A note on Friedlander’s paper “On the class numbers of certain quadratic extensions”

by

A. Mallik (Nottingham)

1. J. B. Friedlander [1] has recently proved the following

**Theorem.** Let $K$ be an algebraic number field of degree $n$ for which $\zeta_K(\frac{1}{2}) = 0$. Let $E$ be a quadratic extension of $K$ having discriminant $d_E$ and Dedekind zeta function $\zeta_E(s) = \zeta_K(s)L(s, \chi)$. Let $\varepsilon > 0$ be arbitrary. Then,

$$L(1, \chi) \gg |d_E|^{-1/2(\log |d_E|)^{2-\varepsilon}},$$

where $\gg$ indicates an effectively computable constant depending (at most) on $\varepsilon$ and $K$.

**Corollary.** Assume in addition that $K$ is a totally real field and that $E$ is a totally imaginary quadratic extension of $K$. Then, if $h(E)$ denotes the class number of $E$,

$$h(E) \gg (\log |d_E|)^{2-\varepsilon}.$$

In a remark at the end of [1] he suggests it should be possible to improve the exponent of $\log |d_E|$ occurring in the above results. We show here that using an old method of Hecke a substantial improvement of the above Theorem is possible, viz.

**Theorem’.** Under the same assumptions of the Theorem above,

$$L(1, \chi) \gg |d_E|^{-1/4},$$

where $\gg$ indicates here (and below) an effectively computable constant depending on $K$ at most.

**Corollary’.** Again under the same assumptions of the Corollary above,

$$h(E) \gg |d_E|^{1/4}.$$

For the case $K = \mathbb{Q}$ Hecke proved that if $L(s, \chi) \neq 0$, for $1 - \frac{\sigma}{\log |d|} < s < 1$, then $h(d) \gg \frac{|d|^{1/2}}{\log |d|}$. A proof appears in [2].

Using this method we are able to prove Theorem’.
2. Proofs. Let \( \kappa(F) \), \( \kappa(K) \) be the residues of \( \zeta_F(s) \), \( \zeta_K(s) \) respectively at \( s = 1 \). Then since 
\[
\zeta_F(s) = \zeta_K(s) L(s, \chi)
\]
we have
\[
L(1, \chi) = \frac{\kappa(F)}{\kappa(K)}.
\]
Under the assumptions of the Theorem, \( L(s, \chi) \) is an entire function, it follows that if \( \zeta_K(\frac{1}{2}) = 0 \) then \( \zeta_F(\frac{1}{2}) = 0 \). We use this fact to obtain a lower bound for \( \kappa(F) \), and since an upper bound for \( \kappa(K) \) is easily got we can prove Theorem 1.

**Lemma 1.** If \( K \) is an algebraic number field of degree \( n \geq 2 \), then
\[
\kappa(K) \leq 2^{2n-n^2} \pi e (1.3)^{n+1} (\log |d_K|)^{n-1}.
\]
And if \( K \) is a totally real field, then
\[
\kappa(K) \leq 2^{n} \pi e (1.3)^{n+1} (\log |d_K|)^{n-1}.
\]

**Proof.** This is Lemma 2.1 of [4].

**Lemma 2.** If \( \zeta_F(\frac{1}{2}) = 0 \), then
\[
\kappa(F) \geq 2^{-2n(n+1)} e^{-\epsilon_0} |d_F|^{-1/4}.
\]

**Proof.** Take \( s_0 = \frac{1}{2}, N = [F:Q] = 2n \) in Lemma 3, p. 323 of [3]. Thus together Lemmas 1 and 2 give
\[
L(1, \chi) \gg |d_F|^{-1/4},
\]
and under the further assumptions of the Corollary we have from the first part of the proof of Theorem 4.1 of [4] (see (7)) that
\[
L(1, \chi) \leq (2\pi)^n \frac{h(F)}{h(K)} |d_K|^{1/2} |d_F|^{1/2},
\]
and so
\[
h(F) \gg L(1, \chi) |d_K|^{1/2} \gg |d_F|^{1/4}.
\]