ACTA ARITHMETICA
XXXV (1979)

A note on Friedlander's paper "On the class numbers of certain quadratic extensions"

by

A. Mallik (Nottingham)

1. J. B. Friedlander [1] has recently proved the following

Theorem. *Let K be an algebraic number field of degree n for which $\zeta_K(\frac{1}{2}) = 0$. Let F be a quadratic extension of K having discriminant d_F and Dedekind zeta function $\zeta_F(s) = \zeta_K(s)L(s,\chi)$. Let $\varepsilon > 0$ be arbitrary. Then,*

$$L(1,\chi) \gg |d_F|^{-1/2}(\log|d_F|)^{2-\varepsilon},$$

where $\gg$ indicates an effectively computable constant depending (at most) on an ε and K.

Corollary. *Assume in addition that K is a totally real field and that F is a totally imaginary quadratic extension of K. Then, if $h(F)$ denotes the class number of F,*

$$h(F) \gg (\log|d_F|)^{2-\varepsilon}.$$

In a remark at the end of [1] he suggests it should be possible to improve the exponent of $\log|d_F|$ occurring in the above results. We show here that using an old method of Hecke a substantial improvement of the above Theorem is possible, viz.

Theorem′. *Under the same assumptions of the Theorem above,*

$$L(1,\chi) \gg |d_F|^{-1/4},$$

where $\gg$ indicates here (and below) an effectively computable constant depending on K at most.

Corollary′. *Again under the same assumptions of the Corollary above,*

$$h(F) \gg |d_F|^{1/4}.$$

For the case $K = Q$ Hecke proved that if $L(s,\chi) \neq 0$, for $1 - \frac{c}{\log|d|} < s < 1$, then $h(d) \gg \frac{|d|^{1/2}}{\log|d|}$. A proof appears in [2].

Using this method we are able to prove Theorem′.

2. Proofs. Let $\varkappa(F)$, $\varkappa(K)$ be the residues of $\zeta_F(s)$, $\zeta_K(s)$ respectively at $s=1$. Then since $\zeta_F(s) = \zeta_K(s)L(s, \chi)$ we have

$$L(1, \chi) = \frac{\varkappa(F)}{\varkappa(K)}.$$

Under the assumptions of the Theorem, $L(s, \chi)$ is an entire function, it follows that if $\zeta_K(\frac{1}{2}) = 0$ then $\zeta_F(\frac{1}{2}) = 0$. We use this fact to obtain a lower bound for $\varkappa(F)$, and since an upper bound for $\varkappa(K)$ is easily got we can prove Theorem′.

LEMMA 1. *If K is an algebraic number field of degree $n \geqslant 2$, then*

$$\varkappa(K) \leqslant 2^{2n}\pi^n\sqrt{e}(1.3)^{n+1}(\log|d_K|)^{n-1}.$$

And if K is a totally real field, then

$$\varkappa(K) \leqslant 2^n\sqrt{e}(1.3)^{n+1}(\log|d_K|)^{n-1}.$$

Proof. This is Lemma 2.1 of [4].

LEMMA 2. *If $\zeta_F(\frac{1}{2}) = 0$, then*

$$\varkappa(F) \geqslant 2^{-2(n+1)}e^{-8\pi n}|d_F|^{-1/4}.$$

Proof. Take $s_0 = \frac{1}{2}$, $N = [F:Q] = 2n$ in Lemma 3, p. 323 of [3]. Thus together Lemmas 1 and 2 give

$$L(1, \chi) \gg |d_F|^{-1/4},$$

and under the further assumptions of the Corollary we have from the first part of the proof of Theorem 4.1 of [4] (see (7)) that

$$L(1, \chi) \leqslant (2\pi)^n\frac{h(F)|d_K|^{1/2}}{h(K)|d_F|^{1/2}},$$

and so

$$h(F) \gg L(1, \chi)|d_F|^{1/2} \gg |d_F|^{1/4}.$$

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MATHEMATICS DEPARTMENT
UNIVERSITY OF NOTTINGHAM
Nottingham, England

Received on 6. 7. 1976
and in revised form on 2. 11. 1976 (861)

ACTA ARITHMETICA
XXXV (1979)

Existence of an indecomposable positive quadratic form in a given genus of rank at least 14

by

G. L. WATSON (London)

0. Introduction. We shall prove the following

THEOREM. *Let f be a positive-definite quadratic form with integer coefficients in $n \geqslant 14$ variables. Then in the genus of f there is at least one class that contains no disjoint form.*

There is also (for $n \geqslant 12$) at least one class that does contain a disjoint form; see [4], pp. 75, 76, Theorem 47.

The constant 14 is best possible; to see this, we define genera each of which consists entirely of classes that contain disjoint forms. Twelve suitable genera may be defined by

(0.1) $$f \simeq x_1^2+x_2^2+\ldots+x_n^2, \quad 2 \leqslant n \leqslant 11 \text{ or } n = 13,$$

(0.2) $$f \simeq x_1^2+x_2^2+\ldots+x_{11}^2+2x_{12}^2 \quad (n = 12).$$

In a number of papers, references to which may be found in [1], it has been shown that

(0.3) each of (0.1), (0.2) implies $f \sim x_1^2+h$, for some $(n-1)$-ary form $h = h(x_2, \ldots, x_n)$.

Denote by $c(f)$ the class-number of f, that is, the number of classes in the genus of f. In the counter-examples (0.1), (0.2) we have $c(f) = 1$ for $n \leqslant 8$; 2 for $n = 9, 10, 11$; 3 for $n = 13$; 4 for $n = 12$. Many other counter-examples, with $n \leqslant 10$ and $c(f) = 1$, may be found in [5]. For the smaller values of n many examples with $c(f) > 1$ could be given. For example, with $n = 2$ and $f \simeq x_1^2+14x_2^2 \simeq 2x_1^2+7x_2^2$, we have $c(f) = 2$.

We shall use the classical formula, see [2], [3] for the weight of a positive genus. The weight, $w(f)$, of the genus of f is the sum of the weights of its constituent classes. Temporarily, let $w'(f)$ be the sum of the weights of the classes that contain disjoint forms; and define $W(f)$ as $w'(f)/w(f)$. Then trivially $W(f) \leqslant 1$; and the theorem may be expressed as:

(0.4) $W(f) < 1$ for every positive-definite f in $n \geqslant 14$ variables.