2. Proofs. Let $\varkappa(F)$, $\varkappa(K)$ be the residues of $\zeta_F(s)$, $\zeta_K(s)$ respectively at $s=1$. Then since $\zeta_F(s) = \zeta_K(s)L(s,\chi)$ we have

$$L(1,\chi) = \frac{\varkappa(F)}{\varkappa(K)}.$$

Under the assumptions of the Theorem, $L(s,\chi)$ is an entire function, it follows that if $\zeta_K(\frac{1}{2}) = 0$ then $\zeta_F(\frac{1}{2}) = 0$. We use this fact to obtain a lower bound for $\varkappa(F)$, and since an upper bound for $\varkappa(K)$ is easily got we can prove Theorem′.

Lemma 1. *If K is an algebraic number field of degree $n \geqslant 2$, then*

$$\varkappa(K) \leqslant 2^{2n}\pi^n\sqrt{e}(1.3)^{n+1}(\log|d_K|)^{n-1}.$$

And if K is a totally real field, then

$$\varkappa(K) \leqslant 2^n\sqrt{e}(1.3)^{n+1}(\log|d_K|)^{n-1}.$$

Proof. This is Lemma 2.1 of [4].

Lemma 2. *If $\zeta_F(\frac{1}{2}) = 0$, then*

$$\varkappa(F) \geqslant 2^{-2(n+1)}e^{-8\pi n}|d_F|^{-1/4}.$$

Proof. Take $s_0 = \frac{1}{2}$, $N = [F:Q] = 2n$ in Lemma 3, p. 323 of [3]. Thus together Lemmas 1 and 2 give

$$L(1,\chi) \gg |d_F|^{-1/4},$$

and under the further assumptions of the Corollary we have from the first part of the proof of Theorem 4.1 of [4] (see (7)) that

$$L(1,\chi) \leqslant (2\pi)^n \frac{h(F)\,|d_K|^{1/2}}{h(K)\,|d_F|^{1/2}},$$

and so

$$h(F) \gg L(1,\chi)\,|d_F|^{1/2} \gg |d_F|^{1/4}.$$

References

[1] J. B. Friedlander, *On the class numbers of certain quadratic extensions*, Acta Arith. 28 (1976), pp. 391–393.

[2] E. Landau, *Über die Klassenzahl imaginär-quadratischer Zahlkörper*, Nachr. Akad. Wiss., Gottingen, Maths. Phys. Kl. II (1918), pp. 285–295.

[3] S. Lang, *Algebraic Number Theory*, Addison-Wesley, Reading 1970.

[4] J. S. Sunley, *Class numbers of totally imaginary quadratic extensions of totally real fields*, Trans. Amer. Math. Soc. 175 (1973), pp. 209–232.

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Existence of an indecomposable positive quadratic form in a given genus of rank at least 14

by

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0. Introduction. We shall prove the following

Theorem. *Let f be a positive-definite quadratic form with integer coefficients in $n \geqslant 14$ variables. Then in the genus of f there is at least one class that contains no disjoint form.*

There is also (for $n \geqslant 12$) at least one class that does contain a disjoint form; see [4], pp. 75, 76, Theorem 47.

The constant 14 is best possible; to see this, we define genera each of which consists entirely of classes that contain disjoint forms. Twelve suitable genera may be defined by

(0.1) $$f \simeq x_1^2 + x_2^2 + \ldots + x_n^2, \quad 2 \leqslant n \leqslant 11 \text{ or } n = 13,$$

(0.2) $$f \simeq x_1^2 + x_2^2 + \ldots + x_{11}^2 + 2x_{12}^2 \quad (n = 12).$$

In a number of papers, references to which may be found in [1], it has been shown that

(0.3) each of (0.1), (0.2) implies $f \sim x_1^2 + h$, for some $(n-1)$-ary form $h = h(x_2, \ldots, x_n)$.

Denote by $c(f)$ the class-number of f, that is, the number of classes in the genus of f. In the counter-examples (0.1), (0.2) we have $c(f) = 1$ for $n \leqslant 8$; 2 for $n = 9, 10, 11$; 3 for $n = 13$; 4 for $n = 12$. Many other counter-examples, with $n \leqslant 10$ and $c(f) = 1$, may be found in [5]. For the smaller values of n many examples with $c(f) > 1$ could be given. For example, with $n = 2$ and $f \simeq x_1^2 + 14x_2^2 \simeq 2x_1^2 + 7x_2^2$, we have $c(f) = 2$.

We shall use the classical formula, see [2], [3] for the weight of a positive genus. The weight, $w(f)$, of the genus of f is the sum of the weights of its constituent classes. Temporarily, let $w'(f)$ be the sum of the weights of the classes that contain disjoint forms; and define $W(f)$ as $w'(f)/w(f)$. Then trivially $W(f) \leqslant 1$; and the theorem may be expressed as:

(0.4) $W(f) < 1$ for every positive-definite f in $n \geqslant 14$ variables:

I have shown in [6], p. 182, Theorem 8, that the theorem follows in all cases if proved for f with certain conveniently simple arithmetical properties, without which the estimation of $W(f)$ would be impossibly complicated. See (6.1), (7.1), below.

1. Formulae for $W(f)$. It suffices (by permuting the variables) to consider disjoint forms of the shape

$$g(x_1, \ldots, x_k)+h(x_{k+1}, \ldots, x_n), \quad 0<k<n. \tag{1.1}$$

For brevity we call this form $g+h$, and for symmetry we write l for $n-k$, the rank of h. We define $W_k(f)$ in the same way as $W(f)$, except that we count only those classes, in the genus of f, that contain at least one form $g+h$ with given k but do not (if $k \geqslant 2$) contain any form $g'+h'$ with g' of rank k', $0<k'<k$. Clearly this makes $k \leqslant l$, or $k \leqslant \frac{1}{2}n$, which lessens the symmetry but gives

$$W(f) = W_1(f)+W_2(f)+ \ldots +W_{[\frac{1}{2}n]}(f). \tag{1.2}$$

We now show that, for $1 \leqslant k \leqslant \frac{1}{2}n$,

$$W_k(f) \leqslant {\sum}' \{w(g)w(h)/w(f)\colon \text{rank } g = k,\ g+h \simeq f\}, \tag{1.3}$$

where the accent means that if $2|n$ the term with $k = \frac{1}{2}n$ is to be halved, and the summation is over ordered pairs of genera. To see this, let a, b be the number of integral automorphs (with determinant ± 1) of g, h respectively. Then by definition the weights of g, h (or of their classes) are $1/a$, $1/b$; and that of $g+h$ is at most $1/ab$, since it has trivially at least ab automorphs. Now let g range over a set of representatives of the classes in some fixed k-ary genus, and similarly for h, with the two genera such that $g+h \simeq f$. The sum of the weights of the $g+h$ is at most

$$\sum (1/ab) = \sum (1/a) \sum (1/b) = w(g)w(h);$$

the sum of their contributions to $W_k(f)$ is at most $w(g)w(h)/w(f)$, whence we have (1.3) except for the accent. Now if $k = l = \frac{1}{2}n$ we need not count both of $g+h$, $h+g$ unless $g \sim h$, in which case we may suppose $g = h$. Then $g+h$ is unaltered by interchanging x_i and $x_{i+\frac{1}{2}n}$ for $i = 1, \ldots, \frac{1}{2}n$; so it has at least $2ab$ automorphs, and the above estimate for its weight may be halved, which completes the proof of (1.3).

We conclude this section by showing that (1.3) remains valid if we impose the additional summation conditions

$$W(g)<1 \text{ if } k \geqslant 2, \quad W_r(h)<1 \text{ if } 1 \leqslant r \leqslant k-1. \tag{1.4}$$

To see this, note that if $W(g) = 1$ then every class in the genus of g contains a disjoint form and so is to be disregarded in the above definition of $W_k(f)$. Similarly for h if the second part of (1.4) fails.

2. The weight formula. For the classical formula for the weight of a positive genus, see [2], p. 96, (2), with a factor $\frac{1}{2}$ since Pall there excludes automorphs with determinant -1, and so doubles the weight. He also omits to mention that another factor $\frac{1}{2}$ is needed in the trivial case $n = 1$. I also replace the d in Pall's formula by $\det A$, where

$$A = A(f) = (\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\ldots,n} \tag{2.1}$$

is the matrix of f in the notation I prefer. In Gaussian notation this A is the matrix of $2f$; but clearly $w(f) = w(2f)$, so we have

$$w(f) = \{1+\operatorname{sgn}(n-1)\}\pi^{-\frac{1}{4}n(n+1)} \prod_{i=1}^{n} \Gamma(\tfrac{1}{2}i) \prod_p \{a_p(A)\}^{-1}(\det A)^{\frac{1}{2}n+\frac{1}{2}}. \tag{2.2}$$

Here the second product is taken over all primes p, and the p-adic density $a_p(A)$ of the matrix $A = A(f)$ is defined as in [2].

Now let $(\det A)_p$ be the highest power of p dividing $\det A$, and define

$$\beta_p(f) = \beta_p(A) = (\det A)_p^{-\frac{1}{2}n-\frac{1}{2}} a_p(A). \tag{2.3}$$

It is easily seen that this gives $\beta_p(af) = \beta_p(f)$ for every positive integer a; and we may replace (2.2) by

$$w(f) = \{1+\operatorname{sgn}(n-1)\}\pi^{-\frac{1}{4}n(n+1)} \prod_{i=1}^{n} \Gamma(\tfrac{1}{2}i) \prod_p \{\beta_p(f)\}^{-1}. \tag{2.4}$$

We define also

$$\theta(k, l) = \theta(l, k) = \pi^{\frac{1}{2}kl} \prod_{i=1}^{k} \{\Gamma(\tfrac{1}{2}i)/\Gamma(\tfrac{1}{2}i+\tfrac{1}{2}l)\}, \tag{2.5}$$

$$\theta_p(g, h) = \beta_p(g+h)/\beta_p(g)\beta_p(h), \tag{2.6}$$

$$S(f, k) = \sum \prod_p \{\theta_p(g, h)\colon \text{rank } g = k,\ g+h \simeq f,\ (1.4)\}. \tag{2.7}$$

The sum in (2.7) is over ordered pairs of genera as in (1.3). With these definitions, (1.3) gives, for $1 \leqslant k \leqslant \frac{1}{2}n$, $n \geqslant 3$,

$$W_k(f) \leqslant \tfrac{1}{2}\{1+\operatorname{sgn}(k-1)\}\{1+\operatorname{sgn}(n-2k)\}\theta(k, n-k)S(f, k). \tag{2.8}$$

3. Further definitions. Let d denote a positive square-free integer. When g, h satisfy the summation conditions of (2.7) we must clearly have, for some d,

$$\operatorname{rank} g = k,\ g+h \simeq f, \text{ and } d^{-1}\det\{\tfrac{1}{2}A(g)\} \text{ is a rational square.} \tag{3.1}$$

So if we define

$$S_d(f,k) = \sum\Big\{\prod_p \theta_p(g,h)\colon (1.4),(3.1)\Big\} \tag{3.2}$$

we have

$$S(f,k) = \sum_d S_d(f,k). \tag{3.3}$$

This last sum may be taken over $d = 1, 2, 3, 5, \ldots$; but clearly only finitely many terms are non-zero.

Now for each prime p (3.1) implies, for some integer s, that

$$\operatorname{rank} g = k,\ g+h \underset{p}{\sim} f, \text{ and } d^{-1}p^{-2s}\det\{\tfrac{1}{2}A(g)\} \text{ is the square of a } p\text{-adic unit}; \tag{3.4}$$

here $\underset{p}{\sim}$ denotes equivalence over the ring of p-adic integers.

We define

$$T_d(f,k,p,s) = {\sum}'\{\theta_p(g,h)\colon (3.4)\}. \tag{3.5}$$

Here the summation is over ordered pairs of classes under $\underset{p}{\sim}$; and the accent means that terms for which (3.4) is inconsistent with (1.4) are to be omitted. Next, we define

$$U_d(f,k,p) = \sum_s T_d(f,k,p,s); \tag{3.6}$$

the sum may be taken over a finite set of integers s for which (3.4) is possible; and we have

$$S_d(f,k) \leqslant \prod_p U_d(f,k,p). \tag{3.7}$$

Here strict inequality is to be expected, because there may not exist positive forms g, h satisfying a given set of conditions (3.4), with $p = 2, 3, 5, \ldots$.

Now define

$$M_p(f,k) = \max_{p\nmid d} U_d(f,k,p) + \max_{p\mid d} U_d(f,k,p). \tag{3.8}$$

Then clearly (3.3) implies

$$S(f,k) \leqslant \prod_p M_p(f,k), \tag{3.9}$$

which will be used for $k = 1$ and for $3 \leqslant k \leqslant \tfrac{1}{2}n$; but the product diverges for $k = 2$.

4. Inequalities for $\theta(k, n-k)$. We notice that (2.5) gives

$$\theta(1, n-1) = \pi^{\frac{1}{2}n}/\Gamma(\tfrac{1}{2}n), \tag{4.1}$$

$$\theta(2, n-2)/\theta(1, n-1) = \pi^{\frac{1}{2}(n-3)}/\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}). \tag{4.2}$$

Each of these expressions is a decreasing function of n for $n \geqslant 14$. For the gamma function is logarithmically convex and so

$$\Gamma(x+\tfrac{1}{2}) > \{\Gamma(x-1)\}^{-1/2}\{\Gamma(x)\}^{3/2} = (x-1)^{1/2}\Gamma(x) > \pi^{1/2}\Gamma(x)$$

if $x > \pi+1$. Using the duplication formula $\Gamma(x)\Gamma(x+\tfrac{1}{2}) = 2^{1-2x}\pi^{1/2}\Gamma(2x)$, (4.1) and (4.2) give

$$\theta(2, n-2) = (2\pi)^{n-2}/(n-2)! \tag{4.3}$$

and (2.5) gives

$$\theta(k+2, n-k-2)/\theta(k, n-k) = (2\pi)^{n-2k-2}k!/(n-k-2)! \tag{4.4}$$

For $n \geqslant 19$ we shall later use (4.1), (4.2) and

$$\theta(k+2, n-k-2)/\theta(k, n-k) < \begin{cases} 4/9 & \text{for } n \geqslant \min(19, 2k+4), \\ 1/20 & \text{for } n \geqslant 19 \text{ and } k \leqslant 2. \end{cases} \tag{4.5}$$

To prove (4.5) for fixed $k \geqslant 8$, note that the right member of (4.4) is a decreasing function of n for $n \geqslant 2k+4$, and reduces for $n = 2k+4$ to $(2\pi)^2/(k+1)(k+2) < 40/90$. For fixed $k \leqslant 7$, the right member of (4.4) is decreasing for $n \geqslant 19$, so greatest for $n = 19$. So (4.5) is easily verified.

For $14 \leqslant n \leqslant 18$, we use (4.1)–(4.4) to calculate the following table:

Rounded up values of $\theta(k, n-k)$, see (2.5).

k \ n	14	15	16	17	18
1	4.20	2.87	1.89	1.199	0.740
2	7.91	3.83	1.72	0.719	0.282
3	10.08	3.60	1.15	0.327	0.085
4	10.59	2.93	0.69	0.139	0.025
5	10.26	2.31	0.42	0.063	0.008
6	9.82	1.90	0.28	0.033	0.003
7	9.64	1.70	0.22	0.021	0.002
8	—	—	0.20	0.016	0.001
9	—	—	—	—	0.001

5. Estimation of certain products over primes. For brevity, write

$$P(k,p) = \prod\{1-p^{-2j}\colon j = 0, 1, \ldots, [\tfrac{1}{2}k]\} \tag{5.1}$$

(for prime p and $k \geqslant 0$), and

$$R(k,p) = \min\{P(k_1,p)P(k_2,p)\colon k_1 \geqslant 0,\ k_2 \geqslant 0,\ k_1+k_2 = k\}. \tag{5.2}$$

Clearly we have

$$P(k,p) = (1-p^{-2})^{[\frac{1}{2}k]} \quad \text{for} \quad k \leqslant 3, \tag{5.3}$$

$$R(k,p) = (1-p^{-2})^{[\frac{1}{2}k]} \quad \text{for} \quad k \leqslant 5. \tag{5.4}$$

For larger k, the product over $j \geqslant 2$ can be estimated numerically by straightforward calculations; and for $j = 1$ we use the well known $\prod(1-p^{-2}) = 6/\pi^2$, giving

$$(5.5)\qquad \prod_p \{P(k,p)\}^{-1} < 1.65, 1.79, 1.82, 1.83,$$

for $k \leqslant 3, \leqslant 5, \leqslant 7, \geqslant 8$ respectively.

Now, for square-free positive d, define

$$(5.6)\qquad L_d(p) = 1-p^{-1}(-d|p), \quad L_d = \prod_p L_d(p),$$

where the Legendre symbol $(-d|p)$ may be interpreted as 0 if $p|2d$. We shall prove that

$$(5.7)\qquad L_d < 0.36\,d \quad \text{for} \quad d \geqslant 3.$$

There exists a Dirichlet character χ, modulo $4d$, such that $\chi(p) = (-d|p)$ for every p, whence (5.6) gives

$$(5.8)\qquad L_d = \sum\{N^{-1}\chi(N):\ N = 1, 2, \ldots\}.$$

When $d = 3$ we have $\chi(N) = 1$ for $N \equiv 1$ or 5, -1 for $N \equiv 7$ or 11, modulo 12; so we have the case $d = 3$ of (5.7) by a straightforward calculation. For $d \geqslant 5$, define

$$u_m = \sum N^{-1}|\chi(N)|,$$

with summation over $2dm < N \leqslant 2d(m+1)$.

It is easily seen that $L_d \leqslant u_0 - u_1 + u_2 - \ldots$, and that u_m is monotone decreasing. So $L_d < u_0$. Subtract $\frac{1}{2}\log d$ from each side of this inequality, and note that $2|N$ implies $\chi(N) = 0$. We have

$$(5.9)\qquad L_d - \tfrac{1}{2}\log d < 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2d-1} - \frac{1}{2}\log d;$$

and the term 1/5 on the right may be omitted if $d = 5$, giving (5.7).

Allowing d to take non-square-free values, and differencing, the right member of (5.9) is a decreasing function of d, for $d \geqslant 1$, so it is less than 1 for $d \geqslant 6$, and we have

$$(5.10)\qquad d^{-1}L_d < d^{-1} + \tfrac{1}{2}d^{-1}\log d.$$

By differentiating the right member of (5.10) we find that it is a decreasing function for real $d \geqslant 6$, and so is at most $6^{-1}(1+\frac{1}{2}\log 6) < 3^{-1}$, which completes the proof of (5.7).

6. Odd primes. For each odd prime p we shall assume that there exist forms f_1, f_2, with integer coefficients, depending on f, p, such that

$$(6.1)\qquad f \underset{p}{\sim} f_1 + pf_2, \quad p \nmid \det A(f_1)\det A(f_2),$$

where the right member is disjoint as in (1.1) unless f_2 is identically 0. If so, $\det A(f_2)$ is to be interpreted as 1. Defining

$$(6.2)\qquad n_i = n_i(f,p) = \operatorname{rank} f_i \ (i = 1, 2), \quad \text{whence} \quad n_1 + n_2 = n,$$

we shall also assume that

$$(6.3)\qquad n_1(f,p) \geqslant \tfrac{1}{2}n \quad \text{for every } p.$$

As far as odd primes are concerned these are just the assumptions justified by the theorem quoted at the end of § 0.

Now for $i = 1, 2$ we define

$$(6.4)\qquad \varepsilon_i = \begin{cases} 0 & \text{if} \quad 2 \nmid n_i, \\ (-1)^{\frac{1}{2}n_i}\big(\det A(f_i)|p\big) & \text{if} \quad 2|n_i; \end{cases}$$

$$(6.5)\qquad m_i = [\tfrac{1}{2}n_i], \quad P_i = \prod\{1-p^{-2j}:\ j = 0, 1, \ldots, m_i\};$$

$$(6.6)\qquad X_i = 1 + \varepsilon_i p^{-m_i}.$$

Later, if there is any risk of confusion, we may write more explicitly $\varepsilon_i = \varepsilon_i(f,p), \ldots$. We note that $P_i = P(n_i, p)$, see (5.1).

It now becomes convenient to write $\exp_p x$ for p^x, when the exponent x is complicated. With this notation, we show that

$$(6.7)\qquad \alpha_p\big(A(f)\big) = 2\exp_p(\tfrac{1}{2}n_2^2 + \tfrac{1}{2}n_2)P_1P_2(X_1X_2)^{-1}.$$

To prove (6.7) for $n_2 > 0$ we put $s = 2$ in [2], p. 101, (21)–(23) and correct an error; the last $-$ sign in (23) should be $+$.

When $n_2 = 0$, the convention $\det A(f_2) = 1$ gives $P_2 = \varepsilon_2 = 1$, $X_2 = 2$, and the right member of (6.7) reduces to $P_1X_1^{-1}$, agreeing with the case $s = 1$ of the formulae just quoted.

Now (2.3), with $(\det A)_p = \exp_p(n_2)$ by (6.1), (6.2), gives

$$(6.8)\qquad \beta_p(f) = 2\exp_p(-\tfrac{1}{2}n_1n_2)P_1P_2(X_1X_2)^{-1}.$$

We now suppose that

$$(6.9)\qquad f \underset{p}{\sim} g + h, \quad \operatorname{rank} g = k, \quad \operatorname{rank} h = l = n-k.$$

It is clear that each of g, h must be of the shape (6.1). For brevity we write, for $i = 1, 2$,

$$(6.10)\qquad k_i = n_i(g), \quad l_i = n_i(h), \quad \eta_i = \varepsilon_i(g), \quad \zeta_i = \varepsilon_i(h),$$
$$Y_i = X_i(g), \quad Z_i = X_i(h).$$

From (6.9) and (6.10) it follows easily that

(6.11) $k_1+k_2 = k, \quad l_1+l_2 = l, \quad$ and $\quad k_i+l_i = n_i$ for $i = 1, 2$;

also that (with g_i, h_i as in (6.1) with g, h for f)

(6.12) $\big(\det A(f_i)|p\big) = \big(\det A(g_i)|p\big)\big(\det A(h_i)|p\big) \quad (i = 1, 2).$

Now from (2.6), (6.8), and (6.10) we have

(6.13) $\theta_p(g, h) = p^{-\frac{1}{2}t}(2X_1X_2)^{-1}Y_1Y_2Z_1Z_2\prod_i \{P_i(f)/P_i(g)P_i(h)\};$

where the product is over $i = 1, 2$ and, see (6.11),

(6.14) $t = t(g, h, p) = k_1l_2+k_2l_1 = n_1n_2-k_1k_2-l_1l_2.$

7. The prime 2. By the theorem quoted at the end of § 0 we may assume that

(7.1) $f \underset{2}{\simeq} \varphi_0+2\psi_1+2\varphi_1+4\psi_2,$

where each φ_i is a diagonal form with odd coefficients and each ψ_i is a form of even rank of one of the shapes

(7.2) $x_1x_2+x_3x_4+\ldots+x_{2m-1}x_{2m} \quad (m \geqslant 0),$

(7.3) $x_1^2+x_1x_2+x_2^2+x_3x_4+x_5x_6+\ldots+x_{2m-1}x_{2m} \quad (m \geqslant 1).$

One or more of the summands in (7.1) may be identically 0, and no two of them have a variable in common. We define

(7.4) $\nu_i = \operatorname{rank}\varphi_i \ (i = 0, 1), \quad 2m_i = \operatorname{rank}\psi_i \ (i = 1, 2),$ and

$$n_i = 2m_i+\nu_{i-1} \ (i = 1, 2).$$

This gives $n_1+n_2 = n$ and we assume $n_1 \geqslant \frac{1}{2}n$ as in (8.3).

We need to normalize (7.1) so as to make the m_i and ν_i into invariants under $\underset{2}{\simeq}$ and to define invariants $\varepsilon_i = \varepsilon_i(f, 2)$. To do so, we appeal to [8], p. 97, Theorem 3: the normalization is as follows.

(i) A 3-ary φ can be put into the shape (7.1) with summands of ranks 1, 2, 0, 0. So we can have

(7.5) $\nu_i = 0, 1,$ or $2 \quad$ (for $i = 0, 1$);

and this makes the m_i and ν_i invariant.

(ii) For $i = 1, 2$, if $\nu_{2-i} \neq 0$, we can take ψ_i to be of the shape (7.2), and we define $\varepsilon_i = 0$.

(iii) For $i = 1, 2$, if $\nu_{i-1} = 2$ and the diagonal coefficients of φ_i have the same residue (± 1) modulo 4, we can again take ψ_i to be the shape (7.2), and we again define $\varepsilon_i = 0$.

(iv) For $i = 1, 2$, if neither of (ii), (iii) above implies $\varepsilon_i = 0$, we define ε_i to be 1 or -1 according as ψ_i is of the shape (7.2) or (7.3). And now the ε_i and ψ_i are invariant.

(v) For $i = 0, 1$, we can take each term of φ_i to be $\pm Q$ or $\pm 3Q$, where Q denotes a unary φ with coefficient 1. For $\nu_i = 2$ we can go further by using the obvious

(7.6) $aQ+bQ \underset{2}{\simeq} bQ+aQ \underset{2}{\simeq} (a+4)Q+(b+4)Q \quad$ if $\quad 2 \nmid ab.$

(vi) We can further normalize $\varphi_0+2\varphi_1$ by using

(7.7) $aQ+2bQ \underset{2}{\simeq} (a+2b)Q+(b+2a)Q \quad$ if $\quad 2\nmid ab.$

(vii) Now let f, f' be two forms of the shape (7.1), each normalized as above, of the same rank n, and with $\det A(f)/\det A(f')$ the square of a 2-adic unit, whence $n_i(f') = n_i(f)$ for $i = 1, 2$. Suppose also that $m_i(f') = m_i(f)$ for $i = 1, 2$, but $f' \underset{2}{\simeq} f$ is false. Then there is at most one possibility for f' up to $\underset{2}{\simeq}$ when f is given, and we must have

(7.8) $\varepsilon_i(f') = -\varepsilon_i(f) \quad$ for $\quad i = 1, 2.$

I leave it to the reader to verify this assertion; and note that it obviously holds good with an odd p instead of 2, and without the hypothesis about the m_i, which by (6.5) is redundant for odd p.

Assuming that f is of the shape (7.1), normalized as above, we define $P_i = P_i(f, 2)$ and $X_i = X_i(f, 2)$ as in (6.5), (6.6), noticing that the = sign in $(6.5)_1$ has to be replaced by $\leqslant$, see (7.5). Now we further define

(7.9) $v = v(f) = n_1(\frac{1}{2}n_1+\frac{1}{2}+n_2)+n_2(n_2+1),$

(7.10) $q = q(f) = (\operatorname{sgn}\nu_0)n_1+(\operatorname{sgn}\nu_1)n_2+\operatorname{sgn}(\nu_0\nu_1),$

(7.11) $E = E(f) = \exp_2\{-2-\operatorname{sgn}\nu_0-\operatorname{sgn}\nu_1\}X_1X_2;$

and we shall show that, with $A = A(f)$ as in (2.1), we have

(7.12) $\alpha_2(A) = 2^{n+v-q-1}PE^{-1}, \quad$ where $\quad P = P_1P_2.$

Pall's formula for α_2, see [2], p. 105, (47), is very complicated and contains obvious errors, so I refer instead to [8], p. 96, Theorem 1, and p. 105, (13.4). Now we have to specialize, replacing the formally infinite sum treated in [8] by the right member of (7.1). This specialization is straightforward except for the factor $E = \prod E_j$, taken over $-\infty < j < \infty$.

The E_j of [8], p. 96, (2.9) is $\frac{1}{2}X_j$ for $j = 1, 2$ in the present notation, and 1, in case (7.1), for other j except 0, 3. $E_0 = 1$ if $\nu_0 = 0$, $\frac{1}{2}$ if not; and $E_3 = 1$ if $\nu_1 = 0$, $\frac{1}{2}$ if not. So the E of [8] reduces in case (7.1) to that of (7.11), and we have (7.12).

From (2.1), (7.1), $\det A$ is exactly divisible by $\exp_2(n_1+2n_2)$, so from (2.3), (7.12) we have

(7.13) $$\beta_2(f) = P_1P_2E^{-1}\exp_2\{n-q-1-\tfrac{1}{2}n_1n_2\}$$

after a little simplification.

8. The prime 2, continued. Now suppose, cf. (6.9), that

(8.1) $$f \underset{2}{\simeq} g+h, \quad \operatorname{rank} g = k, \quad \operatorname{rank} h = l = n-k.$$

Using the notation of (6.10), (6.11) follows. Obviously g, h have to be of the shape (7.1). By (i) of § 7 we have

(8.2) $$m_i(f)-m_i(g)-m_i(h) = \begin{cases} 0 & \text{if } \nu_{i-1}(g)+\nu_{i-1}(h) \leqslant 2, \\ 1 & \text{otherwise}; \quad \text{for } i = 1, 2. \end{cases}$$

For brevity we define, for $i = 1, 2$,

(8.3) $$\varrho_i = \operatorname{sgn}\nu_{i-1}(f), \quad \sigma_i = \operatorname{sgn}\nu_{i-1}(g), \quad \tau_i = \operatorname{sgn}\nu_{i-1}(h).$$

We note that, by (7.5), the n_i and ν_i determine the $m_i(f)$, and that

(8.4) $$2 \nmid n_i, k_i, l_i \text{ imply respectively } \varrho_i, \sigma_i, \tau_i = 1,$$

(8.5) $$n_i, k_i, l_i = 0 \text{ imply respectively } \varrho_i, \sigma_i, \tau_i = 0,$$

(8.6) $$\varrho_i = 0 \Leftrightarrow \sigma_i = \tau_i = 0;$$

each for $i = 1, 2$. For (8.6) see (8.2), (7.5).

We prove next:

LEMMA 1. *If f and g, of the shape (7.1), are given then for given $m_1(h)$, $m_2(h)$ there is at most one possibility, up to $\underset{2}{\simeq}$, for h satisfying (8.1).*

Proof. We refer to [8], p. 96, Theorem 2. Part (i) of that theorem shows that the result is true if g is of the shape (7.1) with each φ_i null; and part (ii), with the hypothesis about the $m_i(h)$, shows that it is true for $g = aQ$ $(= ax_1^2)$, a odd. By considering the forms adjoint to f, g, h the result follows for $g = 2aQ$. Repeating these arguments, we can split off the summands in the normalized form of g one at a time, and the lemma follows.

A similar result for odd p, with (6.9) for (8.1), is trivial; in it, the $m_i(h)$ need not be supposed given, since they are determined by (6.5) and (6.11).

From (7.13) and (2.6) we see that $\theta_2(g, h)$ is the product of the following four factors:

(8.7) $$2^{-\frac{1}{2}t}P_1(f)P_2(f)/P_1(g)P_2(g)P_1(h)P_2(h),$$
$$\text{with } t = t(g, h, 2) \text{ as in (6.14)};$$

(8.8) $$\exp_2\{1-q(f)+q(g)+q(h)\},$$

(8.9) $$\exp_2\Big\{-1+\sum_{i=1}^{2}(\varrho_i-\sigma_i-\tau_i)\Big\},$$

(8.10) $$(2X_1X_2)^{-1}Y_1Y_2Z_1Z_2.$$

For (8.9), we use (7.11) and (8.3).

We define J by

(8.11) $$\log_2 J = -\varrho_1\varrho_2+\sigma_1\sigma_2+\tau_1\tau_2+ + \sum_{i=1}^{2}\{\sigma_i(k_i-1)+\tau_i(l_i-1)-\varrho_i(n_i-1)\}.$$

It is easily verified that J is the product of the expressions (8.8), (8.9) (eliminate q by using (7.10)). So we have

(8.12) $$\theta_2(g, h) = 2^{-\frac{1}{2}t}J(2X_1X_2)^{-1}Y_1Y_2Z_1Z_2\prod_{i=1}^{2}\{P_i(f)/P_i(g)P_i(h)\}.$$

9. Estimation of $T_d(f, k, p, s)$. It is convenient to begin by replacing (6.13), (8.12) by more convenient inequalities. We notice first that (6.5) and (6.11) give $m_i(f) \geqslant m_i(h)$ for $p \geqslant 3$, and this holds also for $p = 2$ by (8.2). $P_i(f) \leqslant P_i(h)$ follows, and we have

(9.1) $$P_1(f)P_2(f)/P_1(h)P_2(h) \leqslant 1.$$

In one special case we shall need to notice that if for some i, m we have $m_i(f) = m > m_i(h)$ then the 1 on the right of (9.1) may be replaced by $1-p^{-2m}$.

Again referring to (6.5), with the first = sign replaced by $\leqslant$ if $p = 2$, as noted above (before (7.9)), we have $P_i(g) \geqslant P(k_i, p)$ and so by (5.2), (6.11) we have

(9.2) $$P_1(g)P_2(g) \geqslant R(k, p).$$

Now (6.11) gives $k_2 \leqslant n_2$, so if $n_2 = n_2(f, p) = 1$ we have $k_2 \leqslant 1$ and $P_2(g) = 1$ by (6.5), (6.10). Using $k_1 \leqslant k$ to estimate $P_1(g)$, we have, see (5.1),

(9.3) $$P_1(g)P_2(g) \geqslant \begin{cases} P(k, p) & \text{if } n_2 \leqslant 1, \\ P(k, p)(1-p^{-2}) & \text{if } n_2 \leqslant 3. \end{cases}$$

With g of the shape (6.1) or (7.1), with k_2 for n_2, we see that

(9.4) $$(3.4) \Rightarrow k_2 = 2s \text{ if } p \nmid d, \quad 2s+1 \text{ if } p \mid d;$$

from which

(9.5) $$P_1(g)P_2(g) = 1 \quad \text{if} \quad k = 1 \text{ or } k = 2 \text{ and } p|d$$

(because the hypotheses with (9.4) give $\max(k_1, k_2) = 1$). We now have, for $p \geqslant 2$,

(9.6) $$\theta_p(g, h) \leqslant p^{-\frac{1}{2}t} J\{R(k, p)\}^{-1}(2X_1X_2)^{-1} Y_1Y_2Z_1Z_2,$$

with the convention that $J = 1$ if $p > 2$, and subject to obvious improvement if we can use (9.3), (9.5) or the remark following (9.1). We define $V = V(g, h, p)$ by

(9.7) $$V = (2X_1X_2)^{-1}\{Y_1Y_2Z_1Z_2 + (2-Y_1)(2-Y_2)(2-Z_1)(2-Z_2)\}.$$

We now show that, with $s = [\frac{1}{2}k_2]$ by (9.4), we have

(9.8) $$T_d(f, k, p, s) \leqslant \{R(k, p)\}^{-1} V p^{-\frac{1}{2}t} \quad \text{if} \quad p > 2.$$

To prove (9.8), we may obviously suppose that there is at least one pair g, h satisfying (6.9) and (3.4); we consider the possibilities for a different pair satisfying the same conditions; denote such a pair by g', h'. By (9.4) and (6.11), (3.4) determines all the k_i, l_i, also the $P_i(g)$ and t, see (6.14). By the remark following (7.8) there is at most one possibility for g', up to $\underset{p}{\sim}$, and, see (6.10), we have $\varepsilon_i(g') = -\varepsilon_i(g) = -\eta_i$, giving, see (6.6), $X_i(g') = 2 - X_i(g) = 2 - Y_i$.

Now (6.9) and (3.4) imply that h satisfies a condition of the same shape as (3.4); so what has been proved above for g' applies to h' with obvious changes of notation. The pairs g, h' and g', h can be excluded since g determines h by the remark following Lemma 1. Summing over g, h and g', h', we have (9.8); and the possibilities for improving on the R-factor are just the same as for (9.6).

When $p = 2$ we can argue exactly as above if we strengthen (3.4) by fixing the $m_i(g)$ and the $m_i(h)$, or equivalently the σ_i, τ_i; which fixes J. (See (8.3)–(8.6) and the preceding remark, also (8.11).) Then summing over the possibilities for the σ_i, τ_i we have, again with $s = [\frac{1}{2}k_2]$,

(9.9) $$T_d(f, k, 2, s) \leqslant 2^{-\frac{1}{2}t}\{R(k, 2)\}^{-1}\sum\{JV\colon (8.4)\text{–}(8.6)\}.$$

Now the case $k = 2$ requires special treatment. Suppose that $p \nmid 2d$ and $s = 0$, whence $k_1 = 2$, $k_2 = 0$. From (3.4), (5.6)–(6.6) and (6.10) we find $\eta_1 = (-d|p)$, $Y_1 = 1 + p^{-1}\eta_1$, $P_1(g) = (1-p^{-2})$. Similarly, with the convention for $n_2 = 0$ in (6.1), we have $\eta_2 = 1$, $Y_2 = 2$, and trivially $X_2 = Z_2$, since g_2 is null and so f_2, h_2 are equal. The second term on the right of (9.7) is zero since $2 - Y_2 = 0$; and it is easily seen that g' does not exist. So $T_d(f, 2, p, 0)$ is the same as $\theta_p(g, h)$, the sum in (3.5) having just one term. So, see (5.6), we have, from (9.6),

(9.10) $$T_d(f, 2, p, 0) \leqslant p^{-n_2} L_d(p) X_1^{-1} Z_1 \quad \text{if} \quad p \nmid 2d,$$

on noting that (6.14) gives $t = 2l_2$, whence by (6.11) $t = 2n_2$.

Similarly we find that

(9.11) $$T_d(f, 2, p, 1) \leqslant p^{-n_1} L_d(p) X_2^{-1} Z_2 \quad \text{if} \quad p \nmid 2d.$$

For $2 < p|d$, giving $s = 1$ and $t = l_1 + l_2 = l = n - 2$, we find, using (9.5),

(9.12) $$T_d(f, 2, p, 0) \leqslant p^{1-\frac{1}{2}n} V \quad (\text{for } 2 < p|d),$$

and note that $Y_1 = Y_2 = 1$, so that V simplifies.

For $k = p = 2$, we shall use (9.9), with the R-factor $= 4/3$.

10. Estimation of V, see (9.7), for odd p. We shall prove:

LEMMA 2. *For $p \geqslant 3$ and $n \geqslant 14$ we have $V \leqslant 2$. For $p \geqslant 3$ and $k_2 = 0$ we have*

(10.1) $$V \leqslant \begin{cases} 1 + p^{-\frac{1}{2}k} & \text{if } 2|k \ (= k_1) \text{ and } 2 \nmid l_1, \\ 1 + p^{-\frac{1}{2}l_1} & \text{if } 2 \nmid k \text{ and } 2|l_1, \\ (1 - p^{-\frac{1}{2}n_1})^{-1} & \text{if } 2 \nmid kl_1, \\ (1 + p^{-\frac{1}{2}k})(1 + p^{-\frac{1}{2}l_1})(1 + p^{-\frac{1}{2}n_1})^{-1} & \text{if } k \equiv l_1 \equiv 0 \pmod 2. \end{cases}$$

Proof. In the case $k_2 = 0$ we have $Y_2 = 0$ and $X_2 = Z_2$ as in the proof of (9.10), and so $V = X_1^{-1}Y_1Z_1$. If k is even and l_1 odd, then n_1 is odd by (6.11), whence $\varepsilon_i = \zeta_i = 0$ by (6.4), (6.10), giving $X_1 = Z_1 = 1$ and $V = Y_1, = 1 \pm p^{-\frac{1}{2}k}$ by (6.5), (6.6), and the first case of (10.1) follows. The second and third cases are dealt with similarly. In the fourth case $\varepsilon_1, \eta_1, \zeta_1$ are each ± 1, and we have (10.1) unless they are $1, 1, -1$. From (6.4), (6.9) this is clearly impossible. So (10.1) is proved; and with (6.11) it clearly gives $V \leqslant 2$.

It now suffices to prove $V \leqslant 2$ in the case $k_1k_2l_1l_2 \neq 0$ since the case k_1, l_1, or $l_2 = 0$ could be dealt with as above. With this assumption, and $p \geqslant 3$, the Y_i, Z_i all lie between $1 \pm 1/3$, and (9.7) gives, crudely, the first part of

(10.2) $$V \leqslant (2X_1X_2)^{-1}\{(4/3)^4 + (2/3)^4\} \leqslant (17/10)X_2.$$

For the second part of (10.2), we use $n \geqslant 14$, giving $n_1 \geqslant 7$ by (6.3), and so $X_1 \geqslant 80/81$ by (6.5), (6.6).

Now (10.2) gives $V \leqslant 2$ unless $X_2 < 17/20 < 1 - p^{-2}$, which is possible only if $n_2 = k_2 + l_2 \leqslant 2$. That, with the k_i, l_i non-zero, gives $n_2 = 2$, $k_2 = l_2 = 1$, whence $Y_2 = Z_2 = 1$, $X_2 \geqslant 2/3$. We can now replace the fourth powers in (10.2) by squares, and $V \leqslant 2$ easily follows.

11. Estimation of V for $p = 2$. We shall show that

(11.1) $$V \leqslant 2 \quad \text{if} \quad p = 2.$$

We begin by noting, see [4], p. 58, Theorem 34, that

(11.2) $$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 \underset{2}{\sim} -a_1a_2a_3x_1^2 + 2x_2x_3$$
$$\text{if } a_1 \equiv a_2 \equiv -a_3 \equiv \pm 1 \pmod 4.$$

We use (11.2) to prove that

$$\varepsilon_i\eta_i\zeta_i \neq -1 \quad (i = 1, 2). \tag{11.3}$$

Taking $i = 1$ (the same argument works for $i = 2$), we note that f may be supposed to be of the shape (7.1) with $\varphi_1 = \varphi_1(f)$ null, else (ii) of § 7 gives $\varepsilon_1 = 0$; and (iii) of § 7 may be supposed not be applicable to f, else again $\varepsilon_1 = 0$. Now by (iv) of § 7 we have $\varepsilon_1 = 1$ or -1 according as $\psi_1 = \psi_1(f)$ is of the shape (7.2) or (7.3). Similarly with g, η_1, or h, ζ_1, for f, ε_1.

In the first case of (8.2) the foregoing gives us

$$\psi_1(f) \underset{2}{\sim} \psi_1(g)+\psi_1(h), \tag{11.4}$$

and the right member is easily seen to be $\underset{2}{\sim}$ (7.2) if both or neither of its summands are so, $\underset{2}{\sim}$ (7.3) if not. That gives $\varepsilon_1\eta_1\zeta_1 = 1$. In the second case of (8.2) we have to normalize the 3-ary or 4-ary form $\varphi_0(g)+\varphi_0(h)$ as in (i) of § 7. If in so doing we can use (11.2), as it stands or with one more unary summand on each side, then (11.4) holds with one more binary summand (7.2) on the right, which does not affect the argument. If we cannot so use (11.2) then one of $\varphi_0(g), \varphi_0(h)$ is a binary form $\equiv \pm(x_1^2+x_2^2) \pmod 4$; but then by (iii) of § 7 we have η_1 or $\zeta_1 = 0$. So (11.3) is proved.

From (6.6), with $p = 2$, we see that (11.3) gives

$$X_i - 1 = (Y_i-1)(Z_i-1) \quad \text{if} \quad \varepsilon_i\eta_i\zeta_i \neq 0 \text{ and } (8.2)_1 \text{ holds}, \tag{11.5}$$

$$X_i - 1 = \tfrac{1}{2}(Y_i-1)(Z_i-1) \quad \text{if} \quad \varepsilon_i\eta_i\zeta_i \neq 0 \text{ and } (8.2)_2 \text{ holds}, \tag{11.6}$$

$$X_i,\ Y_i \text{ or } Z_i = 1 \quad \text{if} \quad \varepsilon_i\eta_i\zeta_i = 0. \tag{11.7}$$

Suppose now that $\varrho_1 = \varrho_2 = 0$, that is, that f, g, h are of the shape (7.1) with each φ_i null. Then the $\varepsilon_i, \eta_i, \zeta_i$ are all non-zero by (iv) of § 7, and $(8.2)_1$ holds. We multiply the inequality $V \leqslant 2$ by $2X_1X_2$ and use (11.5) to express it as a linear inequality in the Y_i, Z_i, which obviously all lie between 0 and 2. We need only verify it when each of the Y_i, Z_i is 0 or 2. So $V \leqslant 2$ is proved in this case.

Next suppose $\varrho_1 = \varrho_2 = 1$. Then each $\varphi_i(f)$ is non-null, see (8.3) and so each ε_i is 0 by (ii) of § 7, giving $X_1 = X_2 = 1$. (8.6) gives σ_i or $\tau_i = 1$, so one at least of η_i, ζ_i is 0 and one of Y_i, Z_i is 1, for $i = 1, 2$. $V \leqslant 2$ is now trivial.

So to prove $V \leqslant 2$ we may suppose $\varrho_1+\varrho_2 = 1$. By symmetry (provided that we do not use $n_1 \geqslant n_2$) we may suppose $\varrho_1 = 1$, $\varrho_2 = 0$. As above this gives σ_1 or $\tau_1 = 1$ and $X_2 = 1$, Y_2 or $Z_2 = 1$. By symmetry (provided that we do not use $k \leqslant \frac{1}{2}n$) we may suppose $Y_2 = 1$; and then $V \leqslant 2$ can be expressed as

$$Y_1Z_1Z_2+(2-Y_1)(2-Z_1)(2-Z_2) \leqslant 4X_1. \tag{11.8}$$

If we can use (11.5), with $i = 1$, we substitute for X_1 on the right of (11.8), and it suffices to verify the resulting inequality for $Y_1 = 2$, for $Z_1 = 2$ and for $Y_1 = Z_1 = 0$, each of these being trivial.

If we can use (11.6), with $i = 1$, we substitute for X_1 in (11.8), giving an inequality which can be verified as above if we further suppose $1/2 \leqslant Z_2 \leqslant 3/2$; whence we may suppose $Z_2 = 0$ or 2. But from (7.11) and (7.12), with h for f, or otherwise we see that $Z_2 \neq 0$; so $Z_2 = 2$. Now (11.8) reduces to $Y_1+Z_1 \leqslant 3$. From $Z_1 = Z_2 = 2$ it would follow that h is null, which is excluded. So $Z_1 \leqslant 3/2$ and we may suppose that $Y_1 = 2$ and $Z_1 > 1$. From $Z_1 > 1$ and $Z_2 > 1$ it follows, see (ii) of § 7, that h is of the shape (7.1) with each φ_i null; but then the first case of (8.2) holds.

We may therefore assume (11.7), with $i = 1$. If any two of X_1, Y_1, Z_1 are equal to 1, then, with $X_1 \neq 0$ as above and so $\geqslant 1/2$, (11.8) is easily verified. So suppose that exactly one of X_1, Y_1, Z_1 is equal to 1, that is, exactly one of $\varepsilon_1, \eta_1, \zeta_1$ is 0, see (6.6). By (ii)–(iv) of § 7, exactly one of $\varphi_0(f), \varphi_0(g), \varphi_0(h)$ is congruent to $\pm(Q+Q) \pmod 4$ as in (iii). This is trivially impossible if either of the others is null; so suppose not, that is $\sigma_1 = \tau_1 = 1$, giving $Y_2 = Z_2 = 1$ by (ii). Substituting in (11.8), we have an inequality which is easily deduced from (11.7), and the proof of (11.1) is complete.

12. Investigation of J, see (8.11). We shall show that

$$\sum\{J\colon (8.4)\text{–}(8.6)\} \leqslant 1, \quad \text{if} \quad n \geqslant 14, \tag{12.1}$$

the summation variables being the σ_i, τ_i of (8.3). With (11.1) this will enable us to replace the last factor on the right of (9.9) by 2; this simple but crude result will however have to be improved in some important cases.

We begin by noticing that if $\varrho_1 = \varrho_2 = 0$ then by (8.6) the σ_i, τ_i are all zero, so the sum in (12.1) has just one term, which is 1 by (8.11).

Next suppose $\varrho_1 = 1$, $\varrho_2 = 0$. We have $\sigma_2 = \tau_2 = 0$ and σ_1, τ_1 can only be 1, 1; 0, 1; or 1, 0, by (8.6). In these cases $\log_2 J = -1, -k_1, -l_1$. (12.1) follows at once unless $\min(k_1, l_1) = 0$ or 1. If k_1 or $l_1 = 0$ (8.5) excludes two of the three terms; and if k_1 or $l_1 = 1$, (8.5) excludes one of them, and again we have (12.1).

The case $\varrho_1 = 0$, $\varrho_2 = 1$ is dealt with in the same way, so now we suppose $\varrho_1 = \varrho_2 = 1$. The sum may now, see (8.6), have up to nine terms. It is easily verified that $\varrho_1\varrho_2 = 1$ and (8.6) imply

$$\sigma_1\sigma_2+\tau_1\tau_2 = \sigma_1+\tau_1+\sigma_2+\tau_2-2.$$

So (8.11) may be rewritten as

$$\log_2(2J) = \sum_i (\sigma_i k_i+\tau_i l_i - n_i). \tag{12.2}$$

The summand in (12.2) takes the three values $0, -k_i, -l_i$, by (8.6). Allowing also the value $-n_i$, we see that

(12.3) $$2\sum J < (1+2^{-k_1})(1+2^{-k_2})(1+2^{-l_1})(1+2^{-l_2}).$$

The 1 in the first bracket on the right corresponds to $\sigma_1 = 1$, and the second term in the first bracket corresponds to $\sigma_1 = 0$. Similarly in the other brackets. Referring to (8.4), (8.5), we see that any of the factors on the right of (12.3) may be omitted if the exponent k_i or l_i occurring in it is 0 or 1. So, by omitting factors $> 5/4$, we can deduce (12.1) from (12.3) unless at least three of the k_i, l_i are equal to 2; and even then unless the fourth is at most 5. But if so we have $n \leqslant 11$ by (6.11); so (12.1) is proved.

It will be useful later to note that if $k = k_1 = 1$, $k_2 = 0$ then we must have $\varrho_1 = \sigma_1 = 1$ and $\sigma_2 = 0$; and then (8.6) gives $\tau_2 = \varrho_2$. So the sum in (12.1) has at most two terms, with $\tau_1 = 1, 0$ and $\log_2 J = -1, 1-n_1-\varrho_2$. Similarly for $k_1, k_2 = 0, 1$.

13. Simplified estimates for $T_d(f, k, p, s)$. In this section $n \geqslant 14$, $p \geqslant 2$ is fixed, and $n_2 = n_2(f, p) \leqslant \frac{1}{2}n$, $k \leqslant \frac{1}{2}n$ are assumed. By (9.8), with $V \leqslant 2$ by Lemma 2, we have for odd p the second of

(13.1) $$T_d(f, k, p, s) \leqslant \begin{cases} 2\{P(k, p)\}^{-1}p^{-\frac12 t} & \text{if} \quad n_2 \leqslant 1, \\ 2\{R(k, p)\}^{-1}p^{-\frac12 t} & \text{if} \quad n_2 \geqslant 2. \end{cases}$$

For the first part, see (9.2), (9.3). For $p = 2$ see (9.9), (11.1), (12.1), and we again have (13.1). For s and t, see (3.4), (6.11), (6.14) and (9.4).

For odd p and $n_2 = 0$, $p \nmid d$, we have $s = t = 0$, and we must improve on the 2 on the right of (13.1). We define (for odd p)

(13.2) $$V_0(p, k) = (1+\gamma_k p^{-\frac12 k})(1-p^{-\frac12 \min l})^{-1},$$

where $\gamma_k = 1$ if $2 | k$, 0 if not, and the minimum is taken over the even $l \geqslant \max(k, 14-k)$. With $n_2 = 0$ we have $l_2 = 0$, $l_1 = l$, $k_2 = 0$, $k_1 = k$, by (6.11), so we can appeal to Lemma 2; note that $V_0(p, k)$ is an upper bound for the right member of (10.1). So using $V \leqslant V_0(p, k)$ instead of $V \leqslant 2$, we have in place of the case $n_2 = 0$ of (13.1)

(13.3) $$T_d(f, k, p, 0) \leqslant \{P(k, p)\}^{-1} V_0(p, k) \quad \text{if} \quad n_2 = 0 \text{ and } p \nmid d.$$

By defining $V_0(2, k) = 2$, this is valid also for $p = 2$.

In the case $k = 1$ we may write (13.3) more explicitly, noting that $P(1, p) = 1$ by (5.1), as

(13.4) $$T_d(f, 1, p, 0) \leqslant (1-p^{-7})^{-1} \quad \text{if } n_2 = 0 \text{ and } p \nmid 2d.$$

The case $n_2 = 1$ of (13.1) can be improved on by an argument like that leading to (13.3); we shall do so for $k = 1$. Taking $k_1 = 1$, $k_2 = 0$, $l_2 = n_2 = 1$, $l_1 = l-1 = n-2$ and $s = 0$, $t = 1$, we find

(13.5) $$T_d(f, 1, p, 0) \leqslant p^{-\frac12}(1-p^{-6})^{-1} \quad \text{if} \quad n_2 = 1 \text{ and } p \nmid 2d.$$

For $p = 2$ (13.4), (13.5) hold with $2, 2^{1/2}$ on the right; but these estimates are too weak, and will be improved in the next section.

Now take $k = 2$; (13.3) is true but of little use because $V_0(p, 2)$ has a factor $1+p^{-1}$, and the product $\prod_p (1+p^{-1})$ is divergent. Take first the case $p > 2$, $k_2 = 0$; we have $l_1 = n_1 - 2 \geqslant 5$. If n_1 and l_1 are odd, $X_1 = Z_1 = 1$, and (9.10) simplifies. If not, the worst case is $n_1 = 0$, $X_1 = 1 \pm p^{-4}$, $Z_1 = 1 \pm p^{-3}$. In either case (9.10) gives

(13.6) $$T_d(f, 2, p, 0) \leqslant (1-p^{-4})^{-1}(1+p^{-3})p^{-n_2}L_d(p) \quad \text{if} \quad p \nmid 2d.$$

Similarly, but more crudely, (9.11) gives

(13.7) $$T_d(f, 2, p, 1) \leqslant 3p^{-n_1}L_d(p) \quad \text{if} \quad p \nmid 2d.$$

From (9.12) and $V \leqslant 2$,

(13.8) $$T_d(f, 2, p, 0) \leqslant 2p^{1-\frac12 n} \quad \text{if} \quad 2 \leqslant p | d,$$

using (9.5), (9.9), (11.1) and (12.1); these also give

(13.9) $$T_d(f, 2, 2, s) \leqslant (4/3)2^{1-\frac12 t} \quad \text{if} \quad 2 \nmid d,$$

with $s = 0$ and $t = 2n_2$, or $s = 1$ and $t = 2n_1$.

14. A sharper estimate for $T_d(f, 1, 2, 0)$. (13.1), with $R(1, p) = P(1, p) = 1$, and $k_2 = 1$, $t = n_1$, gives

(14.1) $$T_d(f, 1, 2, 0) \leqslant 2^{1-\frac12 n_1} \quad \text{for} \quad 2 | d \text{ and } n \geqslant 14.$$

We shall prove that

(14.2) $$T_d(f, 1, 2, 0) \leqslant 2^{-1-\frac12 n_2} + (1-\varrho_2)2^{-\frac12 n} + 2^{2-n_1-\frac12 n_2} \quad \text{if} \quad 2 \nmid d;$$

and note that the second term on the right is zero if $2 \nmid n_2$, since then $\nu_1 = 1$ by (7.4) and $\varrho_2 = 1$ by (8.3).

With $k_1, k_2 = 1, 0$ and $t = n_2 = l_2$, the right member of (9.9) reduces to $2^{-\frac12 t}\sum JV$, with summation over the two possibilities mentioned at the end of § 12. In the term with $\log_2 J = 1-n_1-\varrho_2 \leqslant 1-n_1$, we use (11.1) to give $V \leqslant 2$, whence the third term on the right of (14.2).

In the other term, $\tau_1 = 1$, $J = \frac12$, and we have $\sigma_1, \sigma_2 = 1, 0$, $\varrho_1 = 1$; from which, by using (6.6), (6.10), (7.4), (7.5) and § 7, (ii) it may be deduced that $Y_1 = 2$, $Y_2 = 1$, $X_2 = 1$, $Z_2 = 1$. Now (9.7) simplifies to $V = X_1^{-1}Z_1$. If $\varrho_2 = 1$, then $\sigma_2 = 0$ and (8.6) give $\tau_2 = 1$ and $X_1 = Z_1 = 1$ follows, giving $V = 1$, and (14.2) follows on using $J = \frac12$.

So we suppose X_1, Z_1 not both 1; and write for brevity $m = m_1(f)$, whence by (6.6) we have $X_1 = 1$ or 1 ± 2^{-m}. Now if we suppose $Z_1 = 1$, (14.2) follows since $t+2m = n_2+2m \geqslant n_2+n_1-2 = n-2$, by (7.4). So we suppose $Z_1 \neq 1$; and $\varrho_2 = 0$, giving $\sigma_2 = \tau_2 = 0$ by (8.6), whence f, g, h are all of the shape (7.1) with φ_1 null; but $\sigma_1 = \tau_1 = 1$ gives $\varrho_1 = 0$, so none of them has φ_0 null.

If $\varphi_0(h)$ is unary we may suppose

$$g = dQ = dx_1^2, \quad h = aQ+2\psi_1+4\psi_2, \quad f = dQ+aQ+2\psi_1+4\psi_2,$$

with $2 \nmid ad$ and h normalized as in § 7. From that normalization it easily follows that $Z_1 = 1 \pm 2^{-m}$ and $X_1 = Z_1$ if $ad \equiv -1 \pmod 4$, $X_1 = 0$ if not. So we have $V \leqslant 1+2^{-m}$, from which, as above, (14.2) follows.

So we suppose $\varphi_0(h)$ binary and

$$g = dQ, \quad h = aQ+bQ+2\psi_1+4\psi_2, \quad 2 \nmid abd. \tag{14.3}$$

On normalizing $f = g+h$ we see that $m_1(h) = m-1$, see (8.2); so, by the remark following (9.1), we see that the weaker inequality $V \leqslant (1-2^{-m})^{-1}$ would suffice to prove (14.2). If $ab \equiv 1 \pmod 4$ we see from § 7, (iii) that $Z_1 = 1$, giving $V \leqslant (1+2^{-m})^{-1}$. So suppose $ab \equiv -1 \pmod 4$. We have $X_1 = 1 \pm 2^{-m}$, $Z_1 = 1 \pm 2^{1-m}$, and we have what is wanted if the signs are the same. This follows from (11.2), with a, b, d for the a_i, on using (14.3) and $ab \equiv -1 \pmod 4$; see (11.4) and the following remark. So the proof of (14.2) is complete.

In it, we have not used $n_2 \leqslant \frac{1}{2}n$; so if $n_1 = n_2 = \frac{1}{2}n$ is odd, then (14.2) holds for $2|d$ with ϱ_1 for ϱ_2, since interchange of g, h then interchanges the cases $2|d$, $2 \nmid d$, see (3.4) and (7.1). (14.1) can also be improved when $\varrho_2 = 0$; for then $f \underset{2}{\sim} g+h$ is obviously impossible if $g = dx_1^2$, $d \equiv 2 \pmod 4$, and so the left member is 0.

15. Estimation of $U_d(f, k, p)$. From (6.11) and (9.4) we see that the sum in (3.6) may be taken over $0 \leqslant 2s \leqslant \min(k, n_2)$ if $p \nmid d$, $0 < 2s+1 \leqslant \min(k, n_2)$ if $p|d$. So, from (3.6),

$$U_d(f, k, p) = T_d(f, k, p, 0) \text{ unless } \min(k, n_2) \geqslant 2 \text{ or } 3 \text{ according as } p \nmid d \text{ or } p|d. \tag{15.1}$$

For $k = 2$, $p \nmid d$, we sum over $s = 0, 1$ and (13.6), (3.7) give

$$U_d(f, 2, p) \leqslant \{(20/19)p^{-n_2}+3p^{-n_1}\}L_d(p) \quad \text{if} \quad 2 < p \nmid d \text{ and } n_2 \geqslant 2. \tag{15.2}$$

From (13.9) we have

$$U_d(f, 2, 2) \leqslant (8/3)(2^{-n_1}+2^{-n_2}) \quad \text{if} \quad 2 \nmid d \text{ and } n_2 = n_2(f, p) \geqslant 2. \tag{15.3}$$

We shall now prove that for $k \geqslant 3$, $n_2 \geqslant 3$, we have

$$\sum_s \{p^{-\frac{1}{2}t}\colon (3.4)\} \leqslant p^{\frac{1}{2}-\frac{1}{2}n}(1+p^{-10}) \quad \text{if} \quad p|d. \tag{15.4}$$

We use (6.11) and (9.4) to express t, defined by (6.14), as $8s^2+2as+b$, where a, b are integers independent of s; and we denote by u the smallest value of t. We note that t is constant modulo 2 and takes no value more than twice. If $4|a$, t is constant modulo 8; if not, t takes no value twice. (The foregoing may be clearer if we first use (6.11), (6.14) to express t as $2k_2^2+a'k_2+b'$, a', b' integers depending only on n, k, l.) Obvious upper bounds for the left member of (15.4), in the two cases just distinguished, are

$$2p^{-\frac{1}{2}u}(1+p^{-4}+p^{-8}+\ldots) < p^{-\frac{1}{2}(u-3)}$$

and $p^{-\frac{1}{2}u}(1+p^{-1}+p^{-2}+\ldots)$, which is smaller.

(15.4) follows at once if $u \geqslant n+2$, so we suppose that $u \leqslant n+1$; that is, that some s gives $t \leqslant n+1$. If $\min(k_1, k_2) \geqslant 3$ then $t \geqslant 3(l_1+l_2) = 3l \geqslant 3n/2 \geqslant n+7$, so we suppose that $t \leqslant n+1$ for some k_1, k_2 with $\min(k_1, k_2) \leqslant 2$ and $2 \nmid k_2$. Straightforward calculation shows that this is the case only if either $n_2 = 3$ or $k = 3$, which we therefore assume.

Now the sum in (15.4) has just two terms, with $k_2 = 1, 3$. For $k = 3$ these terms have $t = n-4+n_2 \geqslant n-1$ and $t = 3n-3n_2 \geqslant n+7$, giving (15.4) with much to spare unless $n_2 = 3$. If so, $t = 3n-3n_2$ gives $t \geqslant n-1+20$, and (15.4) follows. So suppose $n_2 = 3$, $k \geqslant 4$. Now the two possible t are $n+k-4 \geqslant n$ and $3l \geqslant n+7$, which crudely gives (15.4).

From (15.4) and (13.1) we have

$$U_d(f, k, p) \leqslant 2\{R(k, p)\}^{-1}(1+p^{-10})p^{\frac{1}{2}-\frac{1}{2}n} \quad \text{if} \quad 2 \leqslant p|d, \text{ and } n_2 \geqslant 2. \tag{15.5}$$

If $p \nmid d$ then $s = 0$ gives $k_2 = 0$, $t = kn_2$. Omitting this term from the sum on the left of (15.4), we proceed as above and obtain an estimate which with (13.1) gives

$$U_d(f, k, p) \leqslant 2\{R(k, p)\}^{-1}(p^{-\frac{1}{2}kn_2}+p^{-\frac{1}{2}n}) \quad \text{if} \quad 2 \leqslant p \nmid d \text{ and } n_2 \geqslant 2. \tag{15.6}$$

In this section we have used the hypothesis $n \geqslant 14$, and $k \leqslant \frac{1}{2}n$, $n_2 \leqslant \frac{1}{2}n$; we shall do so till the end of the proof.

16. Estimation of $S_d(f, 2)$ and $S(f, 2)$. We continue the special treatment of the case $k = 2$, so as to obtain a substitute for (3.9). It will be convenient (for all k) to define

$$D = D(f) = \det\{\tfrac{1}{2}A(f)\}, \quad D_r(f) = \prod_p \{p\colon n_2(f, p) \geqslant r\}, \tag{16.1}$$

whence $D_{r+1}|D_r$, D_r is square-free, $D_r = 1$ for $r > \frac{1}{2}n$, and $D = D_1D_2\ldots$ Clearly we have, from $g+h \simeq f$,

(16.2) $$D(f) = D(g)D(h), \qquad D_r(f) = \text{l.c.m.}\,\{D_r(g), D_r(h)\}.$$

Restricting k to be 2 for the rest of this section, we first show that

(16.3) $$S_1(f, 2) = S_2(f, 2) = 0.$$

To see this, note that $g = D_2(g)G$ for some G of the shape (6.1), (7.1) with $D(G) = d$ by (3.4). Trivially, $G \sim x_1^2 + dx_2^2$ if $d = 1$ or 2; so G and g are disjoint and (1.4) fails. The right member of (3.5) vanishes, and (16.3) follows, by (3.6), (3.7).

We now express the product in (3.7) as $\Pi_1\Pi_2\Pi_3$, where Π_1 is taken over $p \nmid 2d$, Π_2 over $p|d$, and $\Pi_3 = 1$ if $2|d$, $U(f, 2, 2)$ if not.

For $p \nmid 2d$ we use (15.1) and (13.6) if $n_2(f, p) \leqslant 1$, (15.2) if not. This gives

$$U_d(f, 2, p) \leqslant (1-p^{-4})^{-1}(1+p^{-3})L_d(p)p^{-n_2}$$

in the first case and a stronger estimate in the second. We extend the product

$$\prod \{(1-p^{-4})^{-1}(1+p^{-3})\}$$

taken over $p \nmid 2d$, to all odd p and then estimate it numerically. So, assuming $d \geqslant 3$ and using (5.7), as we may by (16.3), and using also (16.2), we have

(16.4) $$\Pi_1 < 0.385d \prod \{p^{-1}\colon\ p \nmid 2d,\ p|D_1\}.$$

Here $\prod$, on the right, is $D_1^{-1}d$ if D_1 is odd, $2D_1^{-1}d$ if not.

For Π_2 we use (15.1) and (13.8), giving

(16.5) $$\Pi_2 < \tau(d)\,d^{1-\frac{1}{4}n},$$

where $\tau(d) = \prod\{2\colon\ p|d\}$ is the number of divisors of the square-free integer d.

Now Π_3 is 1 by definition unless d is odd. If so, by (13.9), (15.1), (15.3) it is at most $8/3$ if $2 \nmid D_1$, $4/3$ if $2|D_1$ but $2 \nmid D_2$, and considerably smaller than $4/3$ if $2|D_2$. Replace $\Pi_3 = 1$, by $4/3$ for the even d, which are less important; then (16.4) and (16.5) give

(16.6) $$S_d(f, 2) < 1.027D_1^{-1}\tau(d)\,d^{3-\frac{1}{4}n}.$$

By estimating the infinite series $\sum \tau(d)\,d^{-4}$, taken over square-free $d \geqslant 3$, we deduce from (16.6), (16.3), (3.3) that

(16.7) $$S(f, 2) < 0.0342D_1^{-1}\cdot 3^{7-\frac{1}{4}n}.$$

With the obvious $S(f, 2) = S_3(f, 2)$ if $D_1 = 3$, (16.7) gives

(16.8) $$S(f, 2) < 0.0085\cdot 3^{7-\frac{1}{4}n}.$$

The argument for (16.3) shows that if $d = 6$ then g is equivalent to a multiple of $ax_1^2 + 6a^{-1}x_2^2$, $a = 1$ or 2, whence $S_6(f, 2) = 0$ and so we have, from (16.6),

(16.9) $$S(f, 2) < 0.0033,\ 0.0043 \quad \text{for} \quad d = 5, 6.$$

17. Proof of theorem for $n \geqslant 19$. We shall estimate $M_p(f, k)$ defined in (3.8), by using the bounds we have found for $U_d(f, k, p)$. We have to consider the three cases $n_2(f, p) = 0, = 1, \geqslant 2$. In the first of these cases the second term on the right of (3.8) is 0, because $k_2 \leqslant n_2 = 0$ and so, see (9.4), (3.4) is impossible if $p|d$. We shall verify that *the bound is always greatest in the case* $n_2 = 0$.

First suppose that $k = 1$ and $p \geqslant 2$. For $p \nmid d$ we refer to (13.4), (13.5), and (13.1) (with $k_2 = 0$, $t = n_2 = 2$) in the three cases distinguished above. For $p|d$ we refer, in the second and third cases, to (13.1) (with $k_1 = 1$, $t = n_1 \geqslant 7$). And we also need (15.1). It is easily seen that the italicized assertion above is correct and so $M_p(f, 1) \leqslant (1-p^{-7})^{-1}$.

Next suppose $k = 1$ and $p = 2$. We refer to (14.1), (14.2), and (15.1), and we similarly find that $M_2(f, 1) \leqslant 2^{-1} + 2^{-7} + 2^{-12}$.

The product $\prod(1-p^{-7})^{-1}$, taken over all odd p, is easily seen to be less than 1.0005. So from the foregoing inequalities, and (3.9), we have

(17.1) $$S(f, 1) < 0.509.$$

For $k \geqslant 3$, and all p, we proceed similarly, referring to (13.3) and (15.1), $(13.1)_1$ and (15.1), (15.6), (15.5); with $k_2 = 0$ and so $t = k_1l_2 = kn_2 = 0$, k in the first two cases. The obvious $V_0(p, k) > 1$ is enough to show that the italicized assertion above is correct, so, with $V_0(2, k) = 2$,

(17.2) $$M_p(f, k) \leqslant \{P(k, p)\}^{-1}V_0(p, k) \quad \text{for} \quad k \geqslant 3.$$

We now refer to (13.2) and estimate $\prod V_0(p, k)$, taken over $p \geqslant 3$; we also use $(5.5)_1$. The necessary calculations can be made finite by noticing that $V_0(k, p) \leqslant V_0(8, p)$ for $k \geqslant 9$. So we find from (3.9) and (17.2) that

(17.3) $$S(f, k) < 3.31,\ 4.64,\ 3.60,\ 3.90,\ 3.70,\ 3.78$$

for $k = 3, \ldots, 7$ and $k \geqslant 8$ respectively.

The foregoing estimates hold for $n \geqslant 14$ and so can be used later; for the rest of this section we suppose $n \geqslant 19$.

From (1.2), (2.8), (17.1), (16.9) and (17.3) we find, very crudely, that

(17.4) $$W(f) < 0.6\theta(1, n-1) + 0.1\theta(2, n-2) + {} + 10\sum\{\theta(k, n-8)\colon\ 3 \leqslant k \leqslant \tfrac{1}{2}n\}.$$

Using the first part of (4.5), (17.4) gives

(17.5) $$W(f) < 0.6\,\theta(1, n-1) + 0.1\,\theta(2, n-2) + 18\,\theta(3, n-3) + {} + 18\,\theta(4, n-4)$$

(by estimating separately the sums over $k = 5, 7, \ldots$ and $k = 6, 8, \ldots$). Then with the second part of (4.5), (17.5) gives

$$W(f) < 1.5\,\theta(1, n-1) + \theta(2, n-2). \tag{17.6}$$

Now we use (4.2), noting that the right member is greatest for $n = 19$ and then equal to $\pi^8/8! < 10^4/8! < \frac{1}{4}$. So (17.6) gives, with (4.1),

$$W(f) < 1.75\,\theta(1, n-1) = 1.75\,\pi^{\frac12 n}/\Gamma(\tfrac12 n). \tag{17.7}$$

By the remark following (4.2), $\pi^{\frac12 n}/\Gamma(\frac12 n)$ is greatest for $n = 19$, and less than $(\pi/8)^{\frac12}\pi^{\frac12 n-\frac12}/\Gamma(\frac12 n-\frac12)$. So (17.7) gives

$$W(f) < 1.75(\pi/8)^{\frac12}\pi^9/8! < 1.1\,\pi^9/8! < 3.5\pi^8/8!, \tag{17.8}$$

which is less than $7/8$ since $\pi^2 < 10$ and $8! = 40320$.

The proof for $n \geqslant 19$ is now complete; and later, using $n \leqslant 18$, we shall have bounds for k, which will enable us to make more use of (1.4). We shall see that (17.3) is very weak for $3 \leqslant k \leqslant 7$.

18. Further inequalities for $S(f, 1)$. As shown in § 17,

$$M_p(f, 1) \leqslant \begin{cases} (1-p^{-7})^{-1} & \text{for} \quad p \geqslant 3, \\ 2^{-1}+2^{-7}+2^{-12} & \text{for} \quad p = 2. \end{cases} \tag{18.1}$$

When $p|D(f)$, see (16.1), we need to do better. By (15.1) and (3.8), for such p, we have

$$M_p(f, 1) = \max_{p\nmid d} T_d(f, 1, p, 0) + \max_{p|d} T_d(f, 1, p, 0). \tag{18.2}$$

Supposing first that $p \geqslant 3$ and $p|D(f)$, that is, $n_2 = n_2(f, p) \geqslant 1$, we use (9.8), with $R(1, p) = 1$ by (5.2) and with $k_2 = 0, 1$ for $p\nmid d$, $p|d$ respectively, whence by (6.11), (6.14) we have $t = n_2, n_1$. By Lemma 2 the V in (9.8) satisfies (10.1) when $k_2 = 0$ and $V \leqslant 2$ for $k_2 = 1$. In (10.1) we have, with $k_2 = 0$, $k_1 = 1$, $l_1 = n_1 - 1 \geqslant 13 - n$, giving $V \leqslant 1 + p^{-\frac12 l_1}$ if $2|l_1$ and $V^{-1} \geqslant 1 - p^{-\frac12 n_1}$ if not. Crudely, these estimates give

$$M_p(f, 1) \leqslant p^{-\frac12 r}(1+p^{-5})(1+2p^{r-7}) \quad \text{if} \quad 1 \leqslant r \leqslant 3 \text{ and } 2 < p|D_r. \tag{18.3}$$

(Check this first (for $n \geqslant 14$) with the additional assumption $p\nmid D_{r+1}$, that is, $n_2 = r$. Then note that the right member of (18.3) is a decreasing function of r, and that we can obviously do better if $n_2 \geqslant 4$.)

Now taking $p = 2$ we use (14.1) and (14.2), with the remarks at the end of § 14. Considering the cases $n_2 = 1, \ldots, 6$; $n_2 = 7$ and $n = 14$; $n_2 = 7$ and $n \geqslant 15$; and $n_2 = 8$, implying $n \geqslant 16$, separately it will be found that

$$M_2(f, 1) \leqslant 2^{-1-\frac12 r}(1+2^{r-6}+2^{r-12}) \quad \text{if} \quad 1 \leqslant r \leqslant 3 \text{ and } 2|D_r. \tag{18.4}$$

Now from (3.9), and the above estimates, we have an inequality for $S(f,1)$ which we simplify by estimating $\prod(1+p^{-5})$ and $\prod(1+2p^{r-7})$, taken over all odd p. We find that

$$S(f, 1) < \min\{0.509,\ 0.521\,D_1^{-1/2},\ 0.541(D_1D_2)^{-1/2},\ 0.585(D_1D_2D_3)^{-1/2}\}. \tag{18.5}$$

We notice now that if f is of the shape (7.1) with φ_0 null then $f \underset{2}{\sim} dx_1^2 + h$ implies $2|d$ and so the first term on the right of (18.2) is zero. Using (14.1) to estimate the second term, we can crudely multiply the right member of (18.5) by $1/4$.

19. Further inequalities for $S(f, k)$, $3 \leqslant k \leqslant 8$. We shall show that

$$S(f, k) < S^{(0)}(f, k) \quad \text{for} \quad k = 3, \ldots, 8, \tag{19.1}$$

where

$$S^{(0)}(f, 3) = 3.57 D_1^{-3/2}\tau(D_1)(D_1, 2)^{-1}, \tag{19.2}$$

$$S^{(0)}(f, 4) = 5.22 D_1^{-2}\tau(D_1)(D_1, 2)^{-1}, \tag{19.3}$$

$$S^{(0)}(f, 5) = 5.48 D_1^{-5/2}\tau(D_1)(D_1, 2)^{-1}, \tag{19.4}$$

$$S^{(0)}(f, 6) = 3.90 D_1^{-3}\tau(D_1)(D_1, 2)^{-1}\prod\{1+p^{-1}:\ p|D_1\}, \tag{19.5}$$

$$S^{(0)}(f, 7) = 3.70 D_1^{-7/2}\tau^2(D_1)(D_1, 2)^{-1}, \tag{19.6}$$

$$S^{(0)}(f, 8) = 3.78 D_1^{-4}\tau^2(D_1)(D_1, 2)^{-1}. \tag{19.7}$$

Here $D_1 = D_1(f)$, see (16.1), is square-free, τ is the divisor function, and $(D_1, 2)$ is the g.c.d. of $D_1, 2$; so $\tau(D_1)/(D_1, 2) = 2^e$, where e is the number of odd primes dividing D_1.

To prove (19.1) we need

$$M_p(f, k) \leqslant 2\{P(k, p)\}^{-1}(p^{-\frac12 k} + p^{-\frac12 l}) \quad \text{if} \quad k \geqslant 3 \text{ and } p|D_1. \tag{19.8}$$

To prove (19.8) in the case $n_2(f, p) = 1$, we refer to (13.1) and (15.1), taking $k_2 = 0, 1$, which by (6.11) and (6.14) give $t = k, l$ respectively. For $n_2 \geqslant 2$, (15.6) and (15.5) give

$$M_p(f, k) \leqslant 2\{R(k, p)\}^{-1}\{p^{-k} + p^{-\frac12 n} + p^{\frac12-\frac12 n}(1+p^{-10})\},$$

from which, with $R(k, p) > (2/3)P(k, p)$ (as is easily verified), (19.8) follows, crudely.

We compare (19.8) with the inequality (17.2) used in § 17 and so see that (17.3) and (19.8) give, for $k \geqslant 3$,

$$S(f, k) \leqslant S_k D_1^{-\frac12 k}\prod\{2(1+p^{k-\frac12 n})V_0^{-1}(p, k):\ p|D_1\}, \tag{19.9}$$

where S_k is the bound for $S(f, k)$ given by (17.3), and $V_0(p, k) = 2$ if $p = 2$, is given for odd p by (13.2).

The cases $k = 7, 8$ of (19.1) follow at once from (19.9) on using $V_0(p, k) = 1, 2$ for $p = 3$, $p = 2$, and $k - \frac{1}{2}n \leqslant 0$. Similarly for $k = 6$, with $k - \frac{1}{2}n \leqslant -1$ since $n \geqslant 14$. For $k = 3, 5$, the product $\prod$ in (19.9) is at most

$$\tau(D_1)(D_1, 2)^{-1} \prod (1 + p^{k-7});$$

and taking this last product over all p (19.1) follows. For $k = 4$ we have

$$V_0(p, 4) = (1 + p^{-2})(1 - p^{-5})^{-1} > 1 + p^{4-\frac{1}{2}n} \qquad (p \neq 2)$$

and this completes the proof of (19.1), since it makes $\prod$ in (19.9) $\leqslant (1 + 2^{4-\frac{1}{2}n})\tau(D_1)(D_1, 2)^{-1}$.

We shall now show that

$$M_p(f, k) \leqslant 2\{P(k, p)\}^{-1}(p^{-k} + p^{1-\frac{1}{2}n} + p^{k-n})(1 - p^{-2})^{-1} \quad \text{if } 4 \leqslant k \leqslant 7 \text{ and } p | D_2. \tag{19.10}$$

If follows easily from (5.1), (5.2) that $R(k, p) \geqslant (1 - p^{-2}) P(k, p)$ for $4 \leqslant k \leqslant 7$, so (19.10) follows from the second part of (13.1) by the argument used to deduce (19.8) from the first part, if we prove a suitable inequality for $\sum p^{-\frac{1}{2}t}$, with summation over $0 \leqslant k_2 \leqslant \min(k, n_2)$, see (6.11), with t given in terms of k_2 by (6.11), (6.14). Omitting the term with $k_2 = 0$, $t = kn_2$, which is at most p^{-k} since $p | D_2$ implies $n_2 \geqslant 2$, what we need is

$$\sum \{p^{-\frac{1}{2}t}\colon 1 \leqslant k_2 \leqslant \min(k, n_2)\} \leqslant p^{1-\frac{1}{2}n} + p^{k-n} \quad \text{if } p | D_2 \text{ and } 4 \leqslant k \leqslant 7. \tag{19.11}$$

To prove (19.11), note first that if $n_2 = 2$ then the sum has just two terms, with $t = n-2$, $2n - 2k$, and that if $n_2 = 3$ it has three, with $t = n + k - 4$, $n + l - 4$, $3n - 3k$, $\geqslant n$, $n+3$, $n+8$. For $n_2 = 4$ there are four terms, each with $t \geqslant n+2$. For $n_2 > 4$ there are at most seven terms, each with $t \geqslant n+4$, implying $p^{-\frac{1}{2}t} \leqslant 8^{-1}p^{1-\frac{1}{2}n}$. So (19.11) is proved and (19.10) follows.

Estimating the ratio of the right member of (19.10) to that of (19.8), we find that

$$S(f, k) \leqslant S^{(0)}(f, k) D_2^{-\frac{1}{2}k} \prod \{(1 - p^{-2})^{-1}(1 + p^{k+1-\frac{1}{2}n})\colon p | D_2\} \quad \text{if } 4 \leqslant k \leqslant 7. \tag{19.12}$$

20. Use of condition (1.4); proof of theorem for $n = 18$. We shall prove:

LEMMA 3. (i) *Suppose $2 \leqslant k \leqslant 13$, $D(g) = 1$, and $k \neq 12$ if g is primitive. Then either $W_1(g) = 1$ or $k = 8$ and*

$$g \underset{2}{\simeq} 2(x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8).$$

(ii) (i) *holds also with l, k in place of k, g, in hypothesis and conclusion.*

Proof. Obviously we need only prove (i). With $D(g) = 1$, g is of the shape (7.1) with φ_1 and ψ_2 each null; so we write $g \underset{2}{\simeq} \varphi + 2\psi$.

Suppose first that g is primitive, that is, φ is not null. Then, see (vii) of § 7, there are at most two possibilities for g up to $\underset{2}{\simeq}$. We may take these to be

$$g \underset{2}{\simeq} x_1^2 + x_2^2 + \ldots + x_{k-2}^2 + \varepsilon(x_{k-1}^2 + x_k^2), \qquad \varepsilon = \pm 1. \tag{20.1}$$

From $D(g) = 1$ it follows trivially that (20.1) holds also, for either value of ε, for every prime $p > 2$. By [4], p. 72, Theorem 43, two forms f, f' with $f \underset{p}{\simeq} f'$ for every p have signatures with the same residue modulo 8. So from (20.1), with g positive definite, we have $k \equiv k - 2 + 2\varepsilon \pmod 8$, $\varepsilon = 1$. Now (0.3), with g, k for f, n, and $2 \leqslant k \leqslant 13$, $k \neq 12$, shows that g is equivalent to a disjoint form with a unary summand. Trivially, this holds for every g in the genus under consideration, and $W_1(g) = 1$ follows.

Now suppose g imprimitive, that is, $g \underset{2}{\simeq} 2\psi$. Then obviously $2|k$; but if $k \equiv 2 \pmod 4$ then $D(g) \equiv -1 \pmod 4$ gives a contradiction, so $4|k$. Now ψ of the shape (7.3) gives $D(g) \equiv -3 \pmod 8$, so ψ is of the shape (7.2), that is,

$$g \underset{2}{\simeq} 2(x_1x_2 + x_3x_4 + \ldots + x_{k-1}x_k), \qquad k \equiv 0 \pmod 4. \tag{20.2}$$

This is trivially true also with $\underset{p}{\simeq}$ for $\underset{2}{\simeq}$, $p > 2$. So the argument used to exclude $\varepsilon = -1$ in (20.1) gives $k \equiv 0 \pmod 8$, whence with $k < 16$ we have $k = 8$, and the proof is complete.

When $D(f) = 1$, or equivalently $D_1 = D_1(f) = 1$, $f \simeq g + h$ implies $D(g) = D(h) = 1$. So an obvious corollary of the lemma is

$$D_1(f) = 1 \Rightarrow S(f, k) = 0 \quad \text{for} \quad k = 2, \ldots, 7. \tag{20.3}$$

Using (19.1)–(19.6) if $D_1 \geqslant 2$, and (20.3) if not, we have

$$S(f, k) < 1.27,\ 1.31,\ 0.97,\ 0.74,\ 0.66 \quad \text{for} \quad k = 3, \ldots, 7. \tag{20.4}$$

We shall see that (20.4) is very weak, but it suffices to prove the theorem for $n = 18$. For taking $n = 18$ in (1.2) and (2.8), and referring to the fifth column of the table in § 4, we find $W(f) < 0.6$ on using (20.4) for $k = 3, \ldots, 7$, (17.3) for $k = 8, 9$, and (17.1), (16.8) for $k = 1, 2$.

21. Improvement on (19.1)–(19.6). We begin by defining $N_p(f, k)$, for $3 \leqslant k \leqslant 7$, in the same way as $M_p(f, k)$, see (3.8), but with the additional summation condition $p | D(g)$; and we prove

$$N_p(f, k) \leqslant 2\{P(k, p)\}^{-1} p^{\frac{1}{2}k - \frac{1}{2}n}. \tag{21.1}$$

If $p \nmid D(f)$ this is trivial, since $p | D(g)$ is impossible. If $n_2 = n_2(f, p) = 1$, then (21.1) follows from (19.8) by suppressing the term involving $p^{-\frac{1}{2}k}$, which corresponds to $k_2 = 0$, implying $p | D(g)$. For $n_2 \geqslant 2$, we suppress the term in p^{-k}, again corresponding to $k_2 = 0$, in the formula used to prove (19.8) for $n_2 \geqslant 2$.

Next, for any integer $m \geqslant 2$, define $S^{(m)}(f, k)$ in the same way as $S(f, k)$, see (2.7), but with the additional condition $m|D(g)$. Then, cf. (3.9), we have obviously

(21.2) $$S^{(m)}(f, k) \leqslant \prod \{M_p(f, k):\ p\nmid m\} \prod \{N_p(f, k):\ p|m\}.$$

If we compare (21.1) with (19.8), which sufficed to prove (19.1), we see that (21.2) gives

(21.3) $$S^{(m)}(f, k) \leqslant S^{(0)}(f, k) \prod \{p^{k-\frac{1}{2}n}:\ p|m\}.$$

More precisely, the factor $p^{k-\frac{1}{2}n}$ in (21.3) may be replaced by $p^{k-\frac{1}{2}n}(1+p^{k-\frac{1}{2}n})^{-1}$. It is worth while to do so if $k = 6$; for then the factor $1+p^{-1}$ in (19.5), which could have been replaced by $1+p^{k-\frac{1}{2}n}$, can be omitted, for $p|m$. Similarly, if $k = 7$, for some factors 2 in (19.6), which could also be replaced by $1+p^{k-\frac{1}{2}n}$.

Now let $\mathscr{M}$ be a finite set of integers each $\geqslant 2$ such that

(21.4) $$(1.4) \Rightarrow m|D(g) \quad \text{for some } m \in \mathscr{M}$$

(for fixed f, k with $3 \leqslant k \leqslant 7$). Then, because of the condition (1.4) in (2.7) we have

(21.5) $$S(f, k) \leqslant \sum \{S^{(m)}(f, k):\ m \in \mathscr{M}\}.$$

Lemma 3 now shows that (21.4) holds if $\mathscr{M}$ is taken to be the set of primet dividing $D(f)$, or $D_1(f)$; for then, with $D(g)|D(f)$, (21.4) is equivalens to $D(g) > 1$. So we have

(21.6) $$S(f, k) \leqslant S^{(0)}(f, k) \sum \{p^{k-\frac{1}{2}n}:\ p|D_1\}, \quad \text{for} \quad 3 \leqslant k \leqslant 7.$$

We now improve on the case $k = 3$ of (21.6) by proving:

LEMMA 4. *If $k = 3$ and either $D_1(g) \leqslant 6$ or $D_1(g) = 10$, then $W_1(g) = 1$*

Proof. We have to prove that every 3-ary g with $D_1(g) \leqslant 6$ or $= 10$ is equivalent to a disjoint form, with one summand necessarily of rank 1. Obviously, see (16.1), $D(g)|\{D_1(g)\}^3$; but by taking out the divisor of g we may suppose $D(g)|\{D_1(g)\}^2$. By considering the reciprocal of g, we may suppose $4\nmid D(g)$. If $p^2|D(g)$, $p = 3$ or 5, we construct a form G_p, with $D(G_p) = p^{-1}D(g)$, which is equivalent to a disjoint form if and only if g is so. This is done as in [6], pp. 179–181; or [7], p. 2.

We may therefore suppose $D(g) = D_1(g)$. It is well known that $\min g$, the minimum of g, satisfies $(\min g)^3 \leqslant 2D(g) < 27$, so $\min g = 1$ or 2. If $\min g = 1$ we may suppose $g = x_1^2 + a_2x_1x_2 + a_3x_1x_3 + g(0, x_2, x_3)$, with each $a_i = 0$ or 1, and each 0 since g is of the shape (7.1). Then g is disjoint; so suppose $\min g = 2$. Transforming g rationally into $2x_1^2 + g'(x_2, x_3)$, with $D(g') \leqslant 5$, we have $\min g' \equiv 0$ or $\frac{1}{2} (\mathrm{mod}\, 1)$, else g could not be of the shape (7.1), and $\min g' \leqslant (4/3)D(g') \leqslant 20/3$, by a well known inequality. It follows that $\min g' \leqslant 5/2$; whence, if we suppose g reduced, we have at most three possibilities for its leading binary section. Bordering each of these binary forms we find easily that every possible reduced g, of the shape (7.1), is disjoint.

Lemma 4 gives us that, for $k = 3$, (1.4) implies that $D(g)$ is divisible by a prime $p \geqslant 7$, or by 15. So in (21.4) we may take $\mathscr{M}$ to be the set of primes $\geqslant 7$ that divide $D_1 = D_1(f)$, together with 15, if $15|D_1$. Then (21.3) and (21.5) give us that (21.6) remains valid, for $k = 3$, if the factor $\sum$ is modified by excluding the terms with $p \leqslant 5$, but adding a term $15^{3-\frac{1}{2}n}$ if $15|D_1$. The resulting inequality for $S(f, 3)$ is clearly weakest if $n = 14$ and $D_1(f) = 7$; and then, see (19.1) and (19.2), we find

(21.7) $$S(f, 3) < 0.00017,$$

with $S(f, 3) = 0$ if $D_1(f) \leqslant 6$ or $= 10$, by Lemma 4.

We shall pursue this argument further, for $k = 4, \ldots, 7$, after we have proved the theorem for $n \geqslant 16$ and so got rid of the term with $k = 8$ in (1.2). It will be useful later to note that

(21.8) $$N_p(f, k) \leqslant 2\{P(k, p)\}^{-1}(1-p^{-2})^{-1}(p^{1-\frac{1}{2}n}+p^{k-n}) \quad \text{if } 4 \leqslant k \leqslant 7 \text{ and } p|D_2.$$

This follows from (19.11) by the argument used for (19.10), omitting the term with $k_2 = 0$ by the definition of $N_p(f, k)$.

22. Proof of theorem for $n = 16$, 17. The sum in (1.2) has to be taken over $k = 1, \ldots, 8$ and we estimate its terms by (2.8) and columns 3, 4 of the table in § 4. Comparing these two columns we see that (2.8) is always weaker for $n = 16$ than for $n = 17$. From (18.5) we have

(22.1) $$W_1(f) < \min\{0.963,\ 0.985D_1^{-1/2},\ 1.023(D_1D_2)^{-1/2}\}.$$

From (16.8) and (21.7) we have

(22.2) $$W_2(f) + W_3(f) < 0.002.$$

For $k = 4, \ldots, 8$, let S_k be the numerical constant on the right of (19.3), …, (19.7) respectively. To save the labour of estimating $S(f, k)$ and then multiplying by the bound given by the table for $(1+\mathrm{sgn}(n-2k))\,\theta(k, n-k)$, note that

(22.3) $$\{1+\mathrm{sgn}(n-2k)\}\,\theta(k, n-k)S_k < 7.21,\ 4.61,\ 2.19,\ 1.63,\ 0.76 \quad \text{for } k = 4, \ldots, 8.$$

Suppose first that $D_1 \geqslant 3$. We use (21.6) for $k = 4, \ldots, 7$, (19.1) for $k = 8$, also (22.3). The resulting estimate is easily seen to be weakest for $D_1 = 3$, and we calculate that

(22.4) $$W_k(f) < 0.020,\ 0.022,\ 0.025,\ 0.047,\ 0.038 \quad \text{for} \quad k = 4, \ldots,$$

From (22.1), giving $W_1(f) < 0.6$, and (22.2), (22.4), the desired result $W(f) < 1$ follows with much to spare; so we may suppose $D_1 \leqslant 2$.

Next suppose $D_1 = 2$. Estimating $W_k(f)$ for $k = 4, \ldots, 8$ as above, we have in place of (22.4) the weaker

(22.5) $$W_k(f) < 0.113,\ 0.103,\ 0.103,\ 0.147,\ 0.095 \quad \text{for} \quad k = 4, \ldots, 8.$$

This with (22.2) gives $W_2(f) + \ldots + W_8(f) < 0.561$. This estimate is too weak; but crudely, by comparing columns 3, 4 of the table, we could divide it by 4 in case $n = 17$, and then (22.1) would give $W(f) < 1$, so we suppose $n = 16$.

If we suppose $D_1 = D_2 = 2$, we can improve on (22.5) by comparing the estimates (21.1), (21.8), for $p = 2$, $4 \leqslant k \leqslant 7$, $n = 16$. Thereby we see that (21.6) remains valid with a factor $(4/3)(2^{1-\frac{1}{2}k} + 2^{\frac{1}{2}k-8})$ on the right. This factor is $\leqslant \frac{3}{4}$ for $k = 4, 5$, $< \frac{1}{2}$ for $k = 6, 7$; so we find $W_2(f) + \ldots + W_8(f) < 0.382$. With this, the third part of (22.1) gives $W(f) < 0.936$. So we suppose $D_2 = 1$, $D(f) = D_1 = 2$.

With this $f \simeq g + h$ implies $D(g)D(h) = 2$, whence one of $D(g)$, $D(h)$ is 1. By Lemma 3, (1.4) fails for $k = 5, 6, 7$ if $D(g) = 1$, and for $l = n - k = 16 - k = 11, 10, 9$ if $D(h) = 1$; so $W_k(f) = 0$ for $k = 5, 6, 7$. With the cases $k = 4, 8$ of (22.5), this gives $W_2(f) + \ldots + W_8(f) < 0.21$; and with (22.1) we have $W(f) < 0.948$. So the case $D_1 > 1$ is disposed of.

Finally, suppose $D_1 = D(f) = 1$. Then $f \simeq g + h$ implies $D(g) = D(g) = 1$. This, by Lemma 3, contradicts (1.4) for $k = 2, \ldots, 7$; and also, in case $n = 17$, for $k = 8$, $l = 9$. With $W_1(f) < 1$ by (22.1), this gives $W(f) < 1$ if $n = 17$, and $W(f) = W_1(f) + W_8(f)$ if $n = 16$. And we have $W_8(f) < 0.76 < 1$ by (19.1), (19.7), (22.3). Referring again to Lemma 3, we see that (1.4) is impossible for $k = l = 8$ unless $f \simeq_2 2(x_1x_2 + \ldots + x_{15}x_{16})$. As observed at the end of § 18, this makes $f \simeq g + h$ impossible for $k = 1$, giving $W_1(f) = 0$. With $W_1(f)$, $W_8(f)$ each < 1 and not both positive, we have $W(f) < 0.963$ and the proof is complete.

23. Preliminaries for $n = 14, 15$. We begin as in § 22; the sum in (1.2) is over $k = 1, \ldots, 7$, and the first and second columns of the table in § 4 show that (2.8) is weaker, for each k, for $n = 14$ than for $n = 15$. From (18.5) we have

(23.1) $$W_1(f) < \min\{2.20 D_1^{-1/2},\ 2.28(D_1D_2)^{-1/2},\ 2.46(D_1D_2D_3)^{-1/2}\},$$

and from (16.8) and (21.7),

(23.2) $$W_2(f) < \min\{0.135,\ 0.542 D_1^{-1}\}, \qquad W_3(f) < 0.004.$$

In place of (22.3) we have

(23.3) $$\{1 + \operatorname{sgn}(n - 2k)\}\,\theta(k, n-k)\,S_k < 111,\ 113,\ 77,\ 36 \quad \text{for} \quad k = 4, 5, 6, 7.$$

We first show that the theorem is true if D_1 has at least three prime factors; we shall do so by using (23.1), (23.2) for $k \leqslant 3$, (21.6) for $k = 4, 5$, and (19.1) for $k = 6, 7$. The bound resulting is easily seen to be weakest when $D_1 = 2\cdot3\cdot5 = 30$, and we find that $W_k(f)$ is less than 0.422, 0.019, for $k = 1, 2$, 0.082, 0.037, 0.028, 0.008 for $k = 4, \ldots, 7$, which with the second part of (23.2) gives $W(f) < 1$ with much to spare.

We next show that the theorem is true if D_1 is the product of two distinct odd primes; we do so as above except that we use (21.6) instead of (19.1) for $k = 6$. Again the resulting bounds are weakest for the smallest $D_1 = 15$. We find $W_k(f)$ less than 0.593, 0.037, 0.023, 0.078, 0.078, 0.011 for $k = 1, 2, 4, 5, 6, 7$, and $W(f) < 0.824$.

Next, we define $N_p^{(r)}(f, k)$, for $r \geqslant 1$, as in (21.1) if $r = 1$, but with the condition $p | D(g)$ replaced by $p^r | D(g)$, or equivalently by $p | D_r(g)$, see (16.1). We prove that

(23.4) $$N_p^{(2)}(f, k) \leqslant 2\{P(k, p)\}^{-1}(1 - p^{-2})^{-1} p^{k-n} \quad \text{if} \quad 5 \leqslant k \leqslant 7.$$

This is trivially true unless $p^2 | D(f)$, that is, $p | D_2(f)$, or $n_2(f, p) \geqslant 2$; see (16.2). Assuming $n_2 \geqslant 2$, the argument used to deduce (19.10) from (19.11) shows that (23.4) follows from

(23.5) $$\sum \{p^{-\frac{1}{2}t}:\ 2 \leqslant k_2 \leqslant \min(k, n_2)\} \leqslant p^{k-n} \quad \text{if} \quad p | D_2 \quad \text{and} \quad 5 \leqslant k \leqslant 7.$$

This is proved in the same way as (19.11). Similarly but more simply, reducing the sum in (23.5) to a single term with $k_1 = k_2 = 2$, $t = 2l_1 + 2l_2 = 2l = 2n - 8$, we have

(23.6) $$N_p^{(2)}(f, 4) - N_p^{(3)}(f, 4) \leqslant 2\{P(4, p)\}^{-1} p^{4-n} (1 - p^{-2})^{-1}.$$

Now, comparing columns 1, 2 of the table in § 4 again, we note that the numerical constants in (23.1) may be replaced for $n = 15$ by 1.51, 1.56, 1.69. Similarly, the constants in (23.2) may be replaced by 0.066, 0.262; and those in (23.3) by 31, 26, 15, 12.

24. Sharper estimates for $S(f, k)$, $k \geqslant 4$. We need the following improvement on some cases of Lemma 3.

LEMMA 5. *Suppose that* $f \simeq g+h$ *and* (1.4) *holds; then* (i)–(v) *below follow*:

(i) *If* $k=4$ *then* $D(g) \neq 9$ *and* $D(g) \leqslant 6$ *implies that* $D(g) = 4$ *or* 5 *and* $2|g$ (*meaning that* g *has divisor* 2).

(ii) *If* $k=5$ *then* $D(g) \neq 1, 2, 3, 4, 5$ *or* 9.

(iii) *If* $k=6$ *then* $D(g) \leqslant 6$ *implies* $D(g) = 3$ *or* 4 *and* $2|g$.

(iv) *If* $k=7$ *then* $D(g) \neq 1, 3$ *or* 6 *and* $D(g) = 2$ *implies* $2|g$.

(v) (iv) *holds with* l, h *for* k, g; *and if* $l = 8$ *then* $D(h) \neq 2$.

Proof. We shall deduce (i) from Lemma 4; then precisely similar arguments give (ii)–(v) inductively. Suppose therefore $k=4$ and $D(g) \leqslant 6$, or $=9$, and as in Lemma 4, suppose $\min g > 1$. The well known inequality $\min^4 g \leqslant 4D(g)$, with equality only when $(\min g)^{-1} g$ is equivalent to a certain form E_4 which has integer coefficients, gives a contradiction for $D(g) \leqslant 3$; and for $D(g) = 4$ it gives $\min g = 2$, $g \sim 2E_4$. So we may suppose $D(g) = 5$, 6 or 9.

Applying the above inequality to $\operatorname{adj} g$, the adjoint form of g, we have $D(\operatorname{adj} g) = \{D(g)\}^3$, $\min \operatorname{adj} g \leqslant 7$ always, $\leqslant 5$ if $D(g) \leqslant 6$. Now g has a 3-ary section, say g_3, with $D(g_3) = \min \operatorname{adj} g$. Since g is of the shape (6.1), with $p=3$, we have $3|D(g_3)$ if $D(g) = 9$, and so $D(g_3) \leqslant 6$ always. We have

$$g \sim g_3(x_1 + r_1 x_4, \ldots, x_3 + r_3 x_3) + \{D(g)/D(g_3)\} x_4^2, \tag{24.1}$$

where the r_i are rational numbers with each $|r_i| \leqslant \frac{1}{2}$, and with denominators dividing $D(g_3)$. We also have $\min g_3 \geqslant \min g \geqslant 2$; and the product terms in g_3 all have even coefficients.

If we suppose that g_3 is of the shape (7.1), and note that it is trivially of the shape (6.1) for every odd p, since if not then $p^2 | D(g_3) \leqslant 6$, then we can appeal to Lemma 4, and with $\min g_3 \geqslant 2$ it is easy to see that $g = 2x_1^2 + 2(x_2^2 + x_2 x_3 + x_3^2)$ is the only possibility up to equivalence. Then in (24.1) $r_1 = 0$ or $\pm\frac{1}{2}$, $r_i = 0$ or $\pm\frac{1}{3}$ for $i = 2, 3$; and $r_1 = 0$ or $r_2 = r_3 = 0$ makes g disjoint. $r_1 = 1/2$, $r_2 = 1/3$ gives only $D(g) = 5$, $2|g$.

So suppose g_3 not of the shape (7.1). We can only have $g \underset{2}{\sim} 2\psi + 4a x_3^2$, a odd. $\min g_3 \geqslant 2$ gives $g_3 \sim 2E_3$, E_3 an extreme 3-ary form. We may suppose g_3 identically congruent modulo 32 to $2\psi + 12x_3^2$, ψ of the shape (7.3). Then in (24.1) we must have $r_1 = r_2 = 0$, $r_3 = 0, \pm\frac{1}{2}$, or $\pm\frac{1}{4}$, $r_3 \neq 0$, to avoid a disjoint g. The first term on the right of (24.1) is now $\equiv x_3^2$ or $\frac{3}{4}x_3^2 \pmod 2$, and we have to have $D(g) = 4$ or 5 and $2|g$. This disposes of (i); for (ii)–(v) see above.

We now use Lemmas 3, 5 to obtain inequalities for $S(f, k)$, $k = 4, 5, 6, 7$. Since we have disposed of all other cases, in §§ 17, 20, 22, 23, we shall assume $n = 14$ or 15 and

$$D_1(f) = 1, 2, p \text{ or } 2p, \quad p \geqslant 3. \tag{24.2}$$

Suppose first that $k = 4$; assume $f \simeq g+h$ and (1.4) satisfied, and note that (16.2) gives $D(g) | \{D_1(f)\}^4$. Lemma 5(i) shows that $D(f) \neq 1, 2$; but by transforming g as in the proof of Lemma 4 we have also $D(f) \neq 8, 16$. So either $D(g) \equiv 4 \pmod 8$ or $p | D(g)$, the latter possibility being excluded if $D_1(f) = 1$ or 2. It follows, cf. (21.5), that

$$S(f, 4) \leqslant S^{(4)}(f, 4) - S^{(8)}(f, 4) + S^{(p)}(f, 4), \tag{24.3}$$

with the last term omitted if $D_1(f) = 1$ or 2. We notice that for $p=2$, $k=4$, $n \geqslant 14$ the ratio of the right member of (23.6) to that of (19.8) is 1/216. So, by the argument used for (21.6), with (23.6) in place of (21.1), we have

$$S^{(4)}(f, 4) - S^{(8)}(f, 4) \leqslant \frac{1}{216} S^{(0)}(f, 4). \tag{24.4}$$

The left member of (24.4) is obviously zero if $4 \nmid D(f)$, or $2 \nmid D_1(f)$, which makes $4|D(g)$ impossible. For $D_1(f) = 2$ it is zero unless $2|D_3(f)$, because, with $l = 10$ or 11, Lemma 3 gives $D(h) > 1$.

For $D_1(f) = 3$ or 6, we can improve on the third term on the right of (24.3), so we use

$$S^{(p)}(f, 4) \leqslant p^{-3} S^{(0)}(f, 4), \tag{24.5}$$

see (21.3), only for $p \geqslant 5$. When $D_1(f) = 3$, we transform g as above and so see that all possibilities for $D(g)$ are excluded. If $D_1(f) = 6$ then $D(g) = 2^a 3^b$ with $0 \leqslant a, b \leqslant 4$, and if $D(g) = 2^a 3^b$ is possible then so too are $D(g) = 2^{4-a} 3^b$ and $D(g) = 2^a 3^{4-b}$. So with $a = b = 1$ excluded by Lemma 5 we must have $2|ab$, and if $a = 0$ or 4 all the b are excluded. The first two terms on the right of (24.3) take care of $a = 2$. So consider $a = 1$ or 3 and $b = 2$; and we have

$$S(f, 4) \leqslant S^{(4)}(f, 4) - S^{(8)}(f, 4) + S^{(18)}(f, 4) \quad \text{if} \quad D_1(f) = 3 \text{ or } 6, \tag{24.6}$$

with the last term on the right omitted if $D_1(f) = 3$. To make use of (24.6) we have, see (21.3),

$$S^{(18)}(f, 4) \leqslant S^{(6)}(f, 4) \leqslant \frac{1}{216} S^{(0)}(f, 4) \quad \text{if} \quad D_1(f) = 6. \tag{24.7}$$

Combining these results we have

$$S(f, 4)/S^{(0)}(f, 4) \leqslant \begin{cases} 0 & \text{if} \quad D_1(f) = 1 \text{ or } 3, \\ 0 & \text{if} \quad D_1(f) = 2 \text{ and } D_3(f) = 1, \\ 1/216 & \text{if} \quad D_1(f) = 2, \\ 1/108 & \text{if} \quad D_1(f) = 6, \\ p^{-3} & \text{if} \quad D_1(f) = p \geqslant 5, \\ 1/216 + p^{-3} & \text{if} \quad D_1(f) = 2p \geqslant 10. \end{cases} \tag{24.8}$$

There are obviously some possibilities for improvement by further distinction of cases.

The case $k = 5$ is simpler. Transforming g as above, we see that by Lemma 5 (ii) we cannot have $D(g)$ a power of 2 or of 3, so all possibilities are excluded if $D_1(f) \leqslant 3$, and all but those with $6|D(g)$ if $D_1 = 6$, or $p|D(g)$ if $p \geqslant 5$ and $D_1(f) = p$ or $2p$. Taking $\mathcal{M} \doteq \{6\}$ or $\{p\}$ in (21.5), we see as above that

$$S(f,5)/S^{(0)}(f,5) \leqslant \begin{cases} 0 & \text{if} \quad D_1(f) \leqslant 3, \\ 1/36 & \text{if} \quad D_1(f) = 6, \\ p^{-2} & \text{if} \quad D_1(f) = p \text{ or } 2p,\ p \geqslant 5. \end{cases} \tag{24.9}$$

Again, since $l = 9$ or 10 and so Lemma 3 gives $D(h) > 1$, there are possibilities for improvement.

Now take $k = 6$. From (24.2) and Lemma 5(iii) we have either $4|D(g)$ or $p|D(g)$, whence

$$S(f,6) \leqslant S^{(4)}(f,6) + S^{(p)}(f,6). \tag{24.10}$$

By the argument for (24.4), but using (23.4) instead of (23.6), we find

$$S^{(4)}(f,6) \leqslant \frac{1}{36} S^{(0)}(f,6), \tag{24.11}$$

the left member being however zero unless $2|D_2(f)$.

The second term on the right of (24.10) may be omitted unless $D_1(f) = p$ or $2p$. By (21.3) with $m = p$ it is at most p^{-1}; but if $4|D(f)$, or $2|D_2(f)$, we can do better by using (21.2) and estimating the factor $M_2(f,6)$. We note that the ratio of the right member of (19.10) to that of (19.8), with 2, 6 for p, k, is $\frac{1}{4}$, so

$$S^{(p)}(f,6)/S^{(0)}(f,6) \leqslant p^{-1},\ (4p)^{-1} \quad \text{for } 2 \nmid D_2(f),\ 2|D_2(f) \text{ respectively.} \tag{24.12}$$

Lastly, take $k = 7$. From (iv), (v) of Lemma 5, the latter giving $D(h) > 1$ if $n = 14$, we find by taking $\mathcal{M}$ in (21.4), (21.5) to be empty if $D_1(f) = 1$, $= \{2\}$ if $D_1(f) = 2$, that

$$S(f,7)/S^{(0)}(f,7) \leqslant \begin{cases} 0 & \text{if} \quad D_1(f) = 1, \\ 0 & \text{if} \quad D_1(f) = 2,\ n = 14, \text{ and } D_2(f) = 1, \\ \frac{1}{2} & \text{if} \quad D_1(f) = 2, \\ 2^{-5/2} & \text{if} \quad D_1(f) = 2 \text{ and } n = 14. \end{cases} \tag{24.13}$$

The third of these is essentially (21.6), with a factor $\frac{1}{2}$ put in by using the remark following (21.3). To save $1/2\sqrt{2}$ in the fourth case, in which $4|D(g)$ and $2|D_2(f)$, compare (21.1) and (21.8). We shall improve on (24.13) later.

For $k = 7$ it is difficult to improve on (19.1) when $D_1(f) = 2p \geqslant 10$. (21.6), improved as above, gives

$$S(f,7)/S^{(0)}(f,7) \leqslant \begin{cases} \frac{1}{2} & \text{if} \quad D_1(f) = p \geqslant 3, \\ \frac{1}{2}(1-p^{-1})^{-1}p^{-6} & \\ & \text{if also either } n = 14 \text{ or } p|D_2(f). \end{cases} \tag{24.14}$$

When $p = 3$, $D(g) = 3$ is impossible and so with $D_1(f) = 3$ we have to have $9|D(f)$, $3|D_2(f)$; and similarly for $n = 14$ we have $9|D(h)$, $3|D_4(f)$. So (24.13) may for $p = 3$ be replaced by

$$S(f,7)/S^{(0)}(f,7) \leqslant 1/972 \quad \text{if} \quad D_1(f) = 3, \tag{24.15}$$

the left member being 0 unless $D_{32-2n} = 3$.

With $k = 7$ and $D_1 = 6$, Lemma 5 gives $D(g) \neq 1$, 3 and so $D(g) = 2$ or $9|D(g)$, giving

$$S(f,7) \leqslant S^{(2)}(f,7) + S^{(9)}(f,7) \quad \text{if} \quad D_1(f) = 6. \tag{24.16}$$

The last term is zero unless $3|D_2(f)$, and by the methods used above we have

$$S^{(9)}(f,7)/S^{(0)}(f,7) \leqslant 1/48\sqrt{3} \quad \text{if} \quad D_1(f) = 6, \tag{24.17}$$

$$S^{(2)}(f,7)/S^{(0)}(f,7) \leqslant 1/2\sqrt{2},\ 5/48\sqrt{3} \quad \text{for} \quad 2|D_2,\ 3|D_2 \tag{24.18}$$

(and $D_1 = 6$), respectively. The case $D_2(f) = 1$ remains. But with $D_2 = 1$, $D(g)D(h) = 6$, we must have $D(g) = 2$ and then $D(h) = 3$, which is impossible for $l = 7$, giving $S(f,7) = 0$ if $n = 14$, $D_1 = 6$, $D_2 = 1$.

25. Completion of proof for composite $D_1(f)$. In this section we suppose $D_1 = D_1(f) = 2p$, $p \geqslant 3$, and $n = 14$ or 15.

Suppose first $p \geqslant 5$. Then by (23.1) and (23.2) we have $W_1(f) + W_2(f) + W_3(f) < 0.755$. Using (23.3) for $k = 4, \ldots, 7$, we have $W_4(f) < 0.014$ by (24.8) and $W_5(f) < 0.019$ by (24.9). From (24.10)–(24.12), and the remark following (24.11), we have $W_6(f) < 0.056$, and from (19.1) we have $W_7(f) < 0.076$. $W(f) < 0.094 < 1$ follows.

Now suppose $p = 3$, $D_1 = 6$, and consider first the case $n = 15$. We have $W_1(f) < 0.615$ by (23.1), with the smaller constant mentioned at the end of § 23, and $W_2(f) < 0.033$ by (16.9) and the table in § 4. $W_3(f) = 0$ by Lemma 6. For $k = 4, \ldots, 7$, we use (23.3) with 31, 26, 15, 12, again see the end of § 23, in place of 111, ... With (24.8) this gives $W_4(f) < 0.016$; and with (24.9) we have $W_5(f) < 0.017$. From (24.10)–(24.12) we see that $S(f,6)/S^{(0)}(f,6)$ does not exceed $\max(1/3, 1/36 + 1/12) = 1/3$, whence by (19.1), (19.5) we have $W_6(f) < 0.092$. (19.1) and (19.6) give $W_7(f) < 0.182$; and now $W(f) < 0.955 < 1$ follows (and could be much improved by the arguments for $n = 14$ below).

Now suppose $D_1 = 6$ and $n = 14$; we have $D_2 = 1, 2, 3$, or 6 by (16.1). Suppose first that $D_2 = 6$. Then we can argue for $k = 1, \ldots, 5$ as for $n = 15$ and we find $W_k(f) \leqslant 0.380, 0.069, 0, 0.058, 0.071$ for $k = 1, \ldots, 5$. For $k = 6$, (24.10)–(24.12) give $S(f, 6)/S^{(0)}(f, 6) \leqslant 1/9$, which gives $W_6(f) < 0.159$. (24.16)–(24.18) give $W_7(f) < 0.040$, and we have $W(f) < 1$.

If $D_2 = 1$ then $D(f) = D(g)D(h) = 6$. With (1.4) and Lemma 5, this is impossible for $k = 4$ and implies $D(g) = 6, 3, 2$, $D(h) = 1, 2, 3$ for $k = 5, 6, 7$, or $l = 9, 8, 7$, which can be excluded by Lemma 3 and Lemma 5(v). So $W_k(f) = 0$ for $k = 4, \ldots, 7$; and as above also for $k = 3$. On calculating that $W_1(f) < 0.9$ and $W_2(f) < 0.069$ we have $W(f) < 1$.

Next suppose $D_2 = 2$. We have $W_1(f) < 0.658$ by (23.1), and we can estimate $W_k(f)$ for $k = 2, \ldots, 5$ as for $D_2 = 6$, giving $W_1(f) + \ldots + W_5(f) < 0.854$. For $k = 6$, with $D(g) = 3, 6$ excluded by Lemma 5, and $9|D(g)$ by $3|D_2(f)$, (24.10) is valid without the second term on the right, and using (24.11) we find $W_6(f) < 0.040$. Omitting the second term on the right of (24.16), (24.17) gives $W_7(f) \leqslant 1/3\sqrt{3} < 0.193$. If we assume $D_3(f) = 2$, giving $W_1(f) < 0.503$ by the last part of (23.1), then $W(f) < 1$; so suppose $D_3(f) = 1$, giving $D(f) = D(g)D(h) = 12$. With $k = l = 7$ this contradicts (1.4), by Lemma 5; so $W_7(f) = 0$ and $W(f) < 1$.

Finally suppose $D_2 = 3$. We find as above that $W_1(f) < 0.538$ and $W_1(f) + \ldots + W_5(f) < 0.736$. From (24.16)–(24.18) we find $S(f, 7)/S^{(0)}(f, 7) \leqslant 1/8\sqrt{3}$, which gives $W_7(f) < 0.040$. So we need to prove $W_6(f) < 0.224$, whereas (24.10), with the first term on the right zero, and (24.12) give only $W_6(f) < 77/162$. Very crudely, the improvement we need can be obtained by referring to (21.3), with $m = 3$, $k = 6$, and using (21.8) to estimate $N_3(f, 6)$, in place of the much weaker (21.1) which is all we needed for (24.12).

It now remains only to prove the theorem for $D_1(f) = 1$ or a prime, whence by (16.2) $D_2(f) = 1$ or $D_1(f)$, and $n = 14, 15$.

26. The case $D_1(f) = D_2(f) = p \geqslant 3$. In this case, see (23.1) and the end of § 23, we have

$$W_1(f) < 2.28p^{-1},\ 1.51p^{-1} \quad \text{for} \quad n = 14, 15. \tag{26.1}$$

From (3.3), taking $d = 1, p$, (16.3) and (16.6), and referring to the table in § 4, we have

$$W_2(f) < 33p^{-5},\ 16p^{-11/2} \quad \text{for} \quad n = 14, 15. \tag{26.2}$$

From (21.7) we have $W_3(f) < 0.004$, as in (23.2); and $W_3(f) = 0$ by Lemma 4 if $p \leqslant 5$.

For $k = 4, \ldots, 7$ we have (21.4), (21.5) with $\mathscr{M} = \{p\}$. Using (21.2), with $m = p$, and (21.8), we may replace (21.6) by

$$S(f, k)/S^{(0)}(f, k) \leqslant p^{1+\frac{1}{2}k-\frac{1}{4}n}(1+p^{k-1-\frac{1}{2}n})(1-p^{-2})^{-1}(1+p^{k-\frac{1}{2}n})^{-1}. \tag{26.3}$$

The right member of (26.3) is the ratio of that of (21.8) to that of (19.8). It is obviously less than $(1-p^{-2})^{-1}p^{1+\frac{1}{2}k-\frac{1}{4}n}$, but for the larger k we can do better.

Using (23.3), and simplifying (19.3)–(19.6) by putting $D_1 = p$, $\tau(D_1) = 2$, $(D_1, 2) = 1$, we find from (26.3) that (for $n = 14$)

$$W_k(f) < \begin{cases} p^{-6}(1-p^{-2})^{-1}(206+4k) & \text{for} \quad k = 4, 5, \\ 154p^{-6}(1-p^{-2})^{-1}(1+p^{-2}) & \text{for} \quad k = 6, \\ 72p^{-6}(1-p^{-1})^{-1} & \text{for} \quad k = 7. \end{cases} \tag{26.4}$$

Similarly, with the smaller constants given at the end of § 23, we find that for $k = 4, 5$ and $n = 15$, (26.4) holds with $p^{-13/2}$ in place of p^{-6}, and $102-10k$ in place of $206+4k$. For $k = 6$, $n = 15$ we find $W_6(f) < 30p^{-13/2}(1-p^{-1})^{-1}$; and with a little simplification we find $W_7(f) < 24p^{-13/2}$ for $n = 15$; and now (26.4) holds for $n = 15$.

A simple calculation shows that these inequalities imply $W(f) < 0.6$ if $p \geqslant 5$. We therefore suppose $p = 3$. This gives $W_3(f) = 0$ as noted above, and also $W_4(f) = W_5(f) = 0$, since as noted earlier, see (24.9) and the remark following (24.6), $W(g) = 1$ for $k = 4, 5$ if $D(g)$ is a power of 3, as it must be here since $D(f)$ is so. We now find $W(f) < 0.7$ if $n = 15$, so we suppose $n = 14$.

Estimating $W_k(f)$ as above for $k = 2, 6$ we find $W(f) < W_1(f) + 0.136 + 0.265 + W_7(f)$, and $W_7(f) < 0.149$. This last estimate can be greatly improved. For Lemma 5 shows that (1.4) fails, for $k = l = 7$ and $D(g)$, $D(h)$ each a power of 3, unless $9|D(g)$ and $D(h)$, implying that $n_2 = n_2(f, 3) \geqslant 4$. This shows that for $k = 7$, $p = 3$, (21.2) holds with $N_3^{(2)}(f, 7)$ in place of $N_3(f, 7)$ on the right, whence in the argument leading to (26.4) we may work with (23.4) instead of (21.8). Comparing these inequalities we improve the above estimate for $W_7(f)$ by a factor $\frac{1}{4}$, giving $W(f) < W_1(f) + 0.440$.

With $W_1(f) < 0.760$ by (26.1), this is still not sufficient. But if we suppose $n_2(f, 3) \geqslant 3$ we may use (23.1), with $D_1 = D_2 = D_3 = p$, giving $W_1(f) < 0.474$ and $W(f) < 1$. And if f is imprimitive we have crudely $W_1(f) < 0.190$, again giving $W(f) < 1$, by the remark at the end of § 18.

So we suppose $D_3(f) = 1$, giving $W_7(f) = 0$ as noted above, and f primitive. From (6.1) it is clear that there are just two possibilities for the 3-adic class of f, and from § 7, (vii) the same holds for the 2-adic class, giving at most four possibilities for the genus. Two of these may be excluded by the argument used at the beginning of the proof of Lemma 3, and the others are

(26.5) $$f \simeq Q+\ldots+Q+3Q+3Q, \qquad n = 14,$$

(26.6) $$f \simeq Q+\ldots+Q+3Q+2B, \qquad n = 14,$$

where Q is a unary form with coefficient 1, and B is the case $m = 1$ of (7.3). We leave these two cases for the present.

27. The case $D_1(f) = p \geqslant 3$ and $D_2(f) = 1$. In this case (with $n = 14$ or 15) we have $D(f) = p$ and $f \simeq g+h$ gives $D(g)D(h) = p$, $D(g) = 1$ or p. By Lemma 3(i), (1.4) now gives $D(g) = p$, $D(h) = 1$, for $k = 2, \ldots, 7$. Now using Lemma 3(ii), (1.4) gives $l = n-k \neq 7, 9, 10, 11, 13$. Excluding from (1.2) the terms that are zero by this argument,

(27.1) $$W(f) = W_1(f)+(15-n)W_2(f)+(n-14)W_3(f)+ +(15-n)W_6(f)+(n-14)W_7(f);$$

and $D(h) \neq 1$, implying $D(g) = 1$, when $l = 13$, gives

(27.2) $$W_1(f) \leqslant \theta(1, 13)S_1(f, 1) < 4.20S_1(f, 1) \quad \text{for} \quad n = 14,$$

from (2.8) and the table in § 4, with $d = 1$ in (3.3).

Suppose first $n = 15$. Then $S(f, 7) \leqslant p^{-1/2}S^{(0)}(f, 7)$ by (21.6), and $S^{(0)}(f, 7) = 3.70\times 4p^{-7/2}$ by (19.6). Using the improvement on (23.3) given at the end of § 23, $W_7(f) < 48p^{-4}$. We have $W_1(f) < 1.51p^{-1/2}$ by the corresponding improvement on (23.1); and $W_3(f) < 0.004$, see (23.2). For $p \geqslant 5$ these inequalities, with (27.1), give $W(f) < 1$ very crudely; so suppose $p = 3$. Now $W_7(f) = 0$ because, by Lemma 5, we cannot have (1.4) and $D(g) = 3$ when $k = 7$; and again we have $W(f) < 1$ very crudely.

Now take $n = 14$. We have $W_1(f) < 2.20p^{-1/2}$ by (23.1), and $W_2(f) < 33p^{-5}$, since $D_2(f) = p$ was not used to obtain (26.2). Using (23.3) and (19.5), we find $W_6(f) < 154p^{-3}(1+p^{-1})$. Now (27.1) gives $W(f) < 1$ with much to spare if $p \geqslant 11$; so we suppose $p \leqslant 7$, and distinguish two cases.

(i) If f is imprimitive, then (7.1) reduces, with $D(f)$ odd, to $f \underset{2}{\simeq} 2\psi$, whence $f \underset{2}{\simeq} g+h$ is impossible unless g, h are of even rank, giving $W_1(f) = 0$, and $W(f) < \frac{1}{2}$ unless $p = 3$, and also for $p = 3$, unless $W_6(f) > 0$. But then, see Lemmas 3, 5, we have

(27.3) $$f \simeq 2E_6+2E_8,$$

where E_6 and E_8 are extreme forms, with non-Gaussian determinants 3, 1. We return to this case later.

(ii) Suppose f primitive. Then for $k, l = 6, 8$ we must have h imprimitive, as noted above, and so g must be primitive. This is excluded by Lemma 5(iii), giving $W_6(f) = 0$, if $p = 3$ or 5. For $p = 7$ we need a slight improvement on Lemma 5 (iii): if $k = 6$ and $D(g) = 7$ we have $W(g) = 1$ unless g is imprimitive. This does not follow from Lemma 5(ii), by the method used in the proof of Lemma 5, but it can be verified by using Hermite reduction as, e.g., in [5], p. 13, § 9; I leave it to the reader.

Now we may suppose $W_6(f) = 0$, and (27.1) gives $W(f) < 2.20p^{-1/2}+ +33p^{-5}$, whence $W(f) < 1$ for $p = 5, 7$. With $p = 3$, there are just two possible genera, as in § 26:

(27.4) $$f \simeq Q+\ldots+Q+3Q, \qquad n = 14,\ D(f) = 3,$$

(27.5) $$f \simeq Q+\ldots+Q+2B, \qquad n = 14,\ D(f) = 3,$$

with $B = x_{13}^2+x_{13}x_{14}+x_{14}^2$ as in (26.6). We return to these two cases later.

28. The case $n = 14$ or 15 and $D(f)$ a power of 2. Denote by E_k (or E_l) an absolutely extreme form with minimum 1 and rank k (or l). E_4, E_7, E_8 occur as exceptional cases in Lemma 5, and have $D = 4, 2, 1$. It is well known, and is an easy corollary of Lemma 5, that $E_4 \underset{2}{\simeq} (7.3)+2(7.3)$, $E_7 \underset{2}{\simeq} -Q+(7.2)$, $E_8 \underset{2}{\simeq} (7.2)$, with the notation of § 7, and $Q = x_1^2$ as above. We deduce that in the present case the possibilities for the genus of f are given by

(28.1) $$f \simeq Q+\ldots+Q+2Q+\ldots+2Q \qquad (n_2 = \log_2 D(f) \geqslant 0),$$

(28.2) $$f \simeq Q+\ldots+Q+2E_4+\ldots+2E_4 \qquad (2|n_2 \geqslant 2),$$

(28.3) $$f \underset{2}{\simeq} 2Q+\ldots+2Q-2Q-\ldots-2Q+2\sigma \qquad (2|n_1,\ n_2 \geqslant 2),$$

with $[\frac{1}{2}n_2+1]$ $-$ signs, and σ, with rank n_1, of the shape (7.2), and

(28.4) $$f \simeq 2E_7+2E_8 \qquad (n_2 = 1,\ n = 15).$$

To see this we count the possibilities up to $\underset{2}{\simeq}$ for f when n, n_2 and $\varrho_1 = \operatorname{sgn}\operatorname{rank}\varphi_0$, $\varrho_2 = \operatorname{sgn}\operatorname{rank}\varphi_1$ are given, see (7.1). From § 7, (i)–(vii), we find at most 2 possibilities, but only one when $n_2 = 1$ and $n = 15$, and none when $\varrho_1 = \varrho_2 = 0$, for as in the proof of Lemma 3 that would give the contradiction $n = 14$ and $D(f) = 4^eD'$ with $e = \frac{1}{2}n_2$ and $D' \equiv -1 \pmod 4$. Then the only possibilities except (28.1)–(28.4) are derived from (28.1)–(28.3) by changing two $+$ signs to $-$, or *vice versa*; and these cases can be excluded by the argument used to prove $\varepsilon = 1$ in (20.1). We note next that, by (16.3), Lemma 4, and Lemma 5(ii),

(28.5) $$W_k(f) = 0 \quad \text{for} \quad k = 2, 3, 5.$$

(For $k = 5$, consider the reciprocal of g in case $D(g) = 8, 16$ or 32.)

Now suppose (28.4) holds. We have $W_1(f) = 0$ because with $k = 1$ $f \simeq Q+h$ is obviously impossible, and $f \simeq 2Q+h$ would imply $h \underset{2}{\sim} 2\varphi_7$ which we showed above to be impossible. From Lemma 5 it is easily see that $W_6(f) = 0$, and that $f \simeq g+h$ is possible with $k = 7$ only with $g \simeq 2E_7$, $h \simeq 2E_8$. With (28.5), noting that E_7 and E_8 have class-number 1, it follows that the right member of (28.4) is the only disjoint class in the genus of f. So if the theorem is false, then f has class-number 1; which, see [9], gives the contradiction $n \leqslant 10$. So the theorem is true in case (28.4).

O'Meara has shown in [1] that there exists a form f' with $D(f') = D(f)$ which is not equivalent to a disjoint form. If we can prove $f' \simeq f$ the theorem follows. That is, the theorem is true if for given $D(f)$ we have only one possibility for the genus of f. Looking at (28.1)–(28.3), we see that (28.1) (with an empty sum of terms $2Q$) is the only possibility if $D(f) = 1$, or $n_2 = 0$; also if $n = 14$ and n_1, n_2 are both odd. We may therefore assume that one of (28.1)–(28.3) holds, and that

(28.6) $$2|n_2 \geqslant 2 \text{ if } n = 14, \quad n_2 \geqslant 1 \text{ if } n = 15.$$

We next show that

(28.7) $$W_4(f) \leqslant 0,\ 111/864,\ 31/864 \quad \text{for} \quad n_2 \leqslant 2,\ n = 14,\ n = 15$$

respectively. For the first of these note that Lemma 5 and (1.4) imply, for $k = 4$ and $l = 10$ or 11, $4|D(g)$, $2|D(h)$, $8|D(f)$. For the second, we use (23.3) and (24.4); for the third, replace 111 by 36, see end of § 23.

Similarly, using (24.10) and (24.11) in place of (24.4),

(28.8) $$W_6^3(f) = 0 \quad \text{if} \quad n_2 \leqslant n-13,$$

(28.9) $$W_6(f) \leqslant 77/192,\ 15/192 \quad \text{for} \quad n = 14,\ 15 \text{ respectively}.$$

We can however improve on (28.9) if we assume $n_2 \geqslant 3$. For (24.11) comes from (23.4), and so from (23.5); and (23.5) is weak for $n_2 \geqslant 3$. A simple calculation shows that for $k = 6$, $p = 2$ the left member of (23.5) is at most $2^{-10}+2^{-27/2}$, $2^{-10}+2^{-12}+2^{-16}$ for $n = 15, 14$, if we assume $n_2 \geqslant 3$, implying $n_2 \geqslant 4$ in case $n = 14$, see (28.6). Comparing these estimates with (23.5) as it is, we have

(28.10) $$W_6(f) < 0.127,\ 0.043 \quad \text{for} \quad n = 14, 15, \text{ if } n_2 \geqslant 3.$$

For $k = 7$, note first that the proof of (19.1) remains valid if (19.8) is weakened by putting $p^{-\frac{1}{2}k}$ for $p^{-\frac{1}{2}l}$ on the right; a factor $\tau(D_1)$ on the right of (19.6) arises in this way. Putting $k = 7$, $p = 2$, the right member of (19.8) reduces to $2P^{-1}\cdot 2^{-7/2}$, $P = P(7, 2)$. By the argument leading to the case $k = 7$, $p = 2$ of (21.6), we can sharpen (19.1) by a factor $2^{5/2}P$ times the right member of (21.8) (with $p = 2$), if we suppose $2|D_2$, or $n_2 \geqslant 2$. More generally, so as to get a better estimate if $n_2 \geqslant 3$, the argument gives the first of

(28.11) $$S(f, 7)/S^{(0)}(f, 7) \leqslant 2^{5/2} \sum \{2^{-\frac{1}{2}l}\colon k = 7, k_2 \geqslant r\},$$

with $r = 1$ or 2 according as (1.4) is or is not consistent with $D(g) = 2$ (for $k = 7$).

The sum on the right of (28.11) is that of (19.11) or (23.5), with $k = 7$, $p = 2$, according as $r = 1$ or 2. The improvement in case $r = 2$ is obvious; and we have $r = 2$ in case (28.2) since $D(g) = 2$ implies $g \underset{2}{\sim} 2E_7 \underset{2}{\sim} -2Q+ +2(7.2)$, and $-2Q$ cannot split off 2-adically from (28.2).

Now using (23.3), with 12 for 36 in case $n = 15$, we find from (28.11) that, with $S^{(0)}(f, 7) = 2^{-5/2}S_7$ by (19.6),

(28.12) $$W_7(f) \leqslant 12(31-2n) \sum \{2^{-\frac{1}{2}l}\colon k = 7, k_2 \geqslant r\}$$

with $r = 2$ in case (28.2), 1 otherwise.

A simple calculation now shows, using (28.6) to exclude $n_2 = 3$ when $n = 14$, that

(28.13) $$W_7(f) < 0.008,\ 0.043 \quad \text{for} \quad n = 14, 15, \text{ if } n_2 \geqslant 3.$$

We have $W_7(f) = 0$ if $n_2 \leqslant 1$; for all cases with $n_2 \leqslant 1$ have been disposed of except (28.1) with $n_2 = 1$, $n = 15$. In that case $f \simeq g+h$ with $k = 7$ is possible only with $g = 2E_7$, $D(g) = 2$, $D(h) = 1$, h primitive, so $W(h) = 1$ by Lemma 3.

For $k = 1$ we calculate the product over odd p in § 18 more precisely and simply, since no such p divides D_1, and so, using the table in § 4, we have

(28.14) $$W_1(f) < 4.21 M_2(f, 1),\quad 2.88 M_2(f, 1) \quad \text{for} \quad n = 14, 15.$$

We note also, see the remark at the end of § 18, and (14.1), that

(28.15) $$M_2(f, 1) \leqslant 2^{1-\frac{1}{2}n_1} \text{ in case (28.3).}$$

29. The cases (28.1)–(28.3). Suppose first that (28.3) holds, with $n_2 \geqslant 3$. Then by (28.14) and (28.15), with $2|n_1 \geqslant \frac{1}{2}n$ giving $n_1 \geqslant 8$, we have $W_1(f) < 0.527$. With (28.5) and (28.7), this gives $W_1(f)+\ldots+W_5(f) < 0.656$. By (28.10) and (28.13) we have $W_6(f)+W_7(f) < 0.170$, and $W(f) < 1$ follows.

Next suppose that (28.3) holds with $n_2 = 2$ and $n = 15$. We estimate $W_k(f)$ as above for $k = 1, 2, 3, 5$, and we have $W_4(f) = 0$ by (28.7), $W_6(f) < 0.079$ by (28.9). (28.12) gives $W_7(f) < 12(2^{-13/2}+2^{-8})$, and $W(f) < 1$ follows crudely. We may therefore suppose that $n = 14$, $n_2 = 2$ in case (28.3) and then it is easily seen that

(29.1) $$f \simeq 2E_7 + 2E_7 \quad (n = 14,\ D(f) = 4).$$

In case (29.1) we have $W_k(f) = 0$ for $2 \leqslant k \leqslant 5$ as above. For $k = 1$ we obviously cannot have $g = Q$, and we cannot have $g = 2Q$, for if we could we should find $f \underset{2}{\sim} 2Q+2Q+2\sigma$, which we have excluded. So $f \simeq g+h$ is possible only for $k = 6, 7$. We shall return to (29.1) later, and use this remark.

Next suppose that (28.1) or (28.2) holds, with $n_2 \geqslant 3$; whence by (28.6) $n_2 = 4$ or 6 if $n = 14$. Using (28.5), (28.7), (28.10) and (28.13) we have $W_2(f)+ \ldots +W_7(f) < 0.264, 0.122$ for $n = 14, 15$. (18.4), with $r = 3$, gives $M_2(f) < 0.201$. Then if $n = 15$ (28.14) gives $W_1(f) < 0.579$ and $W(f) < 1$ follows. For $n = 14$ we need a slightly better estimate for $M_2(f, 1)$. The argument leading to (18.4) shows that that inequality is valid without the restriction $r \leqslant 3$ if we suppose further that $2 \nmid D_{r+1}$, or $r = n_2$. Then with $r = 4$ we find $M_2(f, 1) \leqslant 2^{-3}+2^{-5}+2^{-11}$, and with $r = 6$ we find a better inequality, and (28.14) gives $W_1(f) < 0.662$, and again $W(f) < 1$.

Now suppose that (28.1) or (28.2) holds with $n_2 \leqslant 2$. Then $W_4(f) = 0$ by (28.7). $W_6(f) = 0$ since with $k = 6$ we could only have $D(g) = 4$, $2|g$, and $D(h) = 1$, giving $2|h \sim 2E_8$ by Lemma 5, whence $2|f$. If we furthe suppose $n = 15$ then for $k = 7$ we can only have $D(g) = 2$ or 4, $D(h) = 2$ or 1, $D(h) = 1$ and $h \sim 2E_8$ by Lemma 5, and $D(g) = 4$, giving $2|g$. So $W_7(f)$ is zero, and with (28.5) we have $W(f) = W_1(f)$. Now if $n_2 = 2$, (18.4) gives $M_2(f, 1) < 0.3$ and (28.14) gives $W(f) < 1$. So

$$f \simeq Q+ \ldots +Q+2Q \quad (n = 15) \tag{29.2}$$

is the only outstanding case with $n > 14$; and for $n = 14$ we have only

$$f \simeq Q+ \ldots +Q+2Q+2Q \quad (n = 14), \tag{29.3}$$

$$f \simeq Q+ \ldots +Q+2E_4 \quad (n = 14). \tag{29.4}$$

Besides these four genera (29.1)–(29.4), we shall have to consider (26.5), (26.6), (27.4) and (27.5). (The argument used for (28.4) disposes of (27.3).)

It will be useful later to notice that when (29.3) holds the two congruences $f(x_1, \ldots, x_{14}) \equiv 1, -1 \pmod 4$ have equally many solutions in integers $x_i \pmod 4$, but when (29.4) holds the second congruence has more solutions than the first. For (29.3) this is clear, since each term $2Q$ is $0, 2 \pmod 4$ equally often. In the other case, see [8], p. 98, Lemma 1 and note that $f \underset{2}{\sim} -Q-Q+2(7.2)+4(7.2)$.

30. Proof for some special cases. We consider first the two cases (26.5), (26.6), each with $n = 14$ and $D(f) = 9$. The arguments of § 26 show that in either case we have $W_k(f) = 0$ for $k = 3, 4, 5, 7$, and $W_1(f) < 0.76$, $W_2(f) < 0.14$, $W_6(f) < 0.265$. It would suffice to improve this last inequality to $W_6(f) < 0.1$. In estimating $W_6(f) < 0.265$ we have

excluded (for $k = 6$) the possibility $D(g) = 1$; and we could exclude $D(g) = 9$. But because of the possibility $g \sim 2E_6$, see Lemma 5(iii), we have not excluded $D(g) = 3$.

Now another possibility with $D(g) = 3$, $k = 6$, is $g \sim G$, where G is the diagonal form $Q+ \ldots +Q+3Q$; and by (1.4) we may exclude this case. It is easily seen that $2E_6$ and G have each class number 1. Further, we have $2E_6 \underset{p}{\sim} G$ for every odd p; this is trivial for $p > 3$ and easily verified for $p = 3$. By using § 7, (i), it may be seen that $2E_6+aQ \underset{2}{\sim} G+aQ$ for every odd a. It follows that $2E_6+h \simeq G+h$ for every primitive h. So if (26.5) or (26.6) holds, and $f \simeq 2E_6+h$, necessarily with primitive h, then we have also $f \simeq G+h$.

By excluding the latter alternative we sharpen the estimate for $W_6(f)$ by a factor $< w(2E_6)/w(G)$. This factor is b/a, where a is the number of integral automorphs of $2E_6$, and b the corresponding number for G. Now $b = 2^6 5!$, by counting permutations and changes of sign; and $a = 144 \cdot 6!$, see [10], p. 325. So $b/a < 1/10$, $W_6(f) < 0.03$, and $W(f) < 1$.

Now take the case (29.1), and denote by F the perfect 9-ary form $\frac{1}{2}(x_1^2+x_2^2+ \ldots +x_9^2)+\frac{1}{2}(x_1+ \ldots +x_9)^2$ with minimum 1, see [9], p. 558, (6.1). The (non-Gaussian) determinant of F is 10, and it is easily verified from (29.1) that f represents $2F$ over the p-adic integers for every p. It follows that the genus (29.1) contains a form, say $2f'$, which has a 9-ary section $\sim 2F$. Now suppose f' equivalent to a disjoint form $g'+h'$. As shown in [9], p. 556, Lemma 3, $g'+h'$ cannot represent F unless one of g', h' represents F. Then, crudely, either g' or h' has rank $\geqslant 9$. By the remark following (29.1), g' and h' have ranks 6 and 8 or 7 and 7; so we have a contradiction which shows that $2f'$ is not equivalent to a disjoint form, and the theorem is proved for case (29.1).

We next consider the case (29.4). By the case $n = 14$, $D(f) = 1$, which we have disposed of above, by reference to [1], there exists a form F which is in the genus of $Q+ \ldots +Q$ and indecomposable; whence trivially $\min F \geqslant 2$. Noting that $Q+Q+Q+Q$ goes into $x_1^2+x_2^2+x_3^2+(x_1+x_2+x_3+2x_4)^2 \sim 2E_4$ by a transformation with determinant 2, we see that by such a transformation we can take F into f satisfying (9.4), obviously with $\min f \geqslant 2$. We suppose this f equivalent to a disjointform $g+h$ and deduce a contradiction.

Clearly $\min g \geqslant 2$, $\min h \geqslant 2$, and we have $D(g)D(h) = 4$. By (28.5), we have $k = 1, 4, 6$ or 7. If $k = 7$ then $l = 7$ and by Lemma 5 we must have $D(g) = D(h) = 2$, and then $2|g$, $2|h$, $2|f$, contradicting (29.4). If $k = 4$ or 6 then Lemma 5 gives $D(g) = 4$, $2|g$, and $D(h) = 1$, $l = 10$ or 8 $l = 8$, $2|h$, and again $2|f$. So $k = 1$, and obviously $g = 2Q$. But then in (7.1) we have φ_1 not null, again contradicting (29.4), and this contradiction completes the proof.

In the four remaining cases (27.4), (27.5), (29.2), (29.3), we shall as above construct the form whose existence is asserted by the theorem but we shall do so more explicitly.

31. The cases (29.2), (29.3). In the first of these cases we begin by defining

$$(31.1)\qquad f = \sum_{i=1}^{12}{}' (y_i+\tfrac12 x_{13}+\tfrac12 x_{14}+\tfrac12 x_{15})^2+(x_{13}+\tfrac12 x_{15})^2+ \\ +(x_{14}+\tfrac12 x_{15})^2+\tfrac12 x_{15}^2,$$

where the accent means that the sign of x_{15} is to be changed in the twelfth term, and the y_i are restricted to satisfy $y_1+\ldots+y_{12}\equiv 0 \pmod 2$. We could write $y_i = x_i$ for $i\leqslant 11$ and $y_{12} = x_1+\ldots+x_{11}+2x_{12}$. It is easily seen that (31.1) implies $D(f) = 2$ and $f\equiv (x_{13}+x_{14})^2 \pmod 2$, identically, whence it is clear that f satisfies (29.2).

The leading 12-ary section of (31.1) takes its minimum value 2 at 132 pairs of points at which $\pm(y_1,\ldots,y_{12})$ is a permutation of $(1, 1, 0, \ldots, 0)$ or of $(1, -1, 0, \ldots, 0)$. With $|x_{15}|\geqslant 2$ we have just one pair of points, one of them with $x_{15} = 2$, $x_{13} = x_{14} = -1$, and the y_i uniquely determined, at which f takes the value 2; and none with $f = 1$. If $x_{15} = 1$ and $f\leqslant 2$, then clearly $x_{13}+x_{14}$ has to be odd, so $f = 1$, and $x_{13}, x_{14} = 0, -1$ or $-1, 0$; and then $y_1, \ldots, y_{12} = 0, \ldots, 0, 1$, contradicting $2|\sum y_i$. So $f\geqslant 3$ when $x_{15} = \pm 1$. If $x_{15} = 0$, $f = 2$ implies $x_{13} = 0$ or ± 1, $x_{14} = 0$ or ± 1, and $x_{13}+x_{14}$ even. So we have one pair of points with $x_{13} = x_{14} = \pm 1$ and each $y_i = -x_{13}$, and one with $x_{13} = -x_{14} = \pm 1$, each y_i zero, and $f = 2$.

We see therefore that f has minimum 2, with 135 pairs of minimum points, and that there is just one homogeneous linear equation, namely $x_{13}+x_{14} = 0$, which is satisfied by exactly 134 of the pairs of minimum points.

Suppose now that $f\sim g+h$. Then obviously $\min g\geqslant 2$, $\min h\geqslant 2$, and $D(g)D(h) = 2$. With g of rank $k\leqslant 7$, Lemma 5 shows that $D(g) = 1$ is impossible, and $D(g) = 2$, giving $D(h) = 1$, is possible only for $k = 1$ or 7, with $g = 2Q$ or $2E_7$. In the latter case $l = 8$ and $D(h) = 1$ makes h decomposable unless $h = 2E_8$, making f imprimitive, see (28.4). So $k = 1$, $g = 2Q$, $f\sim 2Q+h$. Now $2Q+h$ reduces to h, and loses just one pair of minimum points, on putting $x_1 = 0$.

From this, and what we have proved above the minimum points of (31.1), it follows that on putting $x_{14} = -x_{13}$ (31.1) (if decomposable) must reduce to a form equivalent to h. But on making this substitution we see at once that (31.1) reduces to a form with $D = 4$, whereas $D(h) = 1$. So we have a contradiction which completes the proof for case (29.2), and so for $n\geqslant 15$.

For the case (29.3), we define

$$(31.2)\qquad f = \sum_{i=1}^{5}(y_i+\tfrac12 x_{13})^2+(y_6+\tfrac12 x_{13}+x_{14})^2+ \\ +\sum_{i=1}^{5}(z_i+\tfrac12 x_{14})^2+(z_6+x_{13}+\tfrac12 x_{14})^2+\tfrac12 x_{13}^2+\tfrac12 x_{14}^2,$$

with $y_1+\ldots+y_6\equiv z_1+\ldots+z_6\equiv 0 \pmod 2$. It is easily verified that this gives $D(f) = 4$ and f identically congruent to $(x_{13}+x_{14})^2$ modulo 2, whence f satisfies either (29.3) or (29.4). Suppose we give a fixed set of values, with $x_{13}+x_{14}$ odd, to all the variables except y_6, whose parity is thereby determined. The term involving y_6 in (31.2) then takes the shape $\frac14(2w+1)^2$, with w an integer, and is congruent to $\frac14, \frac94 \pmod 4$ equally often; the other terms reduce to constants. So we see that $f\equiv 1, -1 \pmod 4$ have equally many solutions, which by the remark at the end of § 29 gives us that f defined by (31.2) satisfies (29.3).

It is clear from (31.2) that at integer points with f odd we have $x_{13}+x_{14}$ odd and $f\geqslant 3$, so f has minimum 2. If we put $x_{13} = x_{14} = 0$, (31.2) reduces to a disjoint form taking its minimum value 2 at $30+30$ pairs of minimum points, with the y_i a permutation of $\pm 1, \pm 1, 0, \ldots, 0$ and the z_i all 0, or *vice versa*. With $x_{13}, x_{14}\neq 0, 0$ we have $f > 2$ unless $x_{13}+x_{14}$ is even, and one of x_{13}, x_{14} is ± 2, the other 0; and this gives just 2 more pairs of minimum points.

Now there are just two linear homogeneous equations, namely $x_{13} = 0$, $x_{14} = 0$, which are satisfied by exactly 61 of the 62 pairs of minimum points.

Suppose now that f defined by (31.2) is equivalent to $g+h$, obviously with $\min g\geqslant 2$, $\min h\geqslant 2$, and $D(g)D(h) = 4$. One possibility, $g = 2Q$, $l = 13$, $D(h) = 2$, can be excluded by noticing that on putting x_{13} or $x_{14} = 0$ (31.2) reduces to a form with $D = 8$, and so not equivalent to h. Lemma 5 shows that we cannot have $k = 2, 3$, or 5. If $k = 4$, $D(g) = 4$, $l = 10$, $D(h) = 1$, also excluded by Lemma 5. If $k = 6$, $D(g) = 4$, $2|g$, $D(h) = 1$, $l = 8$, and $2|h$, $2|f$, by Lemma 5. If $k = l = 7$ then Lemma 5 gives $D(g) = D(h) = 2$, $2|g$, $2|h$, $2|f$. Since $2|f$ contradicts (29.3), $f\sim g+h$ gives a contradiction which completes the proof of the theorem in case (29.3).

32. The cases (27.4), (27.5). We begin by noticing that there exists a positive form F_8 with $\det A(F_8) = 4$ and

$$(32.1)\qquad F_8\equiv x_1x_2+x_3x_4+x_5x_6+2x_7x_8 \pmod{16},$$

and that the extreme form E_5 may, by a suitable integral unimodular transformation, be supposed to satisfy

$$(32.2)\qquad E_5\equiv x_1x_2+(x_3^2+x_3x_4+x_4^2)-6x_5^2 \pmod{16},$$

both congruences holding identically in the variables. $2F_8$ is equivalent to the leading 8-ary section of (31.1); but to show that F_8 exists, with determinant 4 and satisfying (32.1), it is simpler to appeal to [4], p. 72, Theorem 43, $2E_5$ is the leading 5-ary section of (31.1); by putting $y_5 = -(y_1+\ldots+y_4)$ we obtain a 4-ary section of the shape $2(7.3)$, whence (32.2), and $\det A(E_5) = 2$. We now define

$$(32.3)\qquad f = \tfrac{1}{2}F_8(2x_1, \ldots, 2x_6, x_7, 2x_8+x_{14})+ \\ +\tfrac{1}{2}E_5(2x_9, \ldots, 2x_{12}, x_7+2x_{13}+\tfrac{1}{2}x_{14})+\tfrac{3}{4}x_{14}^2.$$

Straightforward calculation shows that $\det A(f) = 3\cdot 2^{14}$, also that f has integer coefficients and is identically congruent to x_7^2 modulo 2. So $D(f) = 3$ and f is of the shape (7.1) with φ_0 not null. By the arguments of § 27 f satisfies either (27.4) or (27.5); obviously the former, since (32.3) shows that $f \underset{3}{\sim} Q+\ldots+Q+3Q$. If $f(x_1, \ldots, x_{14}) = 1$ for integers x_i then x_7 is odd, the first term on the right of (32.3) is non-zero and so at least $\frac{1}{2}$, so x_{14} must be zero. Then each of the first two terms on the right of (32.3) is a positive integer, so $f > 1$, contradiction, and we have $\min f \geqslant 2$.

Now suppose that f satisfies either of (27.4), (27.5), and $\min f = 2$, and that $f \sim g+h$. As in § 27, $k = \operatorname{rank} g = 1, 2$, or 6. But Lemma 5 gives $D(g) = 3$, if $k = 6$, because $\min g \geqslant 2$, and then $2|g$, $l = 8$, $D(h) = 1$, $2|h$, $2|f$, contradiction. So $k = 1$ or 2. If $k = 1$ then $g = 3Q$ since $\min g > 1$; so $D(h) = 1$, $l = 13$, which is excluded, see Lemma 3. So $k = 2$; but now with $D(g) = 1$ or 3 we can take g to be disjoint, with minimum 1, unless $D(g) = 3$ and $g \sim 2B = 2x_1^2+2x_1x_2+2x_2^2 \underset{3}{\sim} -Q-3Q$, which contradicts (27.4). So in that case (32.3) is indecomposable and the theorem is proved. It remains to dispose of case (27.5) by constructing f, in that genus, with $\min f = 2$, which does not satisfy

$$(32.4)\qquad f \sim 2B+h,\quad \operatorname{rank} h = 12,\quad h \simeq Q+\ldots+Q.$$

We transform the extreme form E_7 so as to make it satisfy

$$(32.5)\qquad E_7 \equiv x_1x_2+x_3x_4+x_5x_6-x_7^2 \pmod 8.$$

We use also a form F_6 with non-Gaussian determinant 4 and

$$(32.6)\qquad F_6 \equiv x_1x_2+x_3x_4-x_5^2-x_6^2 \pmod 8.$$

$2F_6$ is the leading 6-ary section of (31.1); it is also the g with $D(g) = 4$ of Lemma 5(iii). Now we define

$$(32.7)\qquad f = \tfrac{1}{2}E_7(2x_1, \ldots, 2x_6, x_7+x_{14})+ \\ +\tfrac{1}{2}F_6(2x_8, \ldots, 2x_{11}, 2x_{12}+x_{14}, x_7+x_{14}+2x_{13})+\tfrac{3}{2}x_{14}^2.$$

It is easily verified that f has integer coefficients and is identically congruent to x_7^2 modulo 2, and so satisfies (27.5). If $f = 1$ for integers x_i then $x_{14} = 0$, x_7 is odd, and the first term on the right of (32.7) is congruent to $-\frac{1}{2} \pmod 4$ and so > 1. This contradiction gives $\min f = 2$.

It remains only to disprove (32.4); so suppose it true. Transforming h, we may suppose $h \equiv x_3^2 \pmod 2$, identically. Putting $2x_3$ for x_3 and then dividing by 2, (32.4) goes into a form of the shape $B+H$. Similarly, putting $2x_7$ for x_7 and then dividing by 2, (32.7) goes into a form which must be equivalent to $B+H$, and which reduces to E_7+F_6 on putting $x_{14} = 0$. So

$$(32.8)\qquad B+H \text{ represents } E_7+F_6.$$

Now E_7 and F_6 are well known to be perfect forms, each with minimum 1; so from [9], p. 557, Lemma 4, (32.8) implies that either B represents one of E_7, F_6 and H represents the other, or one of B, H represents E_7+E_6. Obviously B, of rank 2, represents neither of E_7, E_6; so H represents E_7+F_6. This gives the contradiction $12 = \operatorname{rank} H = \operatorname{rank} h \geqslant \operatorname{rank}(E_7+F_6) = 13$; which completes the proof of the theorem in the last outstanding case.

33. Conclusion. Looking at (17.7), where the constant factor 1.75 could be greatly improved by what we have done later, we see that for forms of the shape (6.1), (7.1) we have proved that

$$(33.1)\qquad \max\{W(f)\colon \operatorname{rank} f = n\} = o(1)\quad \text{as}\quad n\to\infty.$$

That is, for large n, almost all the weight of a positive n-ary genus arises from classes that do not contain disjoint forms. I think it is true that $W(f)$ does not decrease when a general form is put into the shape (6.1), (7.1) by the transformations of [6]. If so, (33.1) is true for all positive f.

Now suppose that we count the classes instead of weighing them; let $c(f)$ be the number of classes in the genus of f, and let $c'(f) \leqslant c(f)$ be the number of these $c(f)$ classes, that contain disjoint forms. Then define $C(f) = c'(f)/c(f)$. Is it true that

$$(33.2)\qquad \max\{C(f)\colon \operatorname{rank} f = n\} = o(1)\quad \text{as}\quad n\to\infty\,?$$

I do not see how to prove (33.2), with or without the restrictions (6.1), (7.1). It may be false; if so, the reason for (33.1) is that the disjoint forms have many automorphs and low weight.

References

[1] O. T. O'Meara, *The construction of indecomposable positive definite quadratic forms*, Journ. für Math. 276 (1975), pp. 99–123.
[2] G. Pall, *The weight of a genus of positive n-ary quadratic forms*, Proc. Sympos. Pure Math. VIII (1965) (Amer. Math. Soc., Providence, R. I.), pp. 95–105.

[3] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. 36 (1935), pp. 527–606.
[4] G. L. Watson, *Integral quadratic forms*, Cambridge 1960.
[5] — *One-class genera of positive quadratic forms in at least five variables*, Acta Arith. 26 (1975), pp. 309–327.
[6] — *Transformations of a quadratic form which do not increase the class-number*, *II*, ibid., 27 (1975), pp. 171–189.
[7] — *One-class genera of positive ternary quadratic forms – II*, Mathematika 22 (1975), pp. 1–11.
[8] — *The 2-adic density of a quadratic form*, ibid., 23 (1976), pp. 94–106.
[9] — *The class-number of a positive quadratic form*, Proc. London Math. Soc. (3) 13 (1963), pp. 549–576.
[10] E. S. Barnes, *The perfect and extreme senary forms*, Canad. J. Math. 9 (1957), pp. 235–242.

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On the existence of a density

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We shall give the details which demonstrate a formula for a number theoretical density which played a vital role in our paper [2], but doubts about existence and correctness of the formula have been expressed by A. Garsia, H. Moeller, and the editors. In the meantime Everett [1] has used our encoding idea to derive a new proof for one of our assertions.

We shall recall some of the conventions and symbolisms in our paper. We considered a function T, mapping the positive integers into themselves, given by

$$Tn = (3^{X(n)}n + X(n))/2, \tag{1}$$

where $X(n) = 1$ when n is odd and $X(n) = 0$ when n is even.

Given an integer n we considered iterated partities $n, Tn, T^2n, \ldots$ $\ldots, T^kn$ and we agreed to stop the iteration at the very first instance when $T^kn < n$. This *stopping time* was denoted by $\chi(n) = k$. Infinite values for the stopping time were permitted. We also introduced a second stopping time $\tau(n)$ which had a periodicity property. The quantity $\mathrm{P}[\tau = k]$ was defined to be the proportion of integers in $[1, 2^k]$ which satisfy the relation $\tau(n) = k$. The quantities $\mathrm{P}[\tau < k]$ and $\mathrm{P}[\tau \geqslant k]$ were defined similarly in the same block of integers.

If A is a set of positive integers then the *density* of A is defined in terms of the counting function μ to be

$$\delta(A) = \lim_{m\to\infty} (1/m)\mu\{n \leqslant m \mid n \in A\} \tag{2}$$

provided this limit exists. We now set $[\chi = k] = \{n \geqslant 0 \mid \chi(n) = k\}$, and we define $[\tau < k]$ and $[\tau \geqslant k]$ in a similar manner.

THEOREM. *The denisty of the set* $[\chi \geqslant k]$ *exists and is given by*

$$\delta[\chi \geqslant k] = \mathrm{P}[\tau \geqslant k]. \tag{3}$$

Proof. The trick involved is to get this formula without forming any infinite sums. In [2] we established the formula $\delta[\chi = k] = \mathrm{P}[\tau = k]$. Finite additivity of density gives $\delta[\chi < k] = \mathrm{P}[\tau < k]$. Since the sets