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(878)

Brauer's class number relation

by

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The main part of this paper proves R. Brauer's class number relation [1] in a shorter and more natural way. Consequently it is possible to obtain Stark's generalization [8] with no extra effort and to observe that the theorem may be applied using only the units of the occurring fields. Nehrorn's conjecture [6] that there exists a corresponding class group isomorphism is also shown to be correct.

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1. Relation theorems. In this first section are derived some general results to describe relations in torsion modules and in torsion-free modules. All the modules concerned will be finitely generated.

Let \mathfrak{D} be a Dedekind domain contained in a field K of characteristic zero and write $\mathfrak{D}_p = \{a/\beta \in K \mid a \in \mathfrak{D}, \beta \in \mathfrak{D} - p\}$ for its localisation at the prime ideal p . Then a \mathfrak{D} -lattice M is a finitely generated torsion-free \mathfrak{D} -module. M will be identified with its natural embedding in $KM = K \otimes_{\mathfrak{D}} M$ and M_p will be written for $\mathfrak{D}_p \otimes_{\mathfrak{D}} M$.

If M and N are two \mathfrak{D} -lattices of $KM = KN$ then the index $[M:N]$ may be defined through the local indices $[M_p:N_p]$ for the free \mathfrak{D}_p -modules M_p and N_p . Let δ_p be the determinant of a matrix which describes a basis of N_p in terms of one for M_p . Then $[M_p:N_p] = \mathfrak{D}_p \delta_p$ is well-defined and non-zero. By taking free \mathfrak{D} -submodules of M and N with the same rank as M and N it is clear that the δ_p can be chosen equal for almost all p and that the ratio of two δ_p is always in the field of fractions k of \mathfrak{D} . Hence the intersection over all primes p which defines the index, viz.

$$[M:N] = \bigcap_p [M_p:N_p]$$

is the product of an ideal in \mathfrak{D} and an element of K . If M and N are isomorphic then $[M:N] = \mathfrak{D}\delta$ for the determinant $\delta \in K$ of the corresponding automorphism of KM . Thus for $\mathfrak{D} = \mathbb{Z}$ and $K = \mathbb{C}$ this coincides with the usual definition of the index viewed as an ideal, and when $K = k$ the definition coincides with that of Fröhlich [2]. If K/k is a number

field extension with norm N_k^K , \mathcal{D}_k is the ring of integers of k , and a, b are ideals of K then $[a:b] = N_k^K(a^{-1}b)$.

Now let G be a finite group. A $\mathcal{D}[G]$ -lattice is just a \mathcal{D} -lattice which is a $\mathcal{D}[G]$ -module.

THEOREM 1.1. *Suppose $\{e_i\}$ is a finite set of idempotents in $k[G]$, χ_i is the character of $K[G]e_i$, and $\sum a_i \chi_i = 0$ for $a_i \in \mathbb{Z}$. If M and N are isomorphic $\mathcal{D}[G]$ -lattices and $KM = KN$ then*

$$\prod_i [M_i : N_i]^{a_i} = \mathcal{D}$$

where $M_i = M \cap e_i KM$ and $N_i = N \cap e_i KN$.

Proof. Any two $K[G]$ -modules X and Y and a $K[G]$ -automorphism α of X induce an automorphism α_Y of $\text{Hom}_{K[G]}(Y, X)$, namely $\alpha_Y(f) = \alpha \circ f$. Clearly $\det \alpha_Y = \Delta(\alpha, \chi)$ depends only on α and the isomorphism class of Y , which is determined by the character χ of Y . If $Y = Y_1 \oplus Y_2$ then $0 \neq \det \alpha_Y = (\det \alpha_{Y_1})(\det \alpha_{Y_2})$, and so

$$\Delta(\alpha, \chi_1 + \chi_2) = \Delta(\alpha, \chi_1) \Delta(\alpha, \chi_2).$$

Thus $\chi \mapsto \Delta(\alpha, \chi)$ extends to a homomorphism from the additive group of the virtual characters of G into the multiplicative group of K . In particular,

$$(*) \quad \sum a_i \chi_i = 0 \Rightarrow \prod \Delta(\alpha, \chi_i)^{a_i} = 1.$$

Let e be an idempotent of $K[G]$ and χ the character of $Y = K[G]e$. Then there is a K -isomorphism $\beta: \text{Hom}_{K[G]}(Y, X) \xrightarrow{\sim} eX$ given by $f \mapsto f(e)$ with inverse $x \mapsto (f: y \mapsto yx)$. Define α_e as the restriction of α to eX . Then $\alpha_e \circ \beta = \beta \circ \alpha_Y$ from which $\Delta(\alpha, \chi) = \det(\alpha_e)$. If α is chosen so that $\alpha M = N$ then $\mathcal{D}\det(\alpha_e) = [M_i : N_i]$ and $(*)$ proves the theorem.

Remark (J.-J. Payan). From the local definition of index, the theorem still holds if the $\mathcal{D}[G]$ -lattices M and N are just assumed to be in the same genus, i.e. $M_p \cong N_p$ for all p .

THEOREM 1.2. *Let $S = \{e_i\}$ be a finite set of idempotents in $k[G]$ and \mathcal{D}_S the subring of k generated over \mathcal{D} by $|G|^{-1}$ and the coefficients of the $e_i \in S$. Suppose χ_i is the character of $k[G]e_i$ and $\sum a_i \chi_i = \sum b_i \chi_i$ for some non-negative integers a_i and b_i . If M is a finite group and a $\mathcal{D}_S[G]$ -module then there is a \mathcal{D}_S -module isomorphism*

$$\bigoplus_{i=1}^{a_i} e_i M^{(i)} \cong \bigoplus_{i=1}^{b_i} e_i M^{(i)} \quad \text{for} \quad M^{(i)} \cong M.$$

Proof. Again let $M_p = \mathcal{D}_p \otimes_{\mathcal{D}} M$ for each prime p of \mathcal{D} . Then M_p is a $\mathcal{D}_p[G]$ -module which is trivial for almost all p and, in particular, for p dividing the ideal $\mathcal{D}[G]$. As $M \cong \bigoplus_p M_p$ we may assume without loss of generality that $M = M_p$ for some prime p not dividing $\mathcal{D}[G]$.

Let $N_i = \mathcal{D}_p[G]e_i$. Then there is a \mathcal{D}_p -isomorphism $\text{Hom}_{\mathcal{D}_p[G]}(N_i, M) \xrightarrow{\sim} e_i M$ given by $f \mapsto f(e_i)$. Two $\mathcal{D}_p[G]$ -lattices N and N' of the same character satisfy $kN \cong kN'$ and therefore the work of Maranda ([5], Theorem 4) shows that $N \cong N'$. Combining these isomorphisms gives

$$\bigoplus_{i=1}^{a_i} e_i M^{(i)} \cong \text{Hom}\left(\bigoplus_{i=1}^{a_i} N_i, M\right) \cong \text{Hom}\left(\bigoplus_{i=1}^{b_i} N_i, M\right) \cong \bigoplus_{i=1}^{b_i} e_i M^{(i)}.$$

2. Nehrorn's theorem. Let K/k be a normal extension of algebraic number fields with Galois group G . Suppose 1_H^G is the character on G induced from the unit character on a subgroup H , and for a module X on which G acts let HX be the submodule fixed under H . Write \bar{H} for the sum of the elements in H . As usual let us define U to be the group of units in K ; W its subgroup of roots of unity; $w(H)$ the order of HW ; and $w_2(H)$ the 2-component of $w(H)$.

THEOREM 2.1. *Let $\mathcal{C}(HK)$ be the part of the ideal class group of HK formed from classes whose orders are prime to $|G|$. If*

$$\sum_H a(H) 1_H^G = \sum_H b(H) 1_H^G$$

where $a(H)$ and $b(H)$ are non-negative integers then there is a group isomorphism

$$\bigoplus_H \bigoplus_{j=1}^{a(H)} \mathcal{C}(HK)^{(j)} \cong \bigoplus_H \bigoplus_{j=1}^{b(H)} \mathcal{C}(HK)^{(j)} \quad \text{for} \quad \mathcal{C}(HK)^{(j)} \cong \mathcal{C}(HK).$$

Nehrorn indicated in [6] that the above result holds but proved it only for K/k abelian. It is immediate from Theorem 1.2 because of the natural isomorphism $\mathcal{C}(HK) \cong HC(K)$ and because the character 1_H^G corresponds to the idempotent $\bar{H}/|H|$.

LEMMA 2.2. *Suppose M is a finite $\mathbb{Z}[G]$ -module fixed by a normal subgroup N over which G is cyclic. If $\sum a(H) 1_H^G = \sum b(H) 1_H^G$ where $a(H)$ and $b(H)$ are non-negative integers then there is a trivial group isomorphism*

$$\bigoplus_H \bigoplus_{j=1}^{a(H)} HM^{(j)} \cong \bigoplus_H \bigoplus_{j=1}^{b(H)} HM^{(j)} \quad \text{for} \quad M^{(j)} \cong M.$$

Proof. From $\sum 1_H^G(g)g = |H|^{-1} \sum g\bar{H}g^{-1}$ for both sums over $g \in G$ we deduce that $1_H^G(g\bar{N}) = |N| 1_{HN/N}^G(g)$. Hence $\sum a(H) 1_{HN/N}^G = \sum b(H) 1_{HN/N}^G$. By Brauer [1], Satz 2, or Rehm [7], Satz 1, this relation is trivial for G/N cyclic. Thus the stated group isomorphism holds trivially as M is a $\mathbb{Z}[G/N]$ -module with $HM = (HN/N)M$.

THEOREM 2.3 (Brauer [1], § 5). If $\sum a(H)1_H^G = 0$ then

$$\prod_H w(H)^{a(H)} = \prod_H w_2(H)^{a(H)}.$$

Suppose also that W_2 is the group of 2-power roots of unity in K and $k(W_2)/k$ is cyclic. Then

$$\prod_H w_2(H)^{a(H)} = 1.$$

Proof. Let W_p be the Sylow p -subgroup of W . Then, with the possible exception of $p = 2$, $k(W_p)/k$ is cyclic and the theorem is a direct consequence of Lemma 2.2 on taking orders.

3. Brauer's theorem. With the notation of § 2 let us assume also that $k = \mathbb{Q}$; $n(H) = [G:H]$ is the degree of HK over k ; $r_1(H)$ and $r_2(H)$ are the numbers of real and non-real infinite valuations of HK ; $r(H)$ is the rank of HU/HW ; $R(H)$ is the regulator and $h(H)$ the class number of HK ; and $\delta(H) = 2$ or 1 according as HK is totally complex or not. For some fixed embedding of K into the complex numbers \mathbb{C} let \mathcal{O} be the Galois group of the maximal real subfield of K . Thus \mathcal{O} is generated by the automorphism γ which induces complex conjugacy on K .

Let L and L^* be $\mathbb{Z}[G]$ -lattices which make

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G]\tilde{\mathcal{O}} \rightarrow L \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L^* \rightarrow \mathbb{Z}[G]\tilde{\mathcal{O}} \rightarrow \mathbb{Z} \rightarrow 0$$

exact sequences of left $\mathbb{Z}[G]$ -modules. Here the maps from and to \mathbb{Z} are given by $n \mapsto n\tilde{\mathcal{O}}$ and $a\tilde{\mathcal{O}} \rightarrow 1(a)$ respectively for the unit character 1 . Specifically, L and L^* will be identified with $\mathbb{Z}[G]\tilde{\mathcal{O}}/\mathbb{Z}\tilde{\mathcal{G}}$ and $\{a \in \mathbb{Z}[G]\tilde{\mathcal{O}} \mid 1(a) = 0\}$. Denote by a bar the natural maps $U \rightarrow U/W$ and $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G]/\mathbb{Z}\tilde{\mathcal{G}}$ and define maps $\lambda: \bar{U} \rightarrow CL$ and $\lambda^*: \bar{U} \rightarrow CL^*$ by

$$\lambda(\bar{\varepsilon}) = \sum_{g \in G} \log \|g^{-1}\varepsilon\|_{\bar{g}},$$

$$\lambda^*(\bar{\varepsilon}) = \sum_{g \in G} \log \|g^{-1}\varepsilon\|_g \quad \text{for} \quad \varepsilon \in U$$

where $\| \cdot \|$ is the absolute value of the chosen embedding of K into \mathbb{C} . These are both $\mathbb{Z}[G]$ -homomorphisms and they are injections because the ranks of $\lambda(\bar{U})$, $\lambda^*(\bar{U})$, L , L^* , and \bar{U} are all equal by the next theorem and the Dirichlet unit theorem.

THEOREM 3.1. We have

$$[HL: \lambda \bar{HU}] = \mathbb{Z} n(H) 2^{-r_2(H)} R(H)$$

and

$$[HL^*: \lambda^* \bar{HU}] = \mathbb{Z} \delta(H) 2^{-r_2(H)} R(H).$$

Proof. Let $Hg\mathcal{O}$ denote the sum of the distinct elements in $\{hgc \mid h \in H, c \in \mathcal{O}\}$ and $|Hg\mathcal{O}|$ the number of such elements. If possible

choose $g_0 \in G$ such that $Hg_0\mathcal{O}$ is a single coset of \bar{H} and otherwise take any g_0 . Then $\delta(H) = |Hg_0\mathcal{O}|/|H|$ and the elements $Hg\mathcal{O} - |Hg\mathcal{O}| |Hg_0\mathcal{O}|^{-1} Hg_0\mathcal{O}$ generate HL^* over \mathbb{Z} . For $\varepsilon \in HU$ we have

$$\lambda^*(\bar{\varepsilon}) = \sum \log \|g^{-1}\varepsilon\|_{Hg\mathcal{O}} = \sum \log \|g^{-1}\varepsilon\| (Hg\mathcal{O} - |Hg\mathcal{O}| |Hg_0\mathcal{O}|^{-1} Hg_0\mathcal{O})$$

for sums over double coset representatives $g \in H \setminus G/\mathcal{O}$. Hence

$$[HL^*: \lambda^* \bar{HU}] = \mathbb{Z} \delta(H) 2^{-r_2(H)} R(H).$$

Now $[HL: \lambda \bar{HU}] = [HL: \bar{HL}^*][HL^*: \lambda^* \bar{HU}]$ because $CL^* \rightarrow CL^*$ is an isomorphism. As a basis of HL is given by $\{\overline{Hg\mathcal{O}}\}$ for $g \in H \setminus G/\mathcal{O}$ with $g \notin Hg_0\mathcal{O}$ so $[HL: \bar{HL}^*] = \mathbb{Z} \det(a_{g,\sigma})$ where $a_{g,\sigma} = \delta_{g,\sigma} + |Hg\mathcal{O}|/|Hg_0\mathcal{O}|$ for the Kronecker delta δ . But $\sum_g a_{g,\sigma} = n(H)/\delta(H)$ gives a constant row by which $|Hg\mathcal{O}|/|Hg_0\mathcal{O}|$ may be subtracted from each $a_{g,\sigma}$. Thus $[HL: \bar{HL}^*] = \mathbb{Z} n(H)/\delta(H)$ as required.

LEMMA 3.2. If $\sum a(H)1_H^G = 0$ then

$$\begin{aligned} 0 &= \sum a(H) = \sum a(H)r_1(H) = \sum a(H)r_2(H) = \sum a(H)r(H) \\ &= \sum a(H)n(H). \end{aligned}$$

Proof. The sums are the evaluations of the relation at $\bar{G}/|G|$, γ^* $(1-\gamma)/2$, $\tilde{\mathcal{O}}/|C| - \bar{G}/|G|$, and 1 respectively because

$$r_1(H) = |\{g_x^* \in G \mid g\gamma g^{-1} \in H\}|/|H| = 1_H^G(\gamma).$$

LEMMA 3.3. If $\sum a(H)1_H^G = 0$ and $M, M^* \subset \bar{U}$ are $\mathbb{Z}[G]$ -isomorphic to L, L^* respectively then

$$\prod [R(H)^{-a(H)}] = \prod (n(H)[\bar{HU}:HM])^{a(H)} = \prod (\delta(H)[\bar{HU}:HM^*])^{a(H)}.$$

Proof. M and M^* exist because λ and λ^* are injective homomorphisms so that L, L^* , and \bar{U} all have the same character.

Let $\pi = \prod (n(H)[\bar{HU}:HM]R(H))^{a(H)}$. By Theorem 3.1

$$\mathbb{Z}\pi = \prod [(2^{r_2(H)}[\lambda \bar{HU}: \lambda HM][HL: \lambda \bar{HU}])^{a(H)}].$$

Hence Lemma 3.2 and Theorem 1.1 yield

$$\mathbb{Z}\pi = \prod [HL: \lambda HM]^{a(H)} = \prod [HL: H\lambda M]^{a(H)} = \mathbb{Z}$$

from $L \cong \lambda M$. The other relation holds similarly.

Application of the functional equation to the residue of the zeta function $\zeta_{HK}(s)$ at $s = 1$ gives the well-known result

$$\lim_{s \rightarrow 1} s^{-r(H)} \zeta_{HK}(s) = -h(H)R(H)/w(H)$$

while the interpretation of $\zeta_{HK}(s)$ as the Artin L -series $L(s, 1_H^G, K/Q)$ shows that $\sum a(H)1_H^G = 0$ implies $\prod \zeta_{HK}(s)^{a(H)} = 1$. Equating values at $s = 0$ and using Lemma 3.2 yields (Kuroda [3])

$$\prod_H (h(H)R(H)/w(H))^{a(H)} = 1.$$

Comparing this with the limit of $\prod \zeta_{HK}(s)^{a(H)} = 1$ as $s \rightarrow 1$ shows that the corresponding product of discriminants is also 1. However, combining it with Lemma 3.3 and Theorem 2.3 immediately provides Brauer's theorem ([1], Satz 4):

THEOREM 3.4. *If the submodules M and M^* of \bar{U} are $Z[G]$ -isomorphic to L and L^* respectively and if $\sum a(H)1_H^G = 0$ then*

$$\begin{aligned} \prod_H h(H)^{a(H)} &= \prod_H (n(H)w_2(H)[\overline{HU}:HM])^{a(H)} \\ &= \prod_H (\delta(H)w_2(H)[\overline{HU}:HM^*])^{a(H)}. \end{aligned}$$

Remark 3.5. Set $S = \{H \mid a(H) \neq 0\}$ and let U_S be the group generated over $Z[G]$ by $\{HU \mid H \in S\}$. Then $\overline{U_S}$ may have smaller rank than \bar{U} so that more units need to be calculated to obtain a module M . However, suppose L_S is a $Z[G]$ -module satisfying $HL \subset L_S \subset L$ for all $H \in S$ and $M' \subset \bar{U}$ is the corresponding submodule of M . As $HM = HM'$ for all $H \in S$ we may replace M by M' in the theorem and by Theorem 1.1 the substitution of any module $M_S \subset \bar{U}$ which is $Z[G]$ -isomorphic to L_S is also valid. In particular, the minimal choice of L_S ensures that $M_S \subset \overline{U_S}$. A module M_S^* can be defined analogously. It is therefore possible to apply Theorem 3.4 when only the units of the occurring subfields are known.

Remark 3.6. The full extent of Theorem 1.1 has not yet been exploited but we expect that when the value of

$$o(\chi) = \lim_{s \rightarrow 0} s^{-r(\chi)} L(s, \chi, K/Q)$$

has been calculated for $r(\chi) = \chi(\bar{O}/|O| - \bar{G}/|G|)$ and any character χ then the same techniques will produce a relation similar to Theorem 3.4 (see Lichtenbaum [4]). An intermediate result can be obtained. If the character ρ is irreducible over Q , contains an absolutely irreducible character of degree $d(\rho)$, and $a(H) \in Q$ are chosen to satisfy $\rho = \sum a(H)1_H^G$, then the methods above give

$$\pm \frac{o(\rho)}{2^{e(1-r)/2} [L_\rho : \lambda M_\rho]^{1/d(\rho)}} = \prod_H \left\{ \frac{h(H)}{n(H)w(H)[\overline{HU}:HM]} \right\}^{a(H)}$$

where $L_\rho = L \cap e_\rho CL$ and $M_\rho = M \cap e_\rho CM$ for the central idempotent e_ρ of $Q[G]$ corresponding to ρ . In [8] Stark derives essentially the same formula by generalizing the methods of Brauer.

4. Change of ground field. It remains to interpret Brauer's theorem in terms of the Galois group \mathcal{G} of a relative normal extension \mathcal{K}/\mathcal{K} within K/Q . \mathcal{H} will denote a subgroup of \mathcal{G} , \mathcal{U} the unit group of \mathcal{K} , and $G(\mathcal{K})$ and $G(\mathcal{H})$ the Galois groups of K/\mathcal{K} and K/\mathcal{H} . Then $\mathcal{L} = G(\mathcal{H})L$ is a $Z[\mathcal{G}]$ -module whose precise structure will be determined below.

THEOREM 4.1. *Suppose $\sum a(\mathcal{H})1_{\mathcal{H}}^{\mathcal{G}} = 0$. If the submodule \mathcal{M} of $\bar{\mathcal{U}}$ is $Z[\mathcal{G}]$ -isomorphic to \mathcal{L} then*

$$\prod h(\mathcal{H})^{a(\mathcal{H})} = \prod (n(\mathcal{H})w_2(\mathcal{H})[\overline{\mathcal{H}\mathcal{U}}:\mathcal{H}\mathcal{M}])^{a(\mathcal{H})}$$

for $n(\mathcal{H}) = [\mathcal{H}\mathcal{K}:\mathcal{K}]$.

Proof. Put $H = G(\mathcal{H})\mathcal{K}$. Then $C[\mathcal{G}]\bar{\mathcal{H}}$ and $C[G(\mathcal{H})]\bar{H}$ are $C[\mathcal{G}]$ -isomorphic under $\bar{\mathcal{H}} \leftrightarrow \bar{H}$. So they have the same characters, i.e. $1_{\bar{\mathcal{H}}}^{\mathcal{G}}(g) = 1_{\bar{H}}^{G(\mathcal{H})}(g)$ if $g \in \mathcal{G}$ is the image of $g \in G(\mathcal{H})$. Hence the character relation $\sum a(H/G(\mathcal{H}))1_H^G = 0$ holds and Theorem 3.4 may be applied. Evidently $\overline{HU} = \overline{\mathcal{H}\mathcal{U}}$ and $HM = \mathcal{H}\mathcal{M}$ for $H = G(\mathcal{H})\mathcal{K}$ and $\mathcal{M} = G(\mathcal{H})M$. When these have been substituted Theorem 1.1 allows any $\mathcal{M} \cong \mathcal{L}$ to be chosen because $\mathcal{M} \cong \mathcal{L}$, and Lemma 3.2 permits the new value of $n(\mathcal{H})$.

The generators HgC of $HZ[G]\bar{C}$ may be identified with the normalised infinite valuations

$$v_{HgC}(x) = \|g^{-1}x\|^f \quad (x \in HK)$$

of HK where $f = |HgC|/|H|$ and $\|\cdot\|$ is the absolute value for the chosen embedding of K into C . So the subgroup

$$\mathcal{G}_i = (g_i C g_i^{-1} \cap G(\mathcal{K})) / G(\mathcal{K})$$

of \mathcal{G} which fixes $G(\mathcal{K})g_i C$ is the decomposition group in \mathcal{K}/\mathcal{K} of the corresponding infinite prime. Thus the double coset decomposition

$$\bar{G} = \sum_{i=1}^r G(\mathcal{K})g_i C$$

determines up to conjugacy a decomposition group \mathcal{G}_i for each infinite prime of \mathcal{K} .

The exact sequence defining L restricts to

$$0 \rightarrow Z \rightarrow G(\mathcal{K})Z[G]\bar{C} \rightarrow G(\mathcal{K})L \rightarrow 0.$$

This is also exact as fixing by a subgroup is a left exact functor and any pre-image of an element in $G(\mathcal{K})L$ is necessarily fixed by $G(\mathcal{K})$. How-

ever, $G(\mathcal{X})Z[G]\tilde{C} \cong \bigoplus_{i=1}^r Z[\mathcal{G}]\tilde{\mathcal{C}}_i$ under the $Z[\mathcal{G}]$ -map $\sum x_i G(\mathcal{X})g_i \tilde{C} \mapsto \bigoplus x_i \tilde{\mathcal{C}}_i$ for $x_i \in Z[\mathcal{G}]$. Hence

LEMMA 4.2. If $\{\mathcal{G}_i\}$ is the set of decomposition groups for one prime divisor in \mathcal{X} of each of the r infinite primes in \mathcal{L} then \mathcal{L} satisfies the exact sequence

$$0 \rightarrow Z \rightarrow \bigoplus_{i=1}^r Z[\mathcal{G}]\tilde{\mathcal{C}}_i \rightarrow \mathcal{L} \rightarrow 0$$

where $n \in Z \mapsto n \bigoplus_i \tilde{\mathcal{C}}_i$.

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Kuroda's class number relation*

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Kuroda's class number relation [5] may be derived easily from that of Brauer [2] by eliminating a certain module of units, but the technique is applicable to a much wider class of relations which are obtained from norm relations. The main aim here is to treat the case in which several radicals of the same prime degree are adjoined to the rational field.

1. Norm relations. Let G be the Galois group of a normal extension K/k of algebraic number fields and \tilde{H} the sum of the elements in a subgroup H . Then a relation of the form

$$(1.1) \quad \sum_H b(H)\tilde{H} = 0 \quad (b(H) \in \mathcal{O})$$

is called a *norm relation*. These have been studied by Rehm in [7] and are so-called because Artin has established in [1] that the relation holds precisely when

$$\prod_H (N_{K/HK}(x))^{b(H)} = 1 \quad \text{for all } x \in K^*.$$

Here HK is the subfield fixed by H and N is the relative norm. If 1_H^G denotes the character on G induced by the unit character on H then the equation

$$(1.2) \quad \sum_{g \in G} 1_H^G(g)g = |H|^{-1} \sum_{g \in G} g\tilde{H}g^{-1}$$

may be used to convert the norm relation (1.1) into the character relation

$$(1.3) \quad \sum_H b(H)|H|1_H^G = 0.$$

The most interesting relations satisfy two further conditions:

(1.4) DEFINITION. $\sum b(H)H = 0$ is called a *direct norm relation* if

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