = 0\), \(\sigma_1, \tau_1\) can only be 1, 1; 0, 1; or 1, 0, by (8.8). In these cases \(\log J = -1, -1, -1\) (12.1) follows at once unless \(\min (k_1, l_1) = 0\), if \(k_1, l_1 = 0\) (8.5) excludes both of the two terms; and if \(k_1, l_1 = 1\) (8.5) excludes one of them, and again we have (12.1).

The case \(\sigma_1 = 0, \tau_1 = 1\) is dealt with in the same way, so now we suppose \(\sigma_1 = \tau_1 = 1\). The sum may now, see (8.6), have up to nine terms. It is easily verified that \(\sigma_1 \sigma_2 + \tau_1 \tau_2 = \sigma_1 + \tau_1 + \sigma_2 + \tau_2 = 2\).

So (8.11) may be rewritten as

\[(12.2) \quad \log_{2}(2J) = \sum_{i} (\sigma_i k_i + \tau_i l_i - n_i),\]
The summand in (12.2) takes the three values 0, \( -k, -l \), by (8.6). Allowing also the value \(-n\), we see that

\[
2 \sum J < (1+2^{-b})(1+2^{-b})(1+2^{-b})(1+2^{-b}).
\]

The 1 in the first bracket on the right corresponds to \( \sigma_1 = 1 \), and the second term in the first bracket corresponds to \( \sigma_2 = 0 \). Similarly in the other brackets. Referring to (8.4), (8.5), we see that any of the factors on the right of (12.3) may be omitted if the exponent \( h_1 \) or \( h_2 \) occurring in it is 0 or 1. So, by omitting factors \( > 2 \), we can deduce (12.1) from (12.3) unless at least three of the \( h_1, h_2 \) are equal to 2; and even then unless the fourth is at most 5. But if so we have \( n \leq 11 \) by (6.11); so (12.1) is proved.

It will be useful later to note that if \( k = h_1 = 1, h_2 = 0 \) then we must have \( \sigma_1 = \sigma_2 = 0 \); and then (8.6) gives \( r_5 = e_5 \). So the sum in (12.1) has at most two terms, with \( r_1 = 1, 0 \) and \( \log_2 J = -1, -1 - n - e_2 \). Similarly for \( h_1, h_2 = 0, 1 \).

13. Simplified estimates for \( T_d(f, k, p, s) \). In this section \( n \geq 14 \), \( p \geq 2 \) is fixed, and \( n_2 = n_2(f, p) < \frac{1}{3} n \), \( k < \frac{1}{3} n \) are assumed. By (9.8), with \( V \leq 2 \) by Lemma 2, we have for odd \( p \) the second of

\[
T_d(f, k, p, s) \leq \begin{cases} 2 \{P(k, p)\}^{-1} p^{-\frac{s}{d}} & \text{if } n_2 < 1, \\ 2 \{R(k, p)\}^{-1} p^{-\frac{s}{d}} & \text{if } n_2 \geq 2. 
\end{cases}
\]

For the part first, see (9.2), (9.3). For \( p = 2 \) see (9.9), (11.1), (12.1), and we again have (13.4). For \( s = 1, 2, \sigma = 1 \), (6.11), at the fourth of (9.4). For odd \( p \) and \( n_2 = 0 \), \( p \neq d \), we have \( e = t = 0 \), and we must improve on \( V \) on the right of (13.1). We define (for odd \( p \))

\[
V_\sigma(p, k) = (1 + e_\sigma p^{-\frac{2}{d}}) (1 - p^{-\frac{1}{d} - \min(\sigma, 1 - \min(\sigma, 1))}).
\]

where \( e_\sigma_1 = 1 \) if \( 2 | h_k \), and \( \min(\sigma, 1 - \min(\sigma, 1)) \) is the minimum taken over the even \( \sigma \). With \( n_2 = 0 \) we have \( t = 0 \), \( e_1 = t = k_1 = 0, k_2 = k \), by (6.11), so we can appeal to Lemma 2; note that \( V_\sigma(p, k) \) is an upper bound for the right member of (10.1). So using \( V \leq V_\sigma(p, k) \) instead of \( V \leq 2 \), we have in place of the case \( n_2 = 0 \) of (13.1)

\[
T_d(f, k, p, 0) \leq \{P(k, p)\}^{-1} V_\sigma(p, k) \quad \text{if } n_2 = 0 \text{ and } p \neq d.
\]

By defining \( V_\sigma(2, k) = 2 \), this is valid also for \( p = 2 \).

In the case \( k = 1 \) we may write (13.3) more explicitly, noting that \( P(1, p) = 1 \) by (5.1), as

\[
T_d(f, 1, p, 0) \leq (1 - p^{-\frac{s}{d}}) \quad \text{if } n_2 = 0 \text{ and } p \neq d.
\]

The case \( n_2 = 1 \) of (13.1) can be improved on by an argument like that leading to (13.3); we shall do so for \( k = 1 \). Taking \( k_1 = 1, k_2 = 0 \), \( t = n_2 = 1, t = 1 \) and \( s = 0 \), \( t = 1 \), we find

\[
T_d(f, 1, p, 0) \leq p^{-1} \left(1 - p^{-\frac{s}{d}}\right) \quad \text{if } n_2 = 1 \text{ and } p \neq d.
\]

For \( p = 2 \) (13.4), (13.5) hold with \( 2, 2^{1/2} \) on the right; but these estimates are too weak, and will be improved in the next section.

Now take \( k = 2 \); (13.3) is true but of little use because \( T_d(p, 2) \) has a factor \( 1 + p^{-1} \), and the product \( \int (1 + p^{-1}) \) is divergent. Take first the case \( p > 2, h_1 = 0 \); we have \( h_2 = n_2 - 2 \geq 5 \). If \( n_2 \) and \( t_3 \) are odd, \( X_1 = Z_1 = 1 \), and (9.10) simplifies. If not, the worst case is \( n_2 = 0, X_1 = 1 + p^{-d} \), \( Z_1 = 1 + p^{-d} \). In either case (9.10) gives

\[
T_d(f, 2, p, 0) \leq (1 - p^{-\frac{s}{d}}) (1 - p^{-\frac{1}{d}}) p^{-n_2} L_d(p) \quad \text{if } p \neq d.
\]

Similarly, but more crudely, (9.11) gives

\[
T_d(f, 2, p, 1) \leq 3 p^{-n_2} L_d(p) \quad \text{if } p \neq d.
\]

From (9.12) and \( V \leq 2 \),

\[
T_d(f, 2, p, 0) \leq 2 p^{-\frac{1}{d} - \min(\sigma, 1 - \min(\sigma, 1))} \quad \text{if } 2 \leq p \neq d,
\]

using (9.5), (9.9), (11.1) and (12.1); these also give

\[
T_d(f, 2, 2, s) \leq (6) 2 p^{-\frac{1}{d} - \min(\sigma, 1 - \min(\sigma, 1))} \quad \text{if } 2 \neq d,
\]

with \( s = 0 \) and \( t = 2n_2 \), or \( s = 1 \) and \( t = 2n_1 \).

14. A sharper estimate for \( T_d(f, 1, 2, 0) \). (13.1), with \( B(1, p) = P(1, p) = 1 \), and \( k_1 = 1, t = n_2 \), gives

\[
T_d(f, 1, 2, 0) \leq 2^{1 - \min(\sigma)} \quad \text{for } 2 \neq d \text{ and } n \geq 14.
\]

We shall prove that

\[
T_d(f, 1, 2, 0) \leq 2^{1 - \min(\sigma)} + (1 - e_0) \left(2^{-\min(\sigma)} + 2^{1 - \min(\sigma)}\right)
\]

if \( 2 \neq d \); and note that the second term on the right is zero if \( 2 \neq n_2 \), since then \( n_2 = 1 \) by (7.4) and \( e_0 = 1 \) by (8.3).

With \( k_2 = 1, t = 0 \), \( t = n_2 = 1 \), the right member of (9.9) reduces to \( 2^{1 - \min(\sigma)} \sum J \), with summation over the two possibilities mentioned at the end of §12. In the term with \( \log J = 1 - n_2 - e_2 \leq 1 - n_2 \), we use (11.1) to give \( V \leq 2 \); whence the third term on the right of (14.2).

In the other term, \( t_1 = 1 \), \( J = 1 \), and we have \( e_1, e_2 = 1, 0, e_1 \); from which, by using (6.6), (6.10), (7.4), (7.5) and §7, (ii) it may be deduced that \( Y_1 = 2, Y_2 = 1, X_1 = 1, Z_1 = 1 \). Now (9.7) simplifies to \( V = X_1^{-1} Z_1 \). If \( e_2 = 1 \), then \( e_2 = 0 \) and (8.6) give \( t_2 = 1 \), and \( X_1 = Z_1 = 1 \) follows, giving \( V = 1 \), and (14.2) follows on using \( J \).
So we suppose \( X_1, Z_1 \neq 0 \); and write for brevity \( m = m_1(f) \), whence by (6.8) we have \( X_1 = 1 \) or \( 1 \pm 2^{-m} \). Now if we suppose \( Z_1 = 1 \), (14.2) follows if \( t + 2m = n_1 + 2m \geq n_2 + n_1 - 2 = n - 2 \), by (7.4). So we suppose \( Z_1 \neq 1 \); and we assume \( s_1 = 0 \), giving \( a_1 = t_2 = 0 \) by (8.6), whence \( f, g, h \) are all of the shape (7.1) with \( \varphi_1 \) null; but \( a_1 = t_1 = 1 \) gives \( \varphi_1 = 0 \), so none of them has \( \varphi_1 \) null.

If \( \varphi_0(h) \) is unary we may suppose

\[
g = dQ = da_1, \quad h = aQ + 2p_1 + 4p_2, \quad f = dQ + aQ + 2p_1 + 4p_2,
\]

with \( 2 \) odd and \( h \) normalized as in § 7. From that normalization it easily follows that \( Z_1 = 1 \pm 2^{-m} \) and \( X_1 = Z_1 \), if \( ad = -1 (\mod 4) \), \( X_1 = 0 \) if not. So we have \( V \leq 1 \pm 2^{-m} \), from which, as above, (14.2) follows.

So we suppose \( \varphi_0(h)(b) \) binary and

\[
g = dQ, \quad h = aQ + bQ + 2p_1 + 4p_2, \quad 2 \text{ odd.}
\]

On normalizing \( f = g + h \) we see that \( m_1(h) = m - 1 \), see (8.2); so, by the remark following (9.1), we see that the weaker inequality \( |V| \leq (1 - 2^{-m-1}) \) would suffice to prove (14.2). If \( ab = 1 (\mod 4) \) we see from § 7, (iii) that \( Z_1 = 1 \), giving \( V \leq (1 - 2^{-m-1}) \). So suppose \( ab = -1 (\mod 4) \). We have \( X_1 = 1 \pm 2^{-m} \), \( Z_1 = 1 \pm 2^{-m} \), and we have what is wanted if the signs are the same. This follows from (11.2), with \( a, b, d \) for the \( a_1 \), using (14.3) and \( ab = -1 (\mod 4) \); see (11.4) and the following remark. So the proof of (14.2) is complete.

In it, we have not used \( n \leq n/2 \); so if \( n_1 = n_2 = n/2 \) is odd, then (14.2) holds for \( 2d \) with \( \varphi_1 \) for \( \varphi_2 \), since interchange of \( g, h \) then interchanges the cases \( 2d, d \), see (3.4) and (7.1). (14.1) can also be improved when \( \varphi_2 = 0 \); for then \( f = g + h \) is obviously impossible if \( g = da, d = 2 (\mod 4) \).

So the left member is 0.

15. Estimation of \( U_d(f, k, p) \). From (6.11) and (9.4) we see that the sum in (3.6) may be taken over \( 0 \leq 2s \leq \min(k, n) \) if \( p \mid d \). If \( 2s + 1 \leq \min(k, n) \) if \( p \nmid d \). So, from (3.0),

\[
U_d(f, k, p) = T_d(f, k, p, 0) \text{ unless } \min(k, n) \geq 2 \text{ or } 3 \text{ according as } p \mid d \text{ or } p \nmid d.
\]

For \( k = 2 \), \( p \mid d \), we sum over \( s = 0, 1 \) and (13.6), (3.7) gives

\[
U_d(f, 2, k, p) \leq (20(19)p^{-n_2} + 3p^{-n_2})L_d(p)
\]

if \( 2 < p \mid d \) and \( n_2 \geq 2 \).

From (13.9) we have

\[
U_d(f, 2, 2) \leq (8/3)(2^{-n_1} + 2^{-n_2}) \text{ if } 2 \nmid d \text{ and } n_2 = n_2(f, p) \geq 2.
\]

We shall now prove that for \( k \geq 3, n_2 \geq 3 \), we have

\[
\sum_p \left\{ p^{-1}; (3.4) \right\} \leq p^{1 - 10(1 + p^{-1})} \quad \text{if } p \mid d.
\]

We use (6.11) and (9.4) to express \( t \), defined by (6.14), as \( 8s^2 + 2as + b \), where \( a, b \) are integers independent of \( s \); and we denote by \( n \) the smallest value of \( t \). We note that \( t \) is constant modulo 2 and takes no value more than twice. If \( 4 \mid t \), \( t \) is constant modulo 8; if not, \( t \) takes no value twice. (The foregoing may be clearer if we first use (6.11), (14.4) to express \( t \) as \( 2k_2 + a'k_1 + b' \), \( a', b' \) integers depending only on \( n, k, l, \)). Obvious upper bounds for the left member of (15.4), in the two cases just distinguished, are

\[
2p^{-10(1 + p^{-1})} \leq \left( 1 + p^{-1} + p^{-2} + \cdots \right) < p^{-10(n-1)}
\]

and \( p^{-10(1 + p^{-1})} \), which is smaller.

(15.4) follows at once if \( n \geq n + 1 \); that is, that some \( s \) gives \( t \leq n + 1 \). If \( \min(k_1, k_2) \geq 3 \) then \( \frac{t}{2} \geq 3(n_1 + n_2) = 3k_1 \geq 2n + n_2 \geq 7 \), so we suppose that \( t \leq n + 1 \) for some \( k_1, k_2 \) with \( \min(k_1, k_2) \leq 2 \) and \( 2k_2 \). Straightforward calculation shows that this is the case only if \( k_1 = 3, k_2 = 3 \), which we therefore assume.

Now the sum in (15.4) has just two terms, with \( k_1 = 1, 3 \). For \( k = 3 \) these terms have \( t = n - 4 + n_2 > n - 1 \) and \( t = 3n - 3n_2 > n + 7 \), giving (15.4) with much to spare unless \( n_2 = 3 \). If \( n_2 = 3 \) we have \( t \geq n - 1 - 20 \), and (15.4) follows. So suppose \( n_2 = 3, k = 4 \). Now the two possible \( t \) are \( n - 4 + n_2 > n \) and \( 3n - 3n_2 > n + 7 \), which crudely gives (15.4).

From (15.4) and (13.1) we have

\[
U_d(f, k, p) \leq 2R(k, p)^{-1}(p^{-2k_2} + p^{-1})
\]

if \( 2 \leq p \mid d \) and \( n_2 \geq 2 \).

If \( p \nmid d \) then \( s = 0 \) gives \( k_0 = 0, t = kn_1 \). Omitting this term from the sum on the left of (15.4), we proceed as above and obtain an estimate which with (13.1) gives

\[
U_d(f, k, p) \leq 2R(k, p)^{-1}(p^{-2n_2} + p^{-1})
\]

if \( 2 \leq p \mid d \) and \( n_2 \geq 2 \).

In this section we have used the hypothesis \( n \geq 14, k \leq 2n \), \( n_2 \leq 3n/2 \); we shall do so till the end of the proof.

16. Estimation of \( S_d(f, 2) \) and \( S(f, 2) \). We continue the special treatment of the case \( k = 2 \), so as to obtain a substitute for (3.9). It will be convenient (for all \( k \)) to define

\[
D = D(f) = \det [\frac{1}{1} A(f)], \quad D_1(f) = \prod_p \left\{ \begin{array}{c} p \mid n_2(f, p) \end{array} \right\}
\]
whence $D_{r+1}D_r$, $D_r$ is square-free, $D_r = 1$ for $r \geq \frac{1}{2} n$, and $D = D_1 D_2 \ldots$

Clearly we have, from \( g + h = f \),

\[
D(f) = D(g)D(h), \quad D_r(f) = \text{l.c.m.} \{D_r(g), D_r(h)\}.
\]

Restricting $k$ to be 2 for the rest of this section, we first show that

\[
S_2(f, 2) = S_3(f, 2) = 0.
\]

To see this, note that $g = D_2(g)G$ for some $G$ of the shape $(6.1)$, $(7.1)$ with $D(G) = d$ by $(3.4)$. Trivially, $G \sim a_1^2 + a_2^2$ if $d = 1$ or 2; so $G$ and $g$ are disjoint and $(1.4)$ fails. The right member of $(3.5)$ vanishes, and $(16.3)$ follows by $(3.6)$, $(3.7)$.

We now express the product in $(3.7)$ as $\Pi_1 \Pi_2 \Pi_3$, where $\Pi_1$ is taken over $p \divides d$, $\Pi_2$ over $p \divides d$, and $\Pi_3$ if $2 \divides d$, $\Pi_3 = 1$ if 2 does not divide $d$. We use $(15.1)$ and $(13.6)$ if $n_2(f, p) \leq 1$, $(15.2)$ if not. This gives

\[
U_\nu(f, 2, p) \leq (1 - p^{-1}) (1 - p^{-2}) L_\nu(p) p^{-n_2}
\]

in the first case and a stronger estimate in the second. We extend the product

\[
\prod \{1 - p^{-1} (1 + p^{-1})\}
\]

taken over $p \divides d$, to all odd $p$ and then estimate it numerically. So, assuming $d \geq 3$ and using $(3.7)$, as we may by $(16.3)$, and using also $(16.2)$, we have

\[
\Pi_1 < 0.385d \prod \{p^{-1} : p \divides d, p \divides d_1\}.
\]

Here $\Pi_1$, on the right, is $D_1^{-1}d$ if $D_1$ is odd, $2D_1^{-1}d$ if not.

For $\Pi_2$ we use $(15.1)$ and $(13.8)$, giving

\[
\Pi_2 < \varepsilon(d) d^{-1} \text{ln} d,
\]

where $\varepsilon(d) = \prod \{2 : p \divides d\}$ is the number of divisors of the square-free integer $d$.

Now $\Pi_3$ is 1 by definition unless $d$ is odd. So, by $(13.9)$, $(15.1)$, $(15.3)$ it is at most $8/3$ if $2 \divides D_1$, $4/3$ if $2 \divides D_2$, but $2 \divides D_1$ and considerably smaller than $4/3$ if $2 \divides D_2$. Replace $\Pi_3 = 1$, by $4/3$ for the even $d$, which are less important; then $(16.4)$ and $(16.5)$ give

\[
S_3(f, 2) < 1.027D_1^{-1} \varepsilon(d) d^{-1} \text{ln} d.
\]

By estimating the infinite series $\sum \varepsilon(d) d^{-1}$, taken over square-free $d \geq 3$, we deduce from $(16.6)$, $(16.3)$, $(3.3)$ that

\[
S(f, 2) < 0.0342D_1^{-1} \text{ln} 3^2\text{ln} 16.
\]

With the obvious $S(f, 2) = S_3(f, 2)$ if $D_1 = 3$, $(16.7)$ gives

\[
S(f, 2) < 0.0085 \cdot 3^2 \text{ln} 16.
\]

The argument for $(16.3)$ shows that if $d = 5$ then $g$ is equivalent to a multiple of $a_1^2 + 6a_2 a_3^2$, $a = 1$ or 2, whence $S_4(f, 2) = 0$ and so we have, from $(16.6)$,

\[
S(f, 2) < 0.0033, \quad S(f, 2) < 0.0043 \quad \text{for} \quad d = 5, 6.
\]

**17. Proof of Theorem**

For $n \geq 19$, we shall estimate $M_\nu(f, k)$ defined in $(3.8)$, by using the bounds we have found for $U_\nu(f, k, \nu)$. We have to consider the three cases $n_2(f, p) = 0, 1, \geq 2$. In the first of these cases the second term on the right of $(3.8)$ is 0, because $h_0 \leq n_0 = 0$ and so, see $(9.4)$, $(3.4)$ is impossible if $p \divides d$. We shall verify that the bound is always greatest in the case $n_2 = 0$.

First suppose that $k = 1$ and $p \geq 2$. For $p \divides d$ we refer to $(13.4)$, $(13.5)$, and $(13.1)$ (with $k_1 = 0, t = n_1 = 2$) in the three cases distinguished above. For $p \notin d$ we refer, in the second and third cases, to $(13.1)$ (with $k_1 = 1, t = n_1 = 7$). And we also need $(15.1)$. It is easily seen that the italicized assertion above is correct and so $M_\nu(f, 1) \leq (1 - p^{-1})^{-1}$.

Next suppose $k = 1$ and $p = 2$. We refer to $(14.1), (14.2)$, and $(15.1)$, and we similarly find that $M_\nu(f, 2) \leq 2^{-1} + 2^{-1} + 2^{-1}$.

The product $\prod (1 - p^{-1})^{-1}$, taken over all odd $p$, is easily seen to be less than 1.0005. So from the foregoing inequalities, and $(3.9)$, we have

\[
S(f, 1) < 0.599.
\]

For $k \geq 3$, and all $p_k$, we proceed similarly, referring to $(13.3)$ and $(15.1)$, $(13.1), (15.1), (15.6), (15.5)$; with $k_1 = 0$ and so $t = k$, $h_0 = h_0 = 0, k$ in the first two cases. The obvious $V_\nu(p, k) > 1$ is enough to show that the italicized assertion above is correct, so, with $V_\nu(k, k) = 2$.

\[
M_\nu(f, k) < k!^{-1} V_\nu(p, k) \quad \text{for} \quad k \geq 3.
\]

We now refer to $(13.2)$ and estimate $\prod V_\nu(p, k)$, taken over $p \geq 3$; we also use $(5.5)$. The necessary calculations can be made finite by noticing that $V_\nu(k, p) \leq \max(2, p)$ for $k \geq 9$. So we find from $(3.9)$ and $(17.2)$ that

\[
S(f, k) < 3.31, 4.64, 3.60, 3.90, 3.70, 3.78
\]

for $k = 3, \ldots, 7$ and $k \geq 8$ respectively.

The foregoing estimates hold for $n \geq 14$ and so can be used later; for the rest of this section we suppose $n \geq 19$.

From $(1.2), (2.8), (17.1), (16.9)$ and $(17.3)$ we find, very crudely, that

\[
W(f) < 0.6 (1, n - 1) + 0.1 (2, n - 2) + \quad \quad + 10 \sum \{\theta(k, n - 8) : 3 \leq k < \frac{3}{4} n\}.
\]

Using the first part of $(4.5)$, $(17.4)$ gives

\[
W(f) < 0.6 \theta(1, n - 1) + 0.1 \theta(2, n - 2) + 18 \theta(3, n - 3) + \quad \quad + 18 \theta(4, n - 4)
\]
(by estimating separately the sums over \( k = 5, 7, \ldots \) and \( k = 6, 8, \ldots \)). Then with the second part of (4.5), (17.5) gives

\[(17.6) \quad W(f) < 1.5 \delta(1, n - 1) + \delta(2, n - 2).
\]

Now we use (4.2), noting that the right member is greatest for \( n = 19 \) and then equal to \( \pi^3/8! < 10^4/8! < 4 \). So (17.6) gives, with (4.1),

\[(17.7) \quad W(f) < 1.75 \delta(1, n - 1) = 1.75 \pi^{3n}/I(\frac{1}{2}n).
\]

By the remark following (4.2), \( \pi^{3n}/I(\frac{1}{2}n) \) is greatest for \( n = 19 \), and less than \( (\pi^8/3) \pi^{3n-1}/I(\frac{1}{2}n - \frac{1}{2}) \). So (17.7) gives

\[(17.8) \quad W(f) < 1.75 (\pi^8/3) \pi^{3n}/I(\frac{1}{2}n - \frac{1}{2}) < 1.1 \pi^3/8! < 3.5 \pi^8/8!,
\]

which is less than \( 7/8 \) since \( \pi^3 < 10 \) and \( 8! = 40320 \).

The proof for \( n > 19 \) is now complete; and later, using \( n \leq 18 \), we shall have bounds for \( \delta \), which will enable us to make use of (1.4). We shall see that (17.3) is very weak for \( 3 < k < 7 \).

18. Further inequalities for \( S(f, 1) \). As shown in §17,

\[(18.1) \quad M_p(f, 1) \leq \begin{cases} (1 - p^{-s})^{-1} & \text{for } p \geq 3, \\ 2^{-4 + 2^{-s}} & \text{for } p = 2. \end{cases}
\]

When \( p | D(f) \), see (16.1), we need to do better. By (15.1) and (3.8), for such \( p \), we have

\[(18.2) \quad M_p(f, 1) = \max_{p \in \mathcal{P}} \tau_d(f, 1, p, 0) + \max_{p \in \mathcal{P}} \tau_d(f, 1, p, 0).
\]

Supposing first that \( p \geq 3 \) and \( p | D(f) \), that is, \( n_2 = n_2(f, p) \geq 1 \), we use (9.8), with \( R(1, p) = 1 \) by (5.2) and with \( k_2 = 0, 1 \) for \( p \neq 2 \), \( p | D(f) \) respectively, whence by (6.11), (6.14) we have \( \theta = \eta_1, \eta_1 \). By Lemma 2 the \( V \) in (9.8) satisfies (10.1) when \( k_2 = 0 \) and \( V \leq 2 \) for \( k_2 = 1 \). In (10.1)

\[
(18.3) \quad M_p(f, 1) \leq \pi^{-1} \tau(1 + p^{-1})(1 + 2p^{-1})
\]

if \( 1 \leq r \leq 3 \) and \( 2 < p | D(f) \).

(Check this first for \( n \geq 14 \) with the additional assumption \( p : D(f) \), that is, \( n_2 = r \). Then note that the right member of (18.3) is a decreasing function of \( r \), and that we can obviously do better if \( n_2 \geq 4 \).)

Now taking \( p = 2 \) we use (14.1) and (14.2), with the remarks at the end of §14. Considering the cases \( n_2 = 1, \ldots, 6 \); \( n_2 = 7 \) and \( n = 14 \); \( n_2 = 7 \) and \( n \geq 15 \); and \( n_2 = 8 \), implying \( n \geq 16 \), separately it will be found that

\[(18.4) \quad M_p(f, 1) \leq 2^{-1 - r} (1 + 2^{-r} + 2^{r-12})
\]

if \( 1 \leq r \leq 3 \) and \( 2 | D(f) \).

Now from (3.9), and the above estimates, we have an inequality for \( S(f, 1) \) which we simplify by estimating \( \frac{1}{1+p^{-1}} \) and \( \frac{1}{1+2p^{-1}} \), taken over all odd \( p \). We find that

\[(18.5) \quad S(f, 1) \leq \min \{ 0.509, 0.521D_4^{-12}, 0.541D_4^{-12}, 0.585D_4^{-12} \}.
\]

We notice now that if \( f \) is of the shape (7.1) with \( q \) null then \( f \tau \) is \( \mathcal{A} \) and \( \mathcal{B} \) in (18.2) is zero. Using (14.1) to estimate the second term, we can crudely multiply the right member of (18.5) by \( 1/4 \).

19. Further inequalities for \( S(f, k) \), \( 3 \leq k \leq 8 \). We shall show that

\[(19.1) \quad S(f, k) < S^{(0)}(f, k) \quad \text{for } k = 3, \ldots, 8,
\]

where

\[(19.2) \quad S^{(0)}(f, 3) = 3.57D_4^{-12} \tau (D_4)(D_4, 2)^{-1},
\]

\[(19.3) \quad S^{(0)}(f, 4) = 5.22D_4^{-12} \tau (D_4)(D_4, 2)^{-1},
\]

\[(19.4) \quad S^{(0)}(f, 5) = 4.58D_4^{-12} \tau (D_4)(D_4, 2)^{-1},
\]

\[(19.5) \quad S^{(0)}(f, 6) = 3.90D_4^{-12} \tau (D_4)(D_4, 2)^{-1},
\]

\[(19.6) \quad S^{(0)}(f, 7) = 3.70D_4^{-12} \tau (D_4)(D_4, 2)^{-1},
\]

\[(19.7) \quad S^{(0)}(f, 8) = 3.78D_4^{-12} \tau (D_4)(D_4, 2)^{-1}.
\]

Here \( D_4 = D_4(f) \), see (16.1), is square-free, \( \tau \) is the divisor function, and \( (D_4, 2) \) is the g.c.d. of \( D_4, 2 \); so \( \tau (D_4)(D_4, 2) = 2^\nu \), where \( \nu \) is the number of odd primes dividing \( D_4 \).

To prove (19.1) we need to show

\[(19.8) \quad M_p(f, k) \leq 2 \{ R(k, p); \}^{-1}(p^{-1} + p^{1+4}) \quad \text{if } k \geq 3 \text{ and } p | D(f).
\]

To prove (19.8) in the case \( n_2(f, p) = 1 \), we refer to (13.1) and (13.5), taking \( k_2 = 0, 1 \), which by (6.11) and (6.14) give \( t = k_1 \), respectively. For \( n_2 \geq 2, (15.6) \) and (15.5) give

\[
M_p(f, k) \leq 2 \{ R(k, p); \}^{-1}(p^{-1} + p^{1+4} + p^{1+6} + (1 + p^{-19}),
\]

from which, with \( R(k, p) > (2/3)P(k, p) \) (as is easily verified), (19.8) follows, crudely.

We compare (19.8) with the inequality (17.2) used in §17 and so see that (17.3) and (19.8) give, for \( k \geq 3 \),

\[(19.9) \quad S(f, k) \leq S_0D_4^{-12} \prod_{\nu} \{ 2(1 + p^{1+6}) \} \tau (p, k); \}^{-1}(p, D_4),
\]

where \( S_0 \) is the bound for \( S(f, k) \) given by (17.3), and \( V_0(p, k) = 2 \) if \( p = 2 \) is given for odd \( p \) by (15.2).
The cases \( k = 7, 8 \) of (19.1) follow at once from (19.9) on using \( V_b(p, k) = 1 \) for \( p = 3, p = 2 \), and \( k - \frac{1}{2} n \leq 1 \) since \( n \geq 14 \). For \( k = 3, 5 \), the product \( \prod \) in (19.9) is at most
\[ \tau(D_1)(D_1, 2)^{-1} \prod (1 + p^{b-7}); \]
and taking this last product over all \( p \) (19.1) follows. For \( k = 4 \) we have
\[ V_b(p, 4) = (1 + p^{-7})(1 - p^{-2}) > 1 + p^{-13} \quad (p \neq 2) \]
and this completes the proof of (19.1), since it makes \( \prod \) in (19.9) \( \leq (1 + 2^{-7} + 3^{-7})\tau(D_1)(D_1, 2)^{-1} \).

We shall now show that
\[ M_p(f, k) \leq 2 \left( \frac{L(k, p)}{P(k, p)} \right)^{-1} (1 - p^{-2})^{-1} \]
if \( 4 \leq k \leq 7 \) and \( p|D_2 \).

If follows easily from (5.1), (5.2) that \( R(k, p) \geq (1 - p^{-2})P(k, p) \) for \( 4 \leq k \leq 7 \), so (19.10) follows from the second part of (13.1) by the argument used to deduce (19.8) from the first part, if we prove a suitable inequality for \( \sum p^{-b} \), with summation over \( 0 \leq k_2 \leq \min(k, n_2) \), see (6.11), with \( t \) given in terms of \( k_2 \) by (6.11), (6.14). Omitting the term with \( k_2 = 0 \), \( t = k_2 \), which is at most \( p^{-b} \) since \( p|D_2 \) implies \( n_2 \geq 2 \), what we need is
\[ \sum (p^{-b} : 1 \leq k_2 \leq \min(k, n_2)) \leq p^{-13} + p^{-n} \]
if \( p|D_2 \) and \( 4 \leq k \leq 7 \).

To prove (19.11), note first that if \( n_2 = 2 \) then the sum has just two terms, with \( t = n - 2, 2n - 2k, 3n - 3k, \cdots, n + 3, n + 8 \). For \( n_2 = 4 \) there are four terms, each with \( t \geq n + 3 \). For \( n_2 > 4 \) there are at most seven terms, each with \( t \geq n + 4 \), implying \( p^{-b} \leq \sum p^{-b} \). So (19.11) is proved and (19.10) follows.

Estimating the ratio of the right member of (19.4) to that of (19.8), we find that
\[ S(f, k) \leq S^{(0)}(f, k) D_1^{-1} \left( (1 - p^{-2})^{-1} (1 + p^{b-1+n}) : p|D_2 \right) \]
if \( 4 \leq k \leq 7 \).

20. Use of condition (1.4); proof of theorem for \( n = 18 \). We shall prove:

LEMMA 3. (i) Suppose \( 2 \leq k \leq 13 \), \( D(g) = 1 \), and \( k \neq 12 \) if \( g \) is primitive. Then either \( W_4(g) = 1 \) or \( k = 8 \) and \( g \sim 2(a_1 a_4 + a_3 a_6 + a_5 a_8 + a_2 n_4) \).

(ii) (i) holds also with \( k \) in place of \( k, g \), in hypothesis and conclusion.

Proof. Obviously we need only prove (i). With \( D(g) = 1 \), \( g \) is of the shape (7.1) with \( n_3 \) and \( n_6 \) each null; so we write \( g \sim p \cdot 2 q^\epsilon \).

Suppose first that \( g \) is primitive, that is, \( g \) is not null. Then, see (7) of (7), there are at most two possibilities for \( g \) up to \( \tau \). We may take these to be
\[ \left( \begin{array}{c} 20.1 \end{array} \right) \]
\( g \sim 2 \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{k-2}^2 + \epsilon(\alpha_{k-1}^2 + \alpha_k^2) \), \( \epsilon = \pm 1 \).

From \( D(g) = 1 \) it follows trivially that (20.1) holds also, for either value of \( \epsilon \), for every prime \( p > 2 \). By [4], p. 72, Theorem 43, two forms \( f, f' \) with \( f \sim \tau \) for every \( p \) have signatures with the same residue modulo 8. So from (20.1), with \( g \) positive definite, we have \( k = -2 + 2q(\mod 8) \). \( \epsilon = 1 \). Now (0.3), with \( g, f \) for \( f_1 n_4 \), and \( 2 \leq k \leq 18 \), shows that \( g \) is equivalent to a disjoint form with a unary summand. Trivially, this holds for every \( g \) in the genus under consideration, and \( W_4(g) = 1 \) follows.

Now suppose \( g \) imprimitive, that is, \( g \sim \tau \cdot 2 q^\epsilon \). Then obviously \( 2|k \); but if \( k = 2(\mod 4) \) then \( D(g) = -1(\mod 4) \) gives a contradiction, so \( 4|k \). Now \( g \) of the shape (7.3) gives \( D(g) = -3(\mod 8) \), so \( g \) is of the shape (7.3), that is,
\[ \left( \begin{array}{c} 20.2 \end{array} \right) \] \( g \sim 2 (a_1 a_4 + a_3 a_6 + \cdots + a_{k-1} a_k) \), \( k = 0(\mod 4) \).

This is trivially true also with \( g \sim \tau \). So the argument used to exclude \( \epsilon = -1 \) in (20.1) gives \( k = 0(\mod 8) \), whence with \( k < 16 \) we have \( k = 8 \); and the proof is complete.

When \( D(f) = 1 \), or equivalently \( D_1 = D_1(f) = 1 \), \( f \sim g + h \) implies \( D(g) = D(h) = 1 \). So an obvious corollary of the lemma is
\[ \left( \begin{array}{c} 20.3 \end{array} \right) D_1(f) = 1 \Rightarrow S(f, k) = 0 \quad \text{for} \quad k = 2, \ldots, 7. \]

Using (19.1)–(19.6) if \( D_1 \geq 2 \), and (30.3) if not, we have
\[ \left( \begin{array}{c} 20.4 \end{array} \right) S(f, k) < 1.27, 1.31, 0.97, 0.74, 0.66 \quad \text{for} \quad k = 3, \ldots, 7. \]

We shall see that (20.4) is very weak, but it suffices to prove the theorem for \( n = 18 \). For taking \( n = 18 \) in (1.2) and (2.8), and referring to the fifth column of the table in §4, we find \( W(f) \leq 0.6 \) on using (20.4) for \( k = 3, \ldots, 7, (17.3) \) for \( k = 8, 9, \) and (17.1), (16.8) for \( k = 1, 2 \).

21. Improvement on (19.1)–(19.6). We begin by defining \( N_p(f, k) \), for \( 3 \leq k \leq 7 \), in the same way as \( \mathcal{M}_p(f, k) \), see (3.8), but with the additional summation condition \( p|D(g) \); and we prove
\[ \left( \begin{array}{c} 21.1 \end{array} \right) \]
\( N_p(f, k) \leq 2 (L(k, p))^{k-1} \cdot 1 \).

If \( p|D(f) \) this is trivial, since \( p|D(g) \) is impossible. If \( n_2 = n_2(f, p) = 1 \), then (21.1) follows from (18.8) by suppressing the term involving \( p^{k-1} \), which corresponds to \( k_2 = 0 \), implying \( p|D(g) \). For \( n_2 \geq 2 \), we suppress
the term in $p^{-k}$, again corresponding to $k = 0$, in the formula used to prove (19.8) for $n_2 \geq 2$.

Next, for any integer $m \geq 2$, define $S^{(m)}(f, k)$ in the same way as $S(f, k)$, see (2.7), but with the additional condition $m|D(g)$. Then, cf. (3.9), we have obviously

\begin{equation}
S^{(m)}(f, k) \leq \prod \{M_p(f, k) : p|m\} \prod \{N_p(f, k) : p|m\}.
\end{equation}

If we compare (21.1) with (19.8), which suffices to prove (19.1), we see that (21.2) gives

\begin{equation}
S^{(m)}(f, k) \leq S^{(0)}(f, k) \prod \{p^{-k+i}: p|m\}.
\end{equation}

More precisely, the factor $p^{-k+i}$ in (21.3) may be replaced by $p^{-k+i}(1+p^{-k+i})^{-1}$. It is worth while to do so if $k = 6$; for then the factor $1+p^{-1}$ in (19.5), which could have been replaced by $1+p^{-k+i}$, can be omitted, for $p|m$. Similarly, if $k = 7$, for some factors $2$ in (19.6), which could also be replaced by $1+p^{-k+i}$.

Now let $\mathcal{A}$ be a finitely set of integers $n \geq 2$ such that

\begin{equation}
(1.4) \Rightarrow m|D(g)
\end{equation}

(for fixed $f, k$ with $3 \leq k \leq 7$). Then, because of the condition (1.4) in (2.7) we have

\begin{equation}
S(f, k) \leq \sum \{S^{(m)}(f, k) : m \in \mathcal{A}\}.
\end{equation}

Lemma 3 now shows that (21.4) holds if $\mathcal{A}$ is taken to be the set of primes dividing $D(f)$, or $D(f)$; for then, with $D(g)|D(f)$, (21.4) is equivalent to $D(g) > 1$. So we have

\begin{equation}
S(f, k) \leq S^{(0)}(f, k) \sum \{p^{-k+i}: p|D_1\}
\end{equation}

for $3 \leq k \leq 7$.

We now improve the case $k = 3$ of (21.6) by proving

\textbf{Lemma 4.} If $k = 3$ and either $D_1(g) \leq 6$ or $D_1(g) = 10$, then $W_1(g) = 1$.

Proof. We have to prove that every 3-ary $g$ with $D_1(g) \leq 6$ or $D_1(g) = 10$ is equivalent to a disjoint form, with one summand conversely of rank 1. Obviously, see (16.1), $D(g)|D_1(g)$; but by taking out the divisor of $g$ we may suppose $D(g)|D_1(g)$; by considering the reciprocal of $g$, we may suppose $D(g)|D_1(g)$. If $p^{-1}D(g) = p^{-1}D_1(g)$, which is equivalent to a disjoint form if and only if $g$ is so. This is done as in [6], pp. 179–181; or [7], p. 2.

We therefore suppose $D(g) = D_1(g)$. It is well known that $\min g$, the minimum of $g$, satisfies $\min g \leq 2D(g)$. So, if $g = 1$, we may suppose $g = a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + 1$, with $a_1 = 0$ or 1, and each 0 since $g$ is of the shape (7.1). Then $g$ is disjoint; so suppose $\min g = 2$. Transforming $g$ rationally into $2a_1^2 + a_2^2 + 1$, with $D(g) \leq 5$, we have $\min g = 0$ or $\frac{1}{2}$ (mod 1), else $g$ could not be of the shape (7.1). And $\min g \leq (4.3) D(g) \leq 20/3$, by a well known inequality. It follows that $\min g \leq 5/2$; whence, if we suppose $g$ reduced, we have at most three possibilities for its leading binary sector. Bordering each of these binary forms we find easily that every possible reduced $g$, of the shape (7.1), is disjoint.

Lemma 4 gives us that, for $k = 3$, (1.4) implies that $D(g)$ is divisible by a prime $p \equiv 7$, or by 15. So in (21.4) we may take $\mathcal{A}$ to be the set of primes $\geq 7$ that divide $D_1 = D_1(f)$, together with 15, if $15|D_1$. Then (21.3) and (21.5) give us that (21.6) remains valid, for $k = 3$, if the factor $\sum$ is modified by excluding the terms with $p < 5$, but adding a term $15^{-k+i}$ if $15|D_1$. The resulting inequality for $S(f, 3)$ is clearly weakest if $n = 14$ and $D_1(f) = 7$; and then, see (19.1) and (19.2), we find

\begin{equation}
S(f, 3) < 0.00017,
\end{equation}

with $S(f, 3) = 0$ if $D_1(f) \leq 6$ or $10$, by Lemma 4.

We shall pursue this argument further, for $k = 4, \ldots, 7$, after we have proved the theorem for $n \geq 16$ and got rid of the term with $k = 8$ in (1.2). It will be useful later to note that

\begin{equation}
X_p(f, k) \leq 2\left(\frac{p}{2}\right)^{-1}(1-p^{-2})(p^{k+1}+p^{k-n})
\end{equation}

if $4 \leq k \leq 7$ and $p|D_1$.

This follows from (19.11) by the argument used for (19.10), omitting the term with $k = 0$ by the definition of $N_2(f, k)$.

**22. Proof of theorem for $n = 16$, 17.** The sum in (1.2) has to be taken over $k = 1, \ldots, 8$ and we estimate its terms by (2.8) and columns 3, 4 of the table in § 4. Comparing these two columns we see that (2.8) is always weaker for $n = 16$ than for $n = 17$. From (15.5) we have

\begin{equation}
W_1(f) < \min \{0.963, 0.985D_1^{15}, 1.023(D_1D_2)^{-1}(2)\}.
\end{equation}

From (16.8) and (21.7) we have

\begin{equation}
W_1(f) + W_3(f) < 0.002.
\end{equation}

For $k = 4, \ldots, 8$, let $S_k$ be the numerical constant on the right of (19.3), ..., (19.7) respectively. To save the labour of estimating $S(f, k)$ and then multiplying by the bound given by the table for $(1+\operatorname{sgn}(n-2k))\theta(k, n-k)$, note that

\begin{equation}
(1+\operatorname{sgn}(n-2k))\theta(k, n-k)S_k < 7.21, 4.61, 2.19, 1.63, 0.76
\end{equation}

for $k = 4, \ldots, 8$. 6 — Acta Arithmetica XXXVI
Suppose first that $D_1 \geq 3$. We use (21.6) for $k = 4, \ldots, 7$, (19.1) for $k = 8$, also (22.3). The resulting estimate is easily seen to be weakest for $D_1 = 3$, and we calculate that

$$W_k(f) < 0.020, 0.022, 0.025, 0.047, 0.038 \quad \text{for} \quad k = 4, \ldots,$$

From (22.1), giving $W_4(f) < 0.6$, and (22.2), (22.4), the desired result $W(f) < 1$ follows with much to spare; so we may suppose $D_1 \leq 2$.

Next suppose $D_1 = 2$. Estimating $W_k(f)$ for $k = 4, \ldots, 8$ as above, we have in place of (22.4) the weaker

$$W_k(f) < 0.113, 0.103, 0.103, 0.147, 0.095 \quad \text{for} \quad k = 4, \ldots, 8.$$

This with (22.2) gives $W_4(f) + \cdots + W_8(f) < 0.561$. This estimate is too weak; but crudely, by comparing columns 3, 4 of the table, we could divide it by 4 in case $n = 17$, and then (22.1) would give $W(f) < 1$, so we suppose $n = 16$.

If we suppose $D_1 = D_2 = 2$, we can improve on (22.5) by comparing the estimates (21.1), (21.8), for $p = 2, 4 \leq n \leq 7$, $n = 16$. We generally see that (21.6) remains valid with a factor $(4/3)(2^{1-s} + 3^{1-s})$ on the right. This factor is $\leq \frac{2}{3}$ for $k = 4, 5, \frac{1}{2}$ for $k = 6, 7$; so we find $W_4(f) + \cdots + W_8(f) < 0.382$. With this, the third part of (22.1) gives $W(f) < 0.936$. So we suppose $D_1 = 1, D_2 = D_3 = 2$.

With this $f \simeq g + k$ implies $D(g) D(k) = 2$, whence one of $D(g), D(k)$ is 1. By Lemma 3, (1.4) fails for $k = 5, 6, 7$ if $D(g) = 1$, and for $l = n - k = 16 - k = 11, 10, 9$ if $D(k) = 1$; so $W_k(f) = 0$ for $k = 5, 6, 7$. With the cases $k = 4, 8$ of (22.5), this gives $W_4(f) + \cdots + W_8(f) < 0.21$; and with (22.1) we have $W(f) < 0.948$. So the case $D_1 = 1$ is disposed of.

Finally, suppose $D_1 = D_2 = 1$. Then $f \simeq g + k$ implies $D(g) = D(k) = 1$. This, by Lemma 3, contradicts (1.4) for $k = 2, \ldots, 7$; and also, in case $n = 17$, for $k = 8, l = 9$. With $W_l(f) < 1$ by (22.1), this gives $W(f) < 1$ if $n = 17$, and $W(f) = W_4(f) + W_8(f)$ if $n = 16$. And we have $W_8(f) < 0.76$ by (19.1), (19.7), (22.8). Referring again to Lemma 3, we see that (1.4) is impossible for $k = l = 8$ unless $f \equiv 2(x_1 x_2 + \cdots + x_8 x_9)$, as observed at the end of §18, this makes $f \simeq g + k$ impossible for $k = 1$, giving $W_1(f) = 0$. With $W_4(f), W_8(f)$ each $< 1$ and not both positive, we have $W(f) < 0.963$ and the proof is complete.

23. Preliminaries for $n = 14, 15$. We begin as in §22; the sum in (1.2) is over $k = 1, \ldots, 7$, and the first and second columns of the table in §4 show that (2.8) is weaker, for each $k$, for $n = 14$ than for $n = 15$. From (18.5) we have

$$W_4(f) < \min \{0.290 D_1^{-1/2}, 2.28 (D_1 D_3)^{-1/2}, 2.46 (D_1 D_4 D_5)^{-1/2} \},$$

and from (18.8) and (21.7),

$$W_4(f) < \min \{0.135, 0.542 D_1^{-1} \}, \quad W_5(f) < 0.004.$$

In place of (22.3) we have

$$\{1 + \text{sgn}(a - 2b)\} \theta(k, k - b) S_b < 111, 113, 77, 36 \quad \text{for} \quad k = 4, 5, 6, 7.$$

We first show that the theorem is true if $D_1$ has at least three prime factors; we shall do so by using (23.1), (23.2) for $k \leq 8$, (21.6) for $k = 5, 6, 7$. The bound resulting is easily seen to be weakest when $D_1 = 2 \cdot 3 \cdot 5 = 30$, and we find that $W_k(f)$ is less than 0.422, 0.019, for $k = 1, 2, 0.082, 0.037, 0.028, 0.008$ for $k = 4, 5, 6, 7$, which with the second part of (23.2) gives $W(f) < 1$ with much to spare.

We next show that the theorem is true if $D_1$ is the product of two distinct odd primes; we do so as above except that we use (21.6) instead of (19.1) for $k = 6$ of the smallest $D_1 = 15$. We find $W_8(f)$ less than 0.895, 0.097, 0.039, 0.078, 0.078, 0.011 for $k = 1, 2, 4, 5, 6, 7$, and $W(f) < 0.824$.

Next, we define $\mathcal{N}_p^D(f, k)$, for $r \geq 1$, as in (21.1) if $r = 1$, but with the condition $p^D(g)$ replaced by $p^r D(g)$, or equivalently by $p^r D_1(g)$, see (16.1). We prove that

$$\mathcal{N}_p^D(f, k) \leq 2 \{P(k, p) - 1 \cdot p^{k-n} \} \quad \text{if} \quad 5 \leq k \leq 7.$$

This is trivially true unless $p^D(D(f))$, that is, $p^r D_1(f)$, or $n_k f(p, p)$; see (16.2). Assuming $n_k \geq 2$, the argument used to deduce (19.10) from (19.11) shows that (23.4) follows from

$$\sum \{p^{-H}: 2 \leq k \leq \min(k, n_2) < p^{k-n} \} \quad \text{if} \quad p D_2$$

and $5 \leq k \leq 7$.

This is proved in the same way as (19.11). Similarly but more simply, reducing the sum in (23.5) to a single term with $k_1 = k_2 = 2, t = 21, + 2 = 21 + 2 = 21 - 8$, we have

$$\mathcal{N}_p^D(f, 4) - \mathcal{N}_p^D(f, 4) \leq 2 \{P(4, p) - 1 \cdot p^{4-n} - (1 - p^{-3})^{-1} \}.$$

Now, comparing columns 1, 2 of the table in §4 again, we note that the numerical constants in (23.1) may be replaced by $n = 15$ by 1.51, 1.56, 1.69. Similarly, the constants in (23.2) may be reduced by $0.066, 0.282$; and those in (23.3) by 31, 26, 15, 12.

24. Sharper estimates for $S(f, k), k \geq 4$. We need the following improvement on some cases of Lemma 3.
Lemma 5. Suppose that \( f \simeq g + h \) and (1.4) holds; then (i)-(v) below follow:

(i) If \( k = 4 \) then \( D(g) \neq 9 \) and \( D(g) \leq 6 \) implies that \( D(g) = 4 \) or 5 and \( 2g \) (meaning that \( g \) has divisor 2).

(ii) If \( k = 5 \) then \( D(g) \neq 1, 2, 3, 4, 5 \) or 9.

(iii) If \( k = 6 \) then \( D(g) \leq 6 \) implies \( D(g) = 3 \) or 4 and \( 2g \).

(iv) If \( k = 7 \) then \( D(g) \neq 1, 3 \) or 6 and \( D(g) = 2 \) implies \( 3g \).

(v) (iv) holds with \( i, k \) for \( g, h \); and if \( k = 8 \) then \( D(h) = 2 \).

Proof. We shall deduce (i) from Lemma 4; then precisely similar arguments give (ii)-(v) indutively. Suppose therefore \( k = 4 \) and \( D(g) \leq 6 \), or \( k = 9 \), and as in Lemma 4, suppose \( m g \geq 1 \). The well known inequality \( \min g \leq 4D(g) \), with equality only when \( \min g \) is equivalent to a certain form \( E_n \) which has integer coefficients, gives a contradiction for \( D(g) \geq 6 \); and for \( D(g) = 4 \) it gives \( \min g = 2, g \sim 2E_n \). So we may suppose \( D(g) = 5, 6 \) or 9.

Applying the above inequality to \( D(g) \), the adjoint form of \( g \), we have \( D(D(g)) = D(g)^3 \), \( \min adj g \leq 3 \) always, \( \leq 2 \) if \( D(g) \leq 6 \). Now \( g \) has a 3-ary section, say \( g_{u} \), with \( D(g) = \min adj g \). Since \( g \) is of the shape (6.1), with \( p = 3 \), we have \( 3|D(g) = 9 \); and so \( D(g) \leq 6 \) always.

We have

\[
\rho \sim g_{3}(a_{1} + a_{2}, a_{3} + a_{4}, a_{5} + a_{6}) + D(g)D(g_{3})a_{3}^2,
\]

where the \( r_{i} \) are rational numbers with each \( |r_{i}| \leq \frac{1}{2} \), and with denominators dividing \( D(g_{3}) \). We also have \( \min g_{3} \geq \min g \geq 2 \); and the product terms of \( g_{3} \), all have even coefficients.

If we suppose that \( g_{3} \) is of the shape (7.1), and note that it is trivially of the shape (6.1) for every odd \( p \), since it is not \( \rho \equiv 0 \); then we can appeal to Lemma 4, and with \( \min g_{3} \geq 2 \) it is easy to see that \( g = 2a_{1}^2 + 2(a_{2}^2 + a_{3}^2 + a_{4}^2 + a_{5}^2 + a_{6}^2) \) is the only possibility up to equivalence. Then in (24.1) \( r_{1} \geq 0 \), \( r_{2} = 0 \), \( r_{3} = 0 \), \( r_{4} = 4 \) or \( r_{5} = 3 \) makes \( g \) disjoint. \( r_{1} = 1/2, r_{2} = 1/3 \) gives \( D(g) = 5, 2g \).

So suppose \( g_{3} \) of the shape (7.1). We can only have \( g \sim 2a_{3} + a_{3}^2 \), a odd. \( \min g_{3} \geq 2 \) gives \( g_{3} \sim 2E_{3} \), a 3-ary form. We may suppose \( g_{3} \) identical congruent matrix 32 to \( 2\rho + 12a_{3}^2 \) of the shape (7.3). Then in (24.4) we must have \( r_{1} = r_{2} = 0, r_{3} = 0, \pm r_{4} \) or \( \pm r_{5} \), to avoid a disjoint \( g \). The first term on the right of (24.1) is now \( = 2a_{3}^2 + \frac{1}{2}a_{3}^2 \bullet \bullet \bullet \bullet \) and we have \( D(g) = 4 \) or 5 and \( 2g \). This disposes of (i); for (ii)-(v) see above.

We now use Lemmas 3, 5 to obtain inequalities for \( S(f, k) \), \( k = 4, 5, 6, 7 \). Since we have disposed of all other cases, in §§ 17, 20, 22, 23, we shall assume \( n = 14 \) or 15 and

\[
D(f) = 1, 2, p \text{ or } 2p, \quad p \geq 3.
\]

Suppose first that \( k = 4 \); assume \( f \simeq g + h \) and (1.4) satisfied, and note that (16.3) gives \( D(f) = D\left(\{D(f)\}^{4}\right) \). Lemma 5(i) shows that \( D(f) \neq 1, 2 \); but by transforming \( g \) as in the proof of Lemma 4 we have also \( D(f) \neq 5, 6, 7 \). So either \( D(f) = 4 \text{ (mod } 8) \) or \( p | D(g) \), the latter possibility being excluded if \( D(f) = 1 \) or 2. It follows, cf. (21.5), that

\[
S(f, 4) \leq S^{(0)}(f, 4) - S^{(0)}(f, 4) - S^{(0)}(f, 4),
\]

with the last term omitted if \( D(f) = 1 \) or 2. We notice that for \( p = 2, k \leq 4 \), \( \rho \geq 16 \) the ratio of the right member of (23.6) to that of (19.5) is \( 1/216 \). So, by the argument used for (21.6), with (23.6) in place of (21.1), we have

\[
S^{(1)}(f, 4) = S^{(0)}(f, 4) \leq \frac{1}{216} S^{(0)}(f, 4).
\]

The left member of (24.4) is obviously zero if \( 4 | D(f) \), or \( 2 | D(f) \), which makes \( 4 | D(g) \) impossible. For \( D(f) = 2 \) it is zero unless \( 2 | D(f) \), because, with \( l = 10 \) or 11, Lemma 3 gives \( D(h) \leq 1 \).

For \( D(f) = 3, 6 \), we can improve on the third term on the right of (24.4), so we use

\[
S^{(p)}(f, 4) \leq p^{-2} S^{(0)}(f, 4),
\]

(24.5) see (21.3), only for \( p \geq 5 \). When \( D(f) = 3 \), we transform \( g \) as above and see that all possibilities for \( D(g) \) are excluded. If \( D(f) = 6 \) then \( D(g) = 2a_{3}b \) with \( 0 \leq a, b \leq 4 \), and if \( D(g) = 2a_{3}b \) is possible then too are \( D(g) = 2a_{3}b \) and \( D(g) = 2a_{3}b \). So with \( a = b = 1 \) excluded by Lemma 5 we must have \( 2 | b \), and if \( a = 0 \) or \( 4 \) all the \( b \) are excluded. The first two terms on the right of (24.4) take care of \( a = 2 \). So consider \( a = 1 \) or 3 and \( b = 2 \); and we have

\[
S(f, 4) \leq S^{(0)}(f, 4) - S^{(0)}(f, 4) + S^{(0)}(f, 4), \quad \text{if } D(f) = 3 \text{ or } 6.
\]

(24.6) with the last term on the right omitted if \( D(f) = 3 \). To make use of (24.6) we have, see (21.3).

\[
S^{(2a)}(f, 4) \leq S^{(0)}(f, 4) \leq \frac{1}{216} S^{(0)}(f, 4) \text{ if } D(f) = 6.
\]

(24.7) Combining these results we have

\[
S(f, 4)/S^{(0)}(f, 4) \leq \begin{cases} 1 & \text{if } D(f) = 1 \text{ or } 3, \\ 0 & \text{if } D(f) = 2 \text{ and } D(f) = 1, \\ \frac{1}{126} & \text{if } D(f) = 2, \\ \frac{1}{126} & \text{if } D(f) = 2, \\ \frac{1}{126} & \text{if } D(f) = 2, \quad p \geq 5, \\ 1 + \frac{1}{126} + p^{-2} & \text{if } D(f) = 2p \geq 10. \end{cases}
\]
There are obviously some possibilities for improvement by further distinction of cases.

The case \( k = 5 \) is simpler. Transforming \( g \) as above, we see that by Lemma 5 (ii) we cannot have \( D(g) \) a power of 2 or 3, so all possibilities are excluded if \( D_1(f) \leq 3 \), and all but those with \( 6 | D(g) \) if \( D_1 = 6 \), or \( p | D(g) \) if \( p \geq 5 \) and \( D_1(f) = p \) or \( 2p \). Taking \( \mathcal{M} = \{ 6 \} \) or \( \{ p \} \) in (21.5), we see as above that

\[
S(f, 5)/S^{(0)}(f, 5) \leq \begin{cases} 
0 & \text{if } D_1(f) \leq 3, \\
1/36 & \text{if } D_1(f) = 6, \\
p^{-2} & \text{if } D_1(f) = p \text{ or } 2p, p \geq 5.
\end{cases}
\]

(24.9)

Again, since \( l = 9 \) or 10 and so Lemma 3 gives \( D(h) > 1 \), there are possibilities for improvement.

Now take \( k = 6 \). From (24.2) and Lemma 5(iii) we have either \( 4 | D(g) \) or \( p | D(g) \), whence

\[
S(f, 6) \leq S^{(0)}(f, 6) + S^{(0)}(f, 6).
\]

(24.10)

By the argument for (24.4), but using (23.4) instead of (23.6), we find

\[
S^{(0)}(f, 6) \leq \frac{1}{36} S^{(0)}(f, 6),
\]

(24.11)

the left member being however zero unless \( 2 | D_1(f) \).

The second term on the right of (24.10) may be omitted unless \( D_1(f) = p \) or \( 2p \). By (21.3) with \( m = p \) it is at most \( p^{-1} \); but if \( 4 | D(f) \), or \( D_2(f) \), we can do better by using (21.2) and estimating the factor \( \mathcal{A}_g(f, 6) \).

We note that the ratio of the right member of (19.10) to that of (19.8), with \( s = 0 \) for \( p = 5 \), is \( \frac{1}{4} \), so

\[
S^{(0)}(f, 6)/S^{(0)}(f, 6) \leq p^{-1}, (4p)^{-1}
\]

(24.12) for \( 2 | D_1(f), 2 | D_2(f) \) respectively.

Lastly, take \( k = 7 \). From (iv), (v) of Lemma 5, the latter giving \( D(h) > 1 \), we find by taking \( \mathcal{A} \) in (21.4), (21.5) to be empty if \( D_1(f) = 1, = [2] \) if \( D_1(f) = 2, \) that

\[
S(f, 7)/S^{(0)}(f, 7) \leq \begin{cases} 
0 & \text{if } D_1(f) = 1, \\
0 & \text{if } D_1(f) = 2, n = 14, \text{ and } D_2(f) = 1, \\
\frac{1}{36} & \text{if } D_1(f) = 2, \\
2^{-n} & \text{if } D_1(f) = 2 \text{ and } n = 14.
\end{cases}
\]

(24.13)

The third of these is essentially (21.6), with a factor \( \frac{1}{4} \) put in by using the remark following (21.3). To save \( 1/2 \) in the fourth case, in which \( 4 | D(g) \) and \( 2 | D_2(f) \), compare (21.1) and (21.8). We shall improve on (24.13) later.

For \( k = 7 \) it is difficult to improve on (19.1) when \( D_1(f) = 2p \geq 10 \).

(21.6), improved as above, gives

\[
S(f, 7)/S^{(0)}(f, 7) \leq \begin{cases} 
\frac{1}{36} & \text{if } D_1(f) = p \geq 3, \\
1 & \text{if } \text{either } n = 14 \text{ or } p | D_2(f).
\end{cases}
\]

(24.14)

When \( p = 3 \), \( D(g) = 3 \) is impossible and so with \( D_1(f) = 3 \) we have to have \( 9 | D(f), 3 | D_2(f) \); and similarly for \( n = 14 \) we have \( 9 | D(h), 3 | D_2(f) \).

So (24.13) may for \( p = 3 \) be replaced by

\[
S(f, 7)/S^{(0)}(f, 7) \leq 1/9 \text{ if } D_1(f) = 3,
\]

(24.15)

the left member being \( 0 \) unless \( D_2(f) = 3 \).

With \( k = 7 \) and \( D_1 = 6 \), Lemma 5 gives \( D(g) = 1, 3 \) and so \( D(g) = 2 \) or \( 9 | D(g) \), giving

\[
S(f, 7) \leq S^{(0)}(f, 7) + S^{(0)}(f, 7) \text{ if } D_1(f) = 6.
\]

(24.16) The last term is zero unless \( 3 | D_2(f) \), and by the methods used above we have

\[
S^{(0)}(f, 7)/S^{(0)}(f, 7) \leq 1/48 \text{ if } D_1(f) = 6,
\]

(24.17)

\[
S^{(0)}(f, 7)/S^{(0)}(f, 7) \leq 1/48 \sqrt{5} \text{ if } D_1(f) = 6,
\]

(24.18)

\[
S^{(0)}(f, 7)/S^{(0)}(f, 7) \leq 1/2 \sqrt{5}, 5/48 \sqrt{5} \text{ for } 2 | D_1, 3 | D_2
\]

(24.19) and \( D_1 = 6 \), respectively. The case \( D_2(f) = 1 \) remains. But with \( D_1 = 1, \)

\( D(g) / D(h) = 6 \), we must have \( D(g) = 2 \) and then \( D(h) = 3 \), which is impossible for \( l = 7 \), giving \( S(f, 7) = 0 \) if \( n = 14, D_1 = 6, D_2 = 1 \).

25. Completion of proof for composite \( D_1(f) \). In this section we suppose \( D_1 = D_2(f) = 2p, p \geq 3, \) and \( n = 14 \) or 15.

Suppose first \( p \geq 5 \). Then by (23.1) and (23.2) we have \( W_1(f) + W_2(f) + + W_5(f) < 0.756 \). Using (23.3) for \( n = 4, \ldots, 7 \), we have \( W_4(f) < 0.014 \) by (24.8) and \( W_5(f) < 0.019 \) by (24.9). From (24.10)–(24.13) and the remark following (24.11), we have \( W_4(f) < 0.506 \), and from (19.1) we have \( W_1(f) < 0.076 \). \( W(f) < 0.004 < 1 \) follows.

Now suppose \( p = 3 \), \( D_1 = 6 \), and consider first the case \( n = 15 \).

We have \( W_1(f) < 0.615 \) by (23.1), with the smaller constant mentioned at the end of \$ 33 \), and \( W_2(f) < 0.033 \) by (16.9) and the table in \$ 4 \). \( W_3(f) = 0 \) by Lemma 6. For \( k = 4, \ldots, 7 \), we use (23.3) with \( 31, 26, 15, 12 \), again see the end of \$ 23 \), in place of \( 111, \ldots \). With (24.8) this gives \( W_4(f) < 0.016 \); and with (24.9) we have \( W_5(f) < 0.017 \). From (24.10)–(24.13) we see that \( S(f, 6)/S^{(0)}(f, 6) \) does not exceed \( \max(1/3, 1/36 + 1/12) = 1/3 \), whence by (19.1), (19.5) we have \( W_4(f) < 0.009 \), (19.1) and (19.6) give \( W_5(f) < 0.182 \); and now \( W(f) < 0.955 < 1 \) follows (and could be much improved by the arguments for \( n = 14 \) below).
Now suppose $D_1 = 6$ and $n = 14$; we have $D_2 = 1, 2, 3,$ or 6 by (16.1). Suppose first that $D_2 = 6.$ Then we can argue for $k = 1, \ldots, 5$ as for $n = 15$ and we find $W_k(f) \leq 0.380, 0.209, 0, 0.058, 0.071$ for $k = 1, \ldots, 5.$ For $k = 6,$ (24.10)-(24.12) give $S(f, 6)/S_0(f, 6) \leq 1/30,$ which gives $W_k(f) < 0.359.$ (24.16)-(24.18) give $W_k(f) < 0.040,$ and we have $W(f) < 1.$

If $D_2 = 1$ then $D(f) = D(g)D(h) = 6.$ With (1.4) and Lemma 5, this is impossible for $k = 4$ and implies $D(g) = 6, 3, 2,$ $D(h) = 1, 2, 3$ for $k = 5, 6, 7,$ or $l = 9, 8, 7,$ which can be excluded by Lemma 3 and Lemma 5(v). So $W_k(f) = 0$ for $k = 4, \ldots, 7,$ and as above also for $k = 3.$ On calculating that $W_k(f) < 0.9$ and $W_k(f) < 0.009$ we have $W(f) < 1.$

Next suppose $D_2 = 2.$ We have $W_k(f) < 0.558$ by (23.1), and we can estimate $W_k(f)$ for $k = 2, \ldots, 5,$ as for $D_2 = 6,$ giving $W_2(f) + \ldots + W_k(f) < 0.554.$ For $k = 6,$ with $D(g) = 3, 6$ excluded by Lemma 5, and $9/3D_2(f),$ (24.10) is valid without the second term on the right, and using (24.11) we find $W_k(f) < 0.040.$ Omitting the second term on the right of (24.16), (24.17) gives $W_k(f) \leq 1/3 \sqrt{2} < 0.198.$ If we assume $D_2(f) = 2,$ giving $W_k(f) < 0.503$ by the last part of (23.1), then $W(f) < 1.$ So suppose $D_2(f) = 1,$ giving $D(f) = D(g)D(h) = 12.$ With $k = 1, \ldots, 7$ this contradicts (1.4), by Lemma 5; so $W_k(f) = 0$ and $W(f) < 1.$

Finally suppose $D_2 = 3.$ We find as above that $W_k(f) < 0.530$ and $W_k(f) + \ldots + W_k(f) < 0.730.$ From (24.16)-(24.18) we find $S(f, 7)/S_0(f, 7) \leq 1/3 \sqrt{2},$ which gives $W_k(f) < 0.040.$ So we need to prove $W_k(f) < 0.224,$ where (24.10), with the first term on the right zero, and (24.12) give only $W_k(f) < 77/16.$ Very crudely, the improvement we need can be obtained by referring to (23.3), with $n = 3, k = 6,$ and using (21.8) to estimate $W_6(f, 6),$ in place of the much weaker (21.1) which is all we needed for (24.12).

It now remains only to prove the theorem for $D_2(f) = 1$ or a prime, whence by (16.2) $D_2(f) = 1$ or $D_2(f),$ and $n = 14, 15.$

26. The case $D_2(f) = D_2(f) = p \geq 3.$ In this case, see (23.1) and the end of § 23, we have

\[ W(f) < 2.5p^{-1}, 1.5p^{-1} \quad \text{for} \quad n = 14, 15. \]

From (3.3), taking $d = 1, p,$ (16.3) and (16.6), and referring to the table in § 4, we have

\[ W(f) < 35p^{-3}, 16p^{-11/2} \quad \text{for} \quad n = 14, 15. \]

From (21.7) we have $W(f) < 0.004,$ as in (23.2); and $W(f) = 0$ by Lemma 4 if $p \leq 5.$

For $k = 4, \ldots, 7$ we have (21.4)-(21.5) with $N = \{p\}.$ Using (23.4), with $m = p,$ and (21.8), we may replace (21.6) by

\[ S(f, k)/S_0(f, k) \leq p^{-1+k-1}h(1+p^{-k-1})(1-p^{-1})^{-1}(1+p^{-k-1})^{-1}. \]

The right member of (26.3) is the ratio of that of (21.8) to that of (19.8). It is obviously less than $(1-p^{-1})^{-1}p^{-1+k-1},$ but for the larger $k$ we can do better.

Using (23.3), and simplifying (19.3)-(19.6) by putting $D_1 = p,$ $\tau(D_1) = 2,$ $(D_1, 2) = 1,$ we find from (26.3) that (for $n = 14$)

\[ W_2(f) < \begin{cases} p^{-6}(1-p^{-2})^{-1}(1+4p^{-6}) & \text{for} \quad k = 4, 5, \\
15p^{-4}(1-p^{-2})^{-1}(1+p^{-2}) & \text{for} \quad k = 6, \\
79p^{-4}(1-p^{-2})^{-1} & \text{for} \quad k = 7. \end{cases} \]

Similarly, with the smaller constants given at the end of § 23, we find that for $k = 4, 5$ and $n = 15,$ (26.4) holds with $p^{-12}$ in place of $p^{-6},$ and $102-10k$ in place of $206-4k.$ For $k = 6,$ $n = 15$ we find $W_2(f) < 29n^{-12}(1-p^{-1})^{-1};$ and with a little simplification we find $W_2(f) < 24n^{-12}$ for $n = 15;$ and now (26.4) holds for $n = 15.$

A simple calculation shows that these inequalities imply $W(f) < 0.6$ if $p > 5.$ We therefore suppose $p = 3.$ This gives $W_2(f) = 0$ as noted above, and also $W_2(f) = W_3(f) = 0,$ since as noted earlier, see (24.9) and the remark following (24.6), $W_2(f) = 1$ for $k = 4, 5$ if $D(g)$ is a power of 3, as it must be here since $D(f)$ is so. We now find $W(f) < 0.7$ if $n = 15,$ so we suppose $n = 14.$

Estimating $W_2(f)$ as above for $k = 2,$ $n = 3$ we find $W(f) < W_2(f) + 0.136 + 0.266 + W_2(f),$ and $W(f) < 0.149.$ This last estimate can be greatly improved. For Lemma 5 shows that (1.4) fails, for $k = 7,$ and $D(g), D(h)$ each a power of 3, unless $9/D(g)$ and $D(h),$ implying that $n_2 = n_1(f, 3) \geq 4.$ This shows that for $k = 7,$ $p = 3,$ (21.2) holds with $N_0(f, 7)$ in place of $N_3(f, 7)$ on the right, whence in the argument leading to (26.4) we may work with (23.4) instead of (21.8). Comparing these inequalities we improve the above estimate for $W_2(f)$ by a factor $\frac{1}{4}$ giving $W(f) < W_2(f) + 0.440.$

With $W(f) < 0.766$ by (26.1), this is still not sufficient. But if we suppose $n_2(f, 3) \geq 4$ we may use (23.1), with $D_2 = D_3 = D_0 = p,$ giving $W_2(f) < 0.474$ and $W(f) < 1.$ And if $f$ is imprimitive we have crudely $W_2(f) < 0.160,$ again giving $W(f) < 1,$ by the remark at the end of § 18.

So we suppose $D_2(f) = 1,$ giving $W_2(f) = 0$ as noted above, and $f$ primitive. From (6.1) it is clear that there are just two possibilities for the 3-adic class of $f,$ and from § 7, (vi) the same holds for the 2-adic class, giving at most four possibilities for the genus. Two of these may be ex-
cluded by the argument used at the beginning of the proof of Lemma 3, and the others are

\[(26.5) \quad f \simeq Q + \ldots + Q + 3Q + 3Q, \quad n = 14,\]
\[(26.6) \quad f \simeq Q + \ldots + Q + 3Q + 2B, \quad n = 14,\]

where \(Q\) is a unary form with coefficient 1, and \(B\) is the case \(n = 1\) of (7.3). We leave these two cases for the present.

27. The case \(D_1(f) = p \geq 3\) and \(D_2(f) = 1\). In this case (with \(n = 14\) or 16) we have \(D(f) = p\) and \(f \simeq g + h\) gives \(D(g)D(h) = p, D(g) = 1\) or \(p\). By Lemma 3(i), (1.4) now gives \(D(g) = p, D(h) = 1\), for \(k = 2, \ldots, 7\). Now using Lemma 3(ii), (1.4) gives \(l = n - k \neq 7, 9, 10, 11, 13\). Excluding from (1.3) the terms that are zero by this argument,

\[(27.1) \quad W(f) = W_1(f) + (15 - n)W_2(f) + (n - 14)W_3(f) + + (15 - n)W_4(f) + + (n - 14)W_5(f);\]

and \(D(h) \neq 1\), implying \(D(g) = 1\), when \(l = 13\), gives

\[(27.2) \quad W_1(f) \leq \theta(1, 13)S_7(f, 1) \leq 4.20S_7(f, 1), \quad n = 14,\]

from (2.8) and the table in §4, with \(d = 1\) in (3.3).

Suppose first \(n = 15\). Then \(S(f, 7) \leq p^{-10}S^9(f, 7)\) by (21.6), and \(S^9(f, 7) \leq 3.70 \times 10^{-12}\) by (19.6). Using the improvement on (23.3) given at the end of §23, \(W_1(f) < 48p^{-4}\). We have \(W_1(f) < 1.51p^{-12}\) by the corresponding improvement on (23.1); and \(W_5(f) < 0.004\), see (22.2). For \(p \geq 5\) these inequalities, with (27.1), give \(W(f) < 1.1\) very crudely; so suppose \(p = 3\). Now \(W_1(f) = 0\) because, by Lemma 3, we cannot have (1.4) and \(D(g) = 3\) when \(k = 7\); and again we have \(W(f) < 1.1\) very crudely.

Now take \(n = 14\). We have \(W_1(f) < 2.29p^{-12}\) by (23.1), and \(W_2(f) < 33p^{-3}\), since \(D_1(f) = p\) was not used to obtain (28.2). Using (23.3) and (10.3), we find \(W_2(f) < 1.51p^{-3}(1 + p^{-7})\). Now (27.1) gives \(W(f) < 1\) with much to spare if \(p \geq 11\); so we suppose \(p \leq 7\), and distinguish two cases.

(i) If \(f\) is imprimitive, then (7.1) reduces, with \(D(f)\) odd, to \(f \not\simeq 2q\), whence \(f \not\simeq g + h\) is impossible unless \(g, h\) are of even rank, giving \(W_1(f) = 0\), and \(W(f) < \frac{1}{2}\) unless \(p = 3\), and also for \(p = 3\), unless \(W_2(f) > 0\). But then, see Lemmas 3, 5, we have

\[(27.3) \quad f \simeq 2E_6 + 2E_6,\]

where \(E_6\) and \(E_4\) are extreme forms, with non-Gaussian determinants 3, 1. We return to this case later.

(ii) Suppose \(f\) primitive. Then for \(k, l = 6, 8\) we must have \(k\) imprimitive, as noted above, and so \(g\) must be primitive. This is excluded by Lemma 3(iii), giving \(W_1(f) = 0\), if \(p = 3\) or 5. For \(p = 7\) we need a slight improvement on Lemma 5 (iii); if \(k = 6\) and \(D(g) = 7\) we have \(W(g) = 1\) unless \(g\) is imprimitive. This does not follow from Lemma 3(iii), by the method used in the proof of Lemma 5, but it can be verified by using Hermite reduction as, e.g., in [5], p. 13, §9; I leave it to the reader.

Now we may suppose \(W_1(f) = 0\), and (27.1) gives \(W(f) < 2.29p^{-12} + 33p^{-3}\), whence \(W(f) < 1\) for \(p = 5, 7\). With \(p = 3\), there are just two possible genera, as in §26:

\[(27.4) \quad f \simeq Q + \ldots + Q + 3Q, \quad n = 14, D(f) = 3,\]
\[(27.5) \quad f \simeq Q + \ldots + Q + 2B, \quad n = 14, D(f) = 3,\]

with \(B = a_1^{13} + a_2^{13} + a_3^{13} + a_4^{13}\) as in (26.6). We return to these two cases later.

28. The case \(n = 14\) or 15 and \(D(f)\) a power of 2. Denote by \(E_6\) or \(E_4\) an absolutely extreme form with minimum 1 and rank \(k\) or \(l\). \(E_6, E_4, E_4\) occur as exceptional cases in Lemma 5, and have \(D = 4, 2, 1.1\). It is well known, and is an easy corollary of Lemma 5, that \(E_6 \not\simeq Q + (7.2), E_4 \not\simeq -Q + (7.2), E_4 \not\simeq Q + (7.2),\) with the notation of §7, and \(Q = a_1^7\) as above. We deduce that in the present case the possibilities for the genus of \(f\) are given by

\[(28.1) \quad f \simeq Q + \ldots + Q + 2Q + \ldots + 2Q \quad (n_6 = \log_2 D(f) > 0),\]
\[(28.2) \quad f \simeq Q + \ldots + Q + 2E_6 + \ldots + 2E_6 \quad (2n_6 \geq 2),\]
\[(28.3) \quad f \simeq 2Q + \ldots + 2Q - 2Q - 2Q - 2Q - 2Q - 2Q \quad (2n_6 \geq 2),\]

with \(\frac{1}{2}n_6 + 1\) signs, and \(u_6\) with rank \(n_6\), of the shape (7.2), and

\[(28.4) \quad f \simeq 2E_6 + 2E_6 \quad (u_6 = 1, n = 15).\]

To see this we count the possibilities up to \(\simeq\) for \(f\) when \(n_6, n_4\) and \(u_6, u_4\) are given, see (7.1). From §7, (i)–(vii), we find at most 2 possibilities, but only one when \(n_6 = 1\) and \(n = 15\), and none when \(u_6 = 0\), as in the proof of Lemma 3 that would give the contradiction \(n = 14\) and \(D(f) = 4^kD'\) with \(e = \frac{1}{2}n_6\) and \(D' = -1\) (mod 4). Then the only possibilities except (28.1)–(28.4) are derived from (28.1)–(28.3) by changing two \(\pm\) signs to \(-\), or vice versa, and these cases can be excluded by the argument used to prove \(n = 1\) in (20.1). We note next that, by (16.3), Lemma 4, and Lemma 5(ii)

\[(28.5) \quad W_3(f) = 0 \quad \text{for} \quad k = 2, 3, 5.\]

(For \(k = 5\), consider the reciprocal of \(g\) in case \(D(g) = 8, 16\) or 32.)
Now suppose (28.4) holds. We have \( W_1(f) = 0 \) because with \( k = 1 \) \( f \sim Q + h \) is obviously impossible, and \( f \simeq 2Q + h \) would imply \( h \simeq 2p \), which we showed above to be impossible. From Lemma 5 it is easily seen that \( W_1(f) = 0 \), and that \( f \sim g + h \) is possible with \( k = 7 \) only with \( g \simeq 2E \), \( h \simeq 2E \). With (23.5) noting that \( E \) and \( E \) have class-number 1, it follows that the right member of (28.4) is the only disjoint class in the genus of \( f \). So if the theorem is false, then \( f \) has class-number 1; which, see [9], gives the contradiction \( n \leq 10 \). Therefore the theorem is true in case (28.4).

O'Meara has shown in [1] that there exists a form \( f' \) with \( D(f') = D(f) \) which is not equivalent to a disjoint form. If we can prove \( f' \sim f \) the theorem follows. That is, the theorem is true if given \( D(f) \) we have only one possibility for the genus of \( f \). Looking at (28.1)–(28.3), we see that (28.1) (with an empty sum of terms \( Q \)) is the only possibility if \( D(f) = 1 \), or \( n_2 = 0 \); also if \( n = 14 \) and \( n_1, n_2 \) are both odd. We may therefore assume that one of (28.1)–(28.3) holds, and that

\[
2|n_2 \geq 2 \text{ if } n = 14, \quad n_2 \geq 1 \text{ if } n = 15.
\]

We next show that

\[
W_1(f) \leq 0, \quad 111/364, 31/364 \text{ for } n_2 \leq 2, \quad n = 14, \quad n = 15
\]

respectively. For the first of these note that Lemma 5 and (1.4) imply, for \( k = 4 \) and \( l = 10 \) or 11, 4|\( D(g) \), 2|\( D(h) \), 8|\( D(f) \). For the second, we use (23.3) and (23.4); for the third, replace 111 by 36, see end of § 23.

Similarly, using (24.10) and (24.11) in place of (24.4),

\[
W_2(f) = 0 \text{ if } n_2 \leq n - 13,
\]

(28.9) \( W_2(f) \leq 17/192, 15/192 \text{ for } n = 14, \quad n = 15 \text{ respectively.} \)

We can however improve on (28.9) if we assume \( n_2 \geq 3 \). For (24.11) comes from (23.4), and so from (23.5); and (23.5) is weak for \( n_2 \geq 3 \). A simple calculation shows that for \( k = 6, \quad p = 2 \) the left member of (23.5) is at most \( 2^{-10} - 2^{-10}, h \sim 2^{-15} + 2^{-16} \); for \( n = 15, \quad n = 14 \), if we assume \( n_2 \geq 3, \quad n_2 \geq 4 \) in case \( n = 14 \), see (28.1). Comparing these estimates with (23.5) as it is, we have

\[
W_2(f) < 0.127, 0.043 \text{ for } n = 14, \quad n = 15, \quad n_2 \geq 3.
\]

For \( k = 7, \quad n_2 \geq 3 \) is weakened by putting \( p \sim 1 \) for \( p \sim 1 \) on the right; a factor \( r(D) \) on the right of (19.6) arises in this way. Putting \( k = 7, \quad p = 2 \), the right member of (19.6) reduces to \( 2^{-10} \cdot 2^{-10}, P = P(7, 2) \). By the argument leading to the case \( k = 7, \quad p = 2 \) of (21.6), we can sharpen (19.1) by a factor \( 2^{10} \) times the right member of (21.8) (with \( p = 2 \), if we suppose 2|\( D_1 \), or \( n_2 \geq 2 \). More generally, so as to get a better estimate if \( n_2 \geq 3 \), the argument gives the first of

\[
(28.11) \quad S(f, \tau)|S(t)\left( f, \tau \right) \leq 2^{10} \text{ if } k = 7, \quad n_2 \geq 3,
\]

with \( r = 1 \) or 2 according as (1.4) is or is not consistent with

\[
D(g) = 2 \text{ for } k = 7.
\]

The sum on the right of (28.11) is that of (19.11) or (23.5), with \( k = 7, \quad p = 2 \), according as \( r = 1 \) or 2. The improvement in case \( r = 2 \) is obvious; and we have \( r = 2 \) in case (28.2) since \( D(g) = 2 \) implies \( g \sim 2E \), \( \tau - 2Q + + 2(7, 3) \); and \( -2Q \) cannot split off 2-adically from (28.2).

Now using (23.3), with 12 for 36 in case \( n = 15 \), we find from (28.11) that, with \( S(t)\left( f, \tau \right) = 2^{-10}S, \) by (19.6),

\[
W_1(f) \leq 12(31 - 2n) \sum \text{if } k = 7, \quad n_2 \geq 3,
\]

with \( r = 2 \) in case (28.2), 1 otherwise.

A simple calculation now shows, using (28.6) to exclude \( n_2 = 3 \) when \( n = 14 \), that

\[
W_1(f) < 0.008, 0.043 \text{ for } n = 14, \quad n = 15, \quad n_2 \geq 3
\]

We have \( W_1(f) = 0 \) if \( n_2 \leq 1 \); for all cases with \( n_2 \leq 1 \) have been disposed of except (28.1) with \( n_2 = 1, \quad n = 15 \). In that case \( f \sim g + h \) with \( k = 7 \) is possible only with \( g \sim 2E, \quad D(g) = 2, \quad D(h) = 1, \quad h \) primitive, so \( W(h) = 1 \) by Lemma 3.

For \( k = 1 \) we calculate the product over odd \( p \) in § 18 more precisely and simply, since no such \( p \) divides \( D_1 \), and using the table in § 4, we have

\[
W_1(f) < 4.21M_1(f, 1), 2.88M_1(f, 1) \text{ for } n = 14, \quad n = 15.
\]

We note also, see the remark at the end of § 18, and (14.1), that

\[
M_1(f, 1) \leq 2^{-2018} \text{ in case (28.3)}.
\]

29. The cases (28.1)–(28.3). Suppose first that (28.3) holds, with \( n_2 \geq 3 \). Then by (28.14) and (28.15), with \( 2|n_2 \geq 3 \) giving \( n_2 \geq 8 \), we have \( W_1(f) < 0.527 \). With (28.5) and (28.7), this gives \( W_1(f) + \cdots + W_1(f) < 0.656 \). By (28.19) and (28.13) we have \( W_1(f) + W_1(f) < 0.170 \), and \( W_1(f) < 1 \) follows.

Next suppose that (28.3) holds with \( n_2 = 2 \) and \( n = 15 \). We estimate \( W_1(f) \) as above if \( k = 1, 2, 3, 5 \), and we have \( W_1(f) = 0 \) by (28.7), \( W_1(f) < 0.705 \) by (28.9), (28.12) gives \( W_1(f) < 13(2^{-10} + 2^{-10}) \), and \( W_1(f) < 1 \) follows crudely. We may therefore suppose that \( n = 14, \quad n_2 = 2 \) in case (28.3) and then it is easily seen that

\[
f \simeq 2E + 2E, \quad (n = 14, \quad D(f) = 4).
\]
In case (29.1) we have $W_k(f) = 0$ for $2 \leq k \leq 5$ as above. For $k = 1$ we obviously cannot have $g = Q$, and we cannot have $g = 2Q$, for if we could we should find $f \approx 2Q + 2Q + 2\alpha$, which we have excluded. So $f \approx g + h$ is possible only for $k = 6, 7$. We shall return to (29.1) later, and use this remark.

Next suppose that (28.1) or (28.2) holds, with $n_2 > 3$; whence by (28.6) $n_2 = 4$ or 5 if $n = 14$. Using (28.5), (28.7), (28.10) and (28.13) we have $W_2(f) + \ldots + W_7(f) < 0.024$, 0.122 for $n = 14, 15, 18, 4$, with $r = 3$, gives $M_1(f) < 0.201$. Then if $n = 15$ (28.14) gives $W_{14}(f) < 0.579$ and $W_{15}(f) < 1$ follows. For $n = 14$ we need a slightly better estimate for $M_1(f, 1)$. The argument leading to (18.4) shows that that inequality is valid without the restriction $r < 3$ if we suppose further that $2E_{r+1}$, or $r = n_2$. Then with $r = 4$ we find $M_4(f, 1) < 2^{-3} + 2^{-6} + 2^{-11}$, and with $r = 5$ we find a better inequality, and (28.14) gives $W_{14}(f) < 0.062$, and again $W_{15}(f) < 1$.

Now suppose that (28.1) or (28.2) holds with $n_2 < 2$. Then $W_{15}(f) = 0$ by (28.7). If $n = 14$ we could only have $D(g) = 4$, $2Q$, and $D(h) = 1$, giving $2f + 3E_4$ by Lemma 5, whence 2f. If we further suppose $n = 15$ then for $k = 7$ we can only have $D(g) = 3$, $4$, $D(h) = 1$, $D(h) = 1$, and $k > 2E_4$ by Lemma 5, and $D(g) = 4$, giving $2Q$. So $W_7(f)$ is zero, and (28.5) we have $W_7(f) = W_{15}(f)$. Now if $n_2 = 2$, (18.4) gives $M_4(f, 1) < 0.3$ and (28.14) gives $W_{15}(f) < 1$. So

$$f \approx Q + \ldots + Q + 2Q \quad (n = 15)$$

is the only outstanding case with $n > 14$; and for $n = 14$ we have only

$$f \approx Q + \ldots + Q + 2Q + 2Q \quad (n = 14),$$

$$f \approx Q + \ldots + Q + 2E_4 \quad (n = 14).$$

Besides these four genera (29.1)–(29.4), we shall have to consider (26.5), (26.6), (27.4) and (27.5). (The argument used for (28.4) disposes of (27.3).)

It will be useful later to notice that when (29.3) holds the two congruences $f(x_1, \ldots, x_7) \equiv 1, -1 (\bmod 4)$ have equally many solutions in integers $x_j (\bmod 4)$, but when (29.4) holds the second congruence has more solutions than the first. For (29.3) this is clear, since each term $2Q$ is 0, 2 (mod 4) equally often. In the other case, see [8], p. 98, Lemma 1 and note that $f \approx Q - Q + 2(7.2) + 4(7.2)$.

30. Proof for special cases. We consider first the two cases (26.5), (26.6), each with $n = 14$ and $D(f) = 9$. The arguments of §26 show that in either case we have $W_k(f) = 0$ for $k = 3, 4, 5, 7$, and $W_1(f) < 0.76$, $W_8(f) < 0.14$, $W_{13}(f) < 0.265$. It would suffice to improve this last inequality to $W_7(f) < 0.1$. In estimating $W_6(f) < 0.265$ we have excluded (for $k = 6$) the possibility $D(g) = 1$; and we could exclude $D(g) = 9$. But because of the possibility $g \approx 2E_4$, see Lemma 5(ii), we have not excluded $D(g) = 3$.

Now another possibility with $D(g) = 3$, $k = 6$, is $g \approx Q$, where $Q$ is the diagonal form $Q + \ldots + Q + 2Q$; and by (1.4) we may exclude this case. It is easily seen that $2E_4$ and $G$ have each class number 1. Further, we have $2E_4 \approx Q$ for every odd $p$; this is trivial for $p > 3$ and easily verified for $p = 3$. By using §2, (i), it may be seen that $2E_4 + aQ \approx Q + aQ$ for every odd $a$. It follows that $2E_4 + aQ \approx Q + aQ$ for every primitive $a$. So if (26.5) or (26.6) holds, and $f \approx 2E_4 + h$, necessarily with primitive $h$, then we have also $f \approx Q + h$.

By excluding the latter alternative we sharpen the estimate for $W_7(f)$ by a factor $\sqrt{2}2E_4/\sqrt{2}Q$. This factor is $b/\alpha$, where $\alpha$ is the number of integral automorphs of $2E_4$, and $b$ the corresponding number for $G$. Now $b = 2\phi$, by counting permutations and changes of sign; and $\alpha = 144(2)$, see [10], p. 325. So $b/\alpha < 1/19$, $W_{13}(f) < 0.03$, and $W_7(f) < 1$.

Now take the case (29.1), and denote by $F$ the perfect 9-ary form $\frac{1}{2}(a_1 + a_2 + \ldots + a_9) + \frac{1}{2}(a_1 + a_2 + a_3)^2$ with minimum 1, see [9], p. 559, (6.4). The (non-Gaussian) determinant of $F$ is 10, and it is easily verified from (29.1) that $f$ represents $2F$ over the p-adic integers for every $p$. It follows that the genus (29.1) contains a form, say $2F$, which has a 9-ary section $\approx 2F$. Now suppose $f$ equivalent to a disjoint form $g' + h'$. As shown in [9], p. 556, Lemma 3, $g' + h'$ cannot represent $F$ unless one of $g' , h'$ represents $F$. Then, crudely, either $g'$ or $h'$ has rank $\approx 9$. By the remark following (29.1), $g'$ and $h'$ have ranks 6 and 8 or 7 and 7; so we have a contradiction which shows that $2F$ is not equivalent to a disjoint form, and the theorem is proved for case (29.1).

We now consider the case (29.4). By the case $n = 14$, $D(f) = 1$, which we have disposed of above, by reference to [1], there exists a form $F$ which is in the genus of $Q + \ldots + Q$ and indecomposable; whence trivially $\min F \geq 1$. Noting that $Q + Q + Q + Q$ goes into $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2$ by a transformation with determinant 2, we see that by such a transformation we can take $F$ into $f$ satisfying (9.4), obviously with $\min f \geq 2$. We suppose this $f$ equivalent to a disjoint form $g + h$ and deduce a contradiction.

Clearly using $g \geq 2$, $\min g \geq 2$, and we have $D(g)D(h) = 4$. By (28.5), we have $k = 1, 4, 6$ or 7. If $k = 7$ then $l = 7$ and by Lemma 5 we must have $D(g) = D(h) = 2$, and then $2g, 2h, 2f$, contradicting (29.4). If $k = 4$ or 6 then Lemma 5 gives $D(g) = 4, 2g$, and $D(h) = 1, l = 6$ or 8 $l = 8, 2h$, and again 2f. So $k = 1$, and obviously $g = Q, 2Q$. But then in (7.1) we have $\varphi_7$ not null, again contradicting (29.4), and this contradiction completes the proof.
In the four remaining cases (27.4), (27.5), (29.2), (29.3), we shall as above construct the form whose existence is asserted by the theorem but we shall do so more explicitly.

31. The cases (29.2), (29.3). In the first of these cases we begin by defining

\[ f = \sum_{i=1}^{12} (y_i + \frac{1}{2}a_{12} + \frac{1}{2}a_{14} + \frac{1}{2}a_{10})^2 + (a_{12} + \frac{1}{2}a_{14} + \frac{1}{2}a_{10})^2 + (a_{14} + \frac{1}{2}a_{12})^2 + \frac{1}{2}a_{10}^2, \]

where the accent means that the sign of \( a_{10} \) is to be changed in the twelfth term, and the \( y_i \) are restricted to satisfy \( y_1 + \ldots + y_6 = 0 \pmod{2} \). We could write \( y_i = a_i \) for \( i \leq 11 \) and \( y_{11} = a_{12} + \ldots + a_{14} + 2a_{12} \). It is easily seen that (31.1) implies \( D(f) = 2 \) and \( f = (a_{12} + a_{14})^3 \pmod{2} \), whence \( f \) satisfies (29.2).

The leading 12-ary section of (31.1) takes its minimum value 2 at 132 pairs of points at which \( (y_1, \ldots, y_{12}) \) is a permutation of \((1, 1, 0, \ldots, 0)\) or \((1, -1, 0, \ldots, 0)\). With \( |a_{10}| \geq 2 \) we have just one pair of points, one of them with \( a_{12} = 2, a_{14} = a_{16} = -1 \), and the \( y_i \) uniquely determined, at which \( f \) takes the value 2; and none with \( f = 1 \). If \( a_{12} = 1 \) and \( f < 2 \), then clearly \( a_{12} + a_{14} \) has to be odd, so \( f = 1 \), and \( a_{12}, a_{14} = 0 \). If \( y_1, \ldots, y_6 = 0, 1 \), contradicting 2|\( \sum y_i \). So \( f \geq 3 \) if \( y_i = \pm 1 \) and \( a_{12} = a_{14} = -1 \), and one with \( a_{12} = a_{14} = \pm 1 \), each \( y_i \), zero, and \( f = 2 \).

We see therefore that \( f \) has minimum 2, with 132 pairs of minimum points, and that there is just one homogeneous linear equation, namely \( a_{12} + a_{14} = 0 \), which is satisfied by exactly 134 of the pairs of minimum points.

Suppose now that \( f \sim g + h \). Then obviously \( \min g \geq 2, \min h \geq 2, \) and \( D(g)D(h) = 2 \). With \( g \) of rank \( k \leq 7 \), Lemma 5 shows that \( D(g) = 1 \) is impossible, and \( D(g) = 2, \) giving \( D(h) = 1 \), is possible only for \( k = 1 \) or \( r \). With \( g = 2Q \) or \( 2P \), \( 2Q \). The latter case \( l = 8 \) and \( D(h) = 1 \) makes \( h \) decomposable unless \( h = 2Q, \) making \( f \) imprimitive, see (28.4). So \( k = 1, \) \( g = 2Q \), \( f \sim 2Q + h \). Now \( 2Q + h \) reduces to \( h \), and loses just one pair of minimum points, on putting \( a_{12} = 0 \).

From this, and what we have proved above the minimum points of (31.1), it follows that on putting \( a_{14} = -a_{12} \) (31.1) (if decomposable) must reduce to a form equivalent to \( h \). But on making this substitution we see at once that (31.1) reduces to a form with \( D = 4 \), whereas \( D(h) = 1 \). So we have a contradiction which completes the proof for case (29.2), and so for \( n \geq 15 \).

For the case (29.3), we define

\[ f = \sum_{i=1}^{5} (y_i + \frac{1}{2}a_{12})^2 + (y_6 + \frac{1}{2}a_{12} + a_{13})^2 + \sum_{i=1}^{5} (a_i + \frac{1}{2}a_{12})^2 + (a_{12} + a_{13} + a_{16})^2 + \frac{1}{2}a_{14}^2, \]

with \( y_1 + \ldots + y_6 = a_{12} + \ldots + a_5 = 0 \pmod{2} \). It is easily verified that this gives \( D(f) = 4 \), and \( f \) is identically congruent to \( (a_{12} + a_{14})^3 \pmod{2} \), whence \( f \) satisfies either (29.3) or (29.4). Suppose we give a fixed set of values, with \( a_{12} + a_{14} \) odd, to all the variables except \( y_6 \), whose parity is thereby determined. The term involving \( y_6 \) in (31.2) then takes the shape \( \frac{1}{2}(2a_6 + 1)^2 \), which is congruent to \( \frac{1}{2}, \frac{3}{2} \pmod{4} \) equally often; the other terms reduce to constants. So we see that \( f \sim 1, -1 \pmod{4} \) have equally many solutions, which by the remark at the end of § 29 gives us that \( f \) defined by (31.2) satisfies (29.3).

It is clear from (31.2) that at integer points with \( f \) odd we have \( a_{12} + a_{14} \) odd and \( f = 3 \), so \( f \) has minimum 2. If we put \( a_{12} = a_{14} = 0 \), (31.2) reduces to a disjoint form taking its minimum value 2 at 30 pairs of minimum points, with the \( y_i \) permutations of \( \pm 1, \pm 1, 0, \ldots, 0 \) and the \( a_i \) all 0, or vice versa. With \( a_{12}, a_{14} \neq 0, \) \( f = 2 \) if \( y_i = y_{12} = y_{14} = \pm 1 \), each \( y_i \), zero, and \( f = 2 \).

Now there are just two linear homogeneous equations, namely \( a_{12} = 0, a_{14} = 0 \), which are satisfied by exactly 61 of the 62 pairs of minimum points.

Suppose now that \( f \) defined by (31.2) is equivalent to \( g + h \), obviously with \( \min g \geq 2, \min h \geq 2, \) and \( D(g)D(h) = 4 \). One possibility, \( g = 2Q, l = 13, D(h) = 2, \) can be excluded by noticing that putting \( a_{12} = a_{14} = 0 \) (31.2) reduces to a form with \( D = 8 \), and so not equivalent to \( h \). Lemma 5 shows that we cannot have \( h = 2Q, \) or \( 2P, \) or \( 2Q + h \). Now 2Q + h reduces to h, and loses just one pair of minimum points, on putting \( a_{12} = 0 \).

From this, and what we have proved above the minimum points of (31.1), it follows that on putting \( a_{14} = -a_{12} \) (31.1) (if decomposable) must reduce to a form equivalent to \( h \). But on making this substitution we see at once that (31.1) reduces to a form with \( D = 4 \), whereas \( D(h) = 1 \). So we have a contradiction which completes the proof for case (29.2), and so for \( n \geq 15 \).

32. The cases (27.4), (27.5). We begin by noticing that there exists a positive form \( F_k \) with \( \det A(F_k) = 4 \) and

\[ F_1 = a_{12}a_2 + a_{12}a_4 + a_{14}a_4 + 2a_2a_4 \pmod{16}, \]

and that the extreme form \( F_k \) may, by a suitable integral unimodular transformation, be supposed to satisfy

\[ F_1 = a_1a_2 + (a_1^2 + a_2a_4 + a_4^2) - 6a_2^2 \pmod{16}, \]
both congruences holding identically in the variables. \(2F_6\) is equivalent to the leading 8-ary section of (31.1); but to show that \(F_n\) exists, with determinant 4 and satisfying (33.1), it is simpler to appeal to [4], p. 72. Theorem 33, \(2F_9\) is the leading 6-ary section of (31.1); by putting \(y_i = -(y_1 + \ldots + y_4)\) we obtain a 4-ary section of the shape (7.3), whence (33.2), and det \(A(E_1) = 2\). We now define

\[
\begin{align*}
(32.3) \quad f = & \frac{1}{2} E_6(2x_1, \ldots, 2x_6, x_7, 2x_8 + x_{12}) + \\
& + \frac{1}{2} E_8(3x_9, \ldots, 2x_{11}, x_{12}, 2x_{13} + x_{14}, x_9 + 2x_{14}) + \frac{1}{2} x_{14}^2.
\end{align*}
\]

Straightforward calculation shows that det \(A(f) = 3 \cdot 2^{16}\), also that \(f\) has integer coefficients and is identically congruent to \(x_7^2\) modulo 2. So \(D(f) = 3\) and \(f\) is of the shape (7.1) with \(y_9\) null. By the arguments of \(\S 27\) \(f\) satisfies either (27.4) or (27.5); obviously the former, since (32.3) shows that \(f \sim Q + \ldots + Q\). If \(f(x_1, \ldots, x_{12}) = 1\) for integers \(x_i\), then \(x_i\) is odd, the first term on the right of (32.3) is non-zero and so at least \(\frac{1}{2}\), so \(x_{14}\) must be zero. Then each of the first two terms on the right of (32.3) is a positive integer, so \(f > 1\), contradiction, and we have minf \(\geq 2\).

Now suppose that \(f\) satisfies either of (27.4), (27.5), and minf \(\geq 2\), and that \(f \sim g + h\). As in \(\S 27\), \(k = \text{rank} g = 1, 2, 6\). But Lemma 5 gives \(D(g) = 3\), if \(k = 6\), because minf \(\geq 2\), and then \(2g, 0, 0, 0, D(h) = (1, 2)\), contradiction. So \(k = 1\) or \(2\). If \(k = 1\) then \(g = 3Q\) since minf \(\geq 1\); so \(D(h) = 1, I = 13\), which is excluded, see Lemma 3. So \(k = 2\); but now with \(D(g) = 1\) or 3 we can take \(g\) to be disjoint, with minimum 1, unless \(D(g) = 3\) and \(g \sim 2B = 2x_1^2 + 2x_3 x_9 + 2x_5^2 - Q - 3Q\), which contradicts (27.4). So in that case (32.3) is indecomposable and the theorem is proved. It remains to dispose of case (27.3) by constructing \(f\) in that genus, with minf = 2, which does not satisfy

\[
(32.4) \quad f \sim 2B + h, \quad \text{rank} k = 12, \quad h \sim Q + \ldots + Q.
\]

We transform the extreme form \(B\) so as to make it satisfy

\[
(32.5) \quad E_7 = x_1 x_2 + x_3 x_4 + x_5 x_6 - x_7^2 \pmod{8}.
\]

We use also a form \(F_8\) with non-Gaussian determinant 4 and

\[
(32.6) \quad F_8 = x_1 x_2 + x_3 x_4 - x_5 x_6 - x_7^2 \pmod{8}.
\]

\(2F_8\) is the leading 6-ary section of (31.1); it is also the \(g\) with \(D(g) = 4\) of Lemma 5(iii). Now we define

\[
(32.7) \quad f = \frac{1}{2} F_6(2x_1, \ldots, 2x_6, x_7 + x_{12}) + \\
+ \frac{1}{2} F_8(2x_9, \ldots, 2x_{11}, x_{12} + x_{14}, x_9 + x_{14} + 2x_{14}) + \frac{1}{2} x_{14}.
\]

It is easily verified that \(f\) has integer coefficients and is identically congruent to \(x_7^2\) modulo 2, and so satisfies (37.5). If \(f = 1\) for integers \(x_i\) then \(x_{14} = 0, x_9\) is odd, and the first term on the right of (32.7) is congruent to \(-\frac{1}{2} (\text{mod} 4)\) and so \(> 1\). This contradiction gives minf \(= 2\).

It remains only to disprove (32.4); so suppose it true. Transforming \(h\), we may suppose \(h = x_7^2 \pmod{2}\), identically. Putting \(2x_9\) for \(x_9\) and then dividing by 2, (32.4) goes into a form of the shape \(B + H\). Similarly, putting \(2x_9\) for \(x_9\) and then dividing by 2, (32.7) goes into a form which must be equivalent to \(B + H\), and which reduces to \(E_1 + F_4\) on putting \(x_{14} = 0\). So

\[
(32.8) \quad B + H \text{ represents } E_1 + F_4.
\]

Now \(E_1\) and \(F_4\) are well known to be perfect forms, each with minimum 1; so from [9], p. 557, Lemma 4, (32.8) implies that either \(B\) represents one of \(E_2, F_4\) and \(H\) represents the other, or one of \(B, H\) represents \(E_1 + F_4\). Obviously \(B\), of rank 2, represents neither of \(E_2, F_4\); so \(H\) represents \(E_1 + F_4\). This gives the contradiction 12 = rank \(H = \text{rank} h \geq \text{rank} (E_1 + F_4) = 13\); which completes the proof of the theorem in the last outstanding case.

33. Conclusion. Looking at (17.7), where the constant factor 1.75 could be greatly improved by what we have done later, we see that for forms of the shape (6.1), (7.1) we have proved that

\[
(33.1) \quad \max \{W(f) : \text{rank} f = n\} = o(1) \quad \text{as} \quad n \to \infty.
\]

That is, for large \(n\), almost all the weight of a positive \(n\)-ary genus arises from classes that do not contain disjoint forms. I think it is true that \(W(f)\) does not decrease when a general form is put into the shape (6.1), (7.1) by the transformations of [6]. If so, (33.1) is true for all positive \(f\).

Now suppose that we count the classes instead of weighing them; let \(e(f)\) be the number of classes in the genus of \(f\), and let \(e'(f) \leq e(f)\) be the number of these \(e(f)\) classes, that contain disjoint forms. Then define \(O(f) = e'(f)/e(f)\). Is it true that

\[
(33.2) \quad \max \{O(f) : \text{rank} f = n\} = o(1) \quad \text{as} \quad n \to \infty ?
\]

I do not see how to prove (33.2), with or without the restrictions (6.1), (7.1). It may be false; if so, the reason for (33.1) is that the disjoint forms have many automorphs and low weight.

References

On the existence of a density

by

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We shall give the details which demonstrate a formula for a number theoretical density which played a vital role in our paper [2], but doubts about existence and correctness of the formula have been expressed by A. Garcia, H. Moeller, and the editors. In the meantime Everett [1] has used our encoding idea to derive a new proof for one of our assertions.

We shall recall some of the conventions and symbolism in our paper. We considered a function $T$, mapping the positive integers into themselves, given by

$$T_n = \left(3^{X(n)} + X(n)\right)/2,$$

where $X(n) = 1$ when $n$ is odd and $X(n) = 0$ when $n$ is even.

Given an integer $n$ we considered iterated partitions $\tau(n), T_\tau(n), \ldots, T^n n$ and we agreed to stop the iteration at the very first instance when $T^n n < n$. This stopping time was denoted by $\tau(n) = k$. Infinite values for the stopping time were permitted. We also introduced a second stopping time $\tau(n)$ which had a periodicity property. The quantity $P[\tau = k]$ was defined to be the proportion of integers in $[1, 2^k]$ which satisfy the relation $\tau(n) = k$. The quantities $P[\tau < k]$ and $P[\tau \geq k]$ were defined similarly in the same block of integers.

If $A$ is a set of positive integers then the density of $A$ is defined in terms of the counting function $\mu$ to be

$$\delta(A) = \lim_{n \to \infty} \left(1/n\right) \mu\{n \leq m \mid n \in A\}$$

provided this limit exists. We now set $[\chi = k] = \{n \geq 0 \mid \chi(n) = k\}$, and we define $[\tau < k]$ and $[\tau \geq k]$ in a similar manner.

**Theorem.** The density of the set $[\chi \geq k]$ exists and is given by

$$\delta([\chi \geq k]) = P[\tau \geq k].$$

**Proof.** The trick involved is to get this formula without forming any infinite sums. In [2] we established the formula $\delta([\chi = k]) = P[\tau = k]$. Finite additivity of density gives $\delta([\chi < k]) = P[\tau < k]$. Since the sets