ACTA ARITHMETICA
XXXIV (1979)

# Some integer sequences related to Pisot sequences*

by

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*Dedicated to Ch. Pisot on the occasion of his retirement*

**1. Introduction.** In his 1938 paper [6], Pisot initiated the study of the sequences of integers, now called Pisot sequences, defined by the condition

$$-1/2 < a_{n+1} - a_n^2/a_{n-1} \leqslant 1/2. \tag{1}$$

Given $0 < a_0 \leqslant a_1$, (1) determines a sequence $E(a_0, a_1)$. These sequences are modelled on sequences of real numbers of the type $u_n = \lambda\theta^n$ in the sense that for such a sequence $u_{n+1} = u_n^2/u_{n-1}$. Pisot showed in fact that if $a_1 > a_0 + (2a_0)^{1/2}$ then there are $\theta > 1$ and $\lambda > 0$ determined by $\lim(a_{n+1}/a_n) = \theta$ and $\lim(a_n^{n+1}/a_{n+1}^n) = \lambda$ such that $a_n = \lambda\theta^n + \varepsilon_n$, where $\varepsilon_n$ is a bounded sequence with $\limsup|\varepsilon_n| \leqslant (\theta-1)^{-2}/2$. The countable set of $\theta$ so produced is called $E$.

Although his motivation was the study of the distribution modulo 1 of sequences of the type $\lambda\theta^n$, Pisot also considered the question of whether the numbers in $E$ are algebraic. In this regard, he showed that $E(2, a_1)$ and $E(3, a_1)$ satisfy linear recurrence relations and that $\theta(2, a_1)$ and $\theta(3, a_1)$ are Pisot numbers. Cantor [4] has recently given an extensive generalization of this result.

In general, if $E(a_0, a_1)$ satisfies a linear recurrence relation, then $\theta$ is either a Pisot or Salem number (or $\theta = 1$), as shown by Flor [5]. Cantor [3] proposed using Pisot sequences to search for Salem numbers, but cast doubt on the possibility that all Pisot sequences satisfy linear recurrence relations by his observation that $E(4, 13)$ satisfies no such relation of degree $\leqslant 100$. The question was settled in [1], by showing that the set of $\theta$ in $E$ corresponding to non-recurrent sequences is dense in $[\tau, \infty)$, where $\tau = (5^{1/2}+1)/2$.

* This research was supported in part by a grant from the National Research Council of Canada.

It is possible that the techniques of [1] could yield positive results about the Salem numbers. For example, if an interval $(a, b)$, with $1 \leqslant a < b$, could be found such that $a < \theta(a_0, a_1) < b$ implies $E(a_0, a_1)$ is non-recurrent, then it would follow that $T \cap (a, b) = \emptyset$ and hence that $\inf T > 1$, by a familiar argument. One difficulty with this approach is that the methods of [1] work best when $\theta > \tau$, whereas it is more hopeful to deal with intervals near 1. This is the motivation for the sequences we will introduce here. Now the methods of [1] will apply in a natural way to the entire interval $\theta > 1$.

We model our sequences on $u_n = \lambda\theta^n + \mu\theta^{-n}$. Noting that

$$(u_{n+2} + u_n)/u_{n+1} = \theta + \theta^{-1} = (u_{n+1} + u_{n-1})/u_n,$$

we define the *F-sequence* $F(a_0, a_1, a_2)$ to be the sequence of integers satisfying

$$(2) \qquad -1/2 < a_{n+2} + a_n - (a_{n+1}/a_n)(a_{n+1} + a_{n-1}) \leqslant 1/2, \qquad n \geqslant 1.$$

Thus $a_{n+2}$ is uniquely determined unless $a_n = 0$. If this should happen for $n \geqslant 2$, a natural definition is to take $a_{n+2} = -a_{n-2}$.

It is important to realize that we may take $a_0 = 0$, so for example

$$F(0, 5, 11) = 0, 5, 11, 19, 30, 46, 70, 106, \ldots$$

We call an $F$-sequence with $a_0 = 0$ a *special* $F$-sequence. (This is an appropriate point to mention that Cantor [4] uses the term $F$-sequence for a quite different concept.)

In Section 2, we will show, under fairly general assumptions, that an $F$-sequence satisfies $a_n = \lambda\theta^n + \varepsilon_n$ for some $\lambda > 0, \theta > 1$, where $\varepsilon_n$ is a bounded sequence. Note that the motivating parameter $\mu$ is not identifiable since an arbitrary term of the form $\mu\theta^{-n}$ can be added without affecting the asymptotic behaviour of $\varepsilon_n$. The advantage of (2) over (1), for our purposes, is the appearance of an extra factor $(E - \theta^{-1})$ in the "characteristic inequality" (23) for $\varepsilon_n$.

It should be mentioned that Pisot [6] studied sequences of integers modelled on quite general functions of $r$ parameters. In his case, the parameters are all determinable from the asymptotic behaviour of $a_n$, so our $F$-sequences do not quite fit into his framework.

In Section 4, we discuss the relation between $F$-sequences and the Pisot and Salem numbers. In Section 5, we apply the methods of [1] to deduce a criterion for an $F$-sequence to satisfy a $T$-recurrence; some examples are discussed there. Finally, in Section 6, we list a few other related sequences.

## 2. Geometric $F$-sequences

DEFINITION. We say that an $F$-sequence is *geometric* if $a_n > 0$ eventually, and $\liminf(a_{n+1}/a_n) > 1$.

The following three lemmas generalize results of [6] and [4]:



LEMMA 1. *Let $a_n$ be a geometric F-sequence. Then* $\lim(a_{n+1}/a_n) = \theta > 1$, $\lim(a_n^{n+1}/a_{n+1}^n) = \lambda > 0$, *and* $\varepsilon_n = a_n - \lambda\theta^n$ *is a bounded sequence.*

Proof. Write $\theta_n = a_{n+1}/a_n$ and $\lambda_n = a_n^{n+1}/a_{n+1}^n = a_n\theta_n^{-n} = a_{n+1}\theta_n^{-n-1}$. We begin by showing $b_n = a_{n+1} - a_n^2/a_{n-1}$ is a bounded sequence. Given $1 < \alpha < \liminf\theta_n$, we have $\theta_n \geqslant \alpha$ for $n \geqslant m$, say. By (2), $|b_{n+1} - \theta_{n-1}^{-1}b_n| \leqslant 1/2$ and hence $|b_{n+1}| \leqslant \alpha^{-1}|b_n| + 1/2$ for $n > m$. Iterating this inequality, we have

$$(3) \qquad \begin{aligned} |b_{n+1}| &\leqslant \alpha^{m-n}|b_{m+1}| + \alpha/(2(\alpha-1)) \\ &\leqslant |b_{m+1}| + \alpha/(2(\alpha-1)) = c, \qquad \text{for} \quad n \geqslant m. \end{aligned}$$

But $b_{n+1} = a_{n+1}(\theta_{n+1} - \theta_n)$, hence (3) shows that

$$(4) \qquad |\theta_{n+1} - \theta_n| \leqslant c/a_{n+1} \leqslant c\alpha^{m-n-1}/a_m \qquad \text{for} \quad n \geqslant m.$$

This shows that $\theta_n$ converges to $\theta$, say, where $\theta > 1$ and

$$(5) \qquad |\theta - \theta_n| \leqslant ca_n^{-1}(\alpha-1)^{-1} \qquad \text{for} \quad n \geqslant m.$$

Now write

$$(6) \qquad \begin{aligned} |\lambda_{n+1} - \lambda_n| &= a_{n+1}|\theta_{n+1}^{-n-1} - \theta_n^{-n-1}| \leqslant (n+1)\alpha^{-n-2}a_{n+1}|\theta_{n+1} - \theta_n| \\ &= (n+1)\alpha^{-n-2}|b_{n+1}| \leqslant c(n+1)\alpha^{-n-2}, \end{aligned}$$

so $\lambda_n$ converges to $\lambda$, say, and it will be obvious later that $\lambda > 0$.

We next show that $a_n\theta^{-n} \to \lambda$. Multiply (5) by $a_n\theta^{-n-1}$ to get

$$(7) \qquad |a_n\theta^{-n} - a_{n+1}\theta^{-n-1}| \leqslant c\theta^{-n-1}(\alpha-1)^{-1},$$

which shows $a_n\theta^{-n}$ converges to $l$, say. But $l = \lambda$, since

$$(8) \qquad |a_n\theta^{-n} - a_n\theta_n^{-n}| \leqslant na_n\alpha^{-n-1}|\theta - \theta_n| \leqslant cn\alpha^{-n-1}(\alpha-1)^{-1},$$

by (5). Thus (7) implies

$$(9) \qquad |a_n\theta^{-n} - \lambda| \leqslant c\theta^{-n}(\theta-1)^{-1}(\alpha-1)^{-1},$$

that is

$$(10) \qquad |\varepsilon_n| \leqslant c(\theta-1)^{-1}(\alpha-1)^{-1}.$$

Since $a_n \to \infty$, and $\varepsilon_n$ is bounded, it is clear that $\lambda > 0$.

LEMMA 2. *Suppose $a_1 > 0$ and $a_2 > 2a_1$, then the special F-sequence $F(0, a_1, a_2)$ is geometric. Furthermore, the set of limits $\theta$ from such sequences is dense in $[1, \infty)$.*

Proof. We shall prove by induction that for all $n \geqslant 1$,

$$(11) \qquad a_{n+1} - 2a_n + a_{n-1} \geqslant n \qquad \text{and} \qquad a_{n+1}/a_n \geqslant (n+1)/n.$$

For $n = 1$, this has been assumed. From (2), assuming (11), we have

$$(12) \qquad \begin{aligned} a_{n+2} + a_n &\geqslant (a_{n+1}/a_n)(a_{n+1} + a_{n-1}) - 1/2 \\ &\geqslant (a_{n+1}/a_n)(2a_n + n) - 1/2 \geqslant 2a_{n+1} + n + 1 - 1/2, \end{aligned}$$

which proves $a_{n+2}-2a_{n+1}+a_n \geqslant n+1$, since the left member of (12) is an integer. This in turn implies that

$$a_{n+2}/a_{n+1} \geqslant 2-a_n/a_{n+1}+(n+1)/a_{n+1} > 2-n/(n+1) = (n+2)/(n+1),$$

so the induction is complete.

Summing (11) twice gives

$$a_n \geqslant \binom{n+1}{3}+na_1 > \binom{n+1}{3}. \tag{13}$$

By (2), we have

$$|(a_{n+2}+a_n)/a_{n+1}-(a_{n+1}+a_{n-1})/a_n| \leqslant (2a_{n+1})^{-1}. \tag{14}$$

Summing (14) and using (13) shows that $(a_{n+1}+a_{n-1})/a_n \to \varrho$, with

$$|\varrho-(a_{n+1}+a_{n-1})/a_n| < 3/(2n(n+1)). \tag{15}$$

By (11), we have

$$(a_{n+1}+a_{n-1})/a_n-2 \geqslant n/a_n, \tag{16}$$

so that $\varrho \geqslant 2$. But, if $\varrho = 2$, then (15) and (16) imply that

$$a_n \geqslant 2n^2(n+1)/3 > 4\binom{n+1}{3}. \tag{17}$$

This improvement of (13) can be used as above giving a corresponding improvement of (15), and by induction $a_n > 4^k\binom{n+1}{3}$ for all $k$, which is not possible.

Since $\lim((a_{n+1}+a_{n-1})/a_n) > 2$ and $a_{n-1}/a_n < 1$, it follows that $\liminf(a_{n+1}/a_n) > 1$ so $a_n$ is geometric.

To see that the set of $\theta$ is dense in $[1, \infty)$, it suffices to prove

$$|(\theta+\theta^{-1})-a_2/a_1| < (\log a_1)/(4a_1)+3/(2(a_1-a_1^{1/2})). \tag{18}$$

To do this, we have from (13) that $a_n > \max\left(na_1, \binom{n+1}{3}\right)$. Hence if $m = [a_1^{1/2}]$, (14) implies

$$|\varrho-a_2/a_1| < \sum_{n=2}^{m}(2na_1)^{-1}+\sum_{n=m+1}^{\infty}3/((n+1)n(n-1)) \tag{19}$$
$$< \log m/(2a_1)+3/(2m(m+1)),$$

which implies (18).

LEMMA 3. *Suppose that* $a_2-2a_1+a_0 > (2a_1)^{1/2}$ *and* $a_1 \geqslant a_0$. *Then* $F(a_0, a_1, a_2)$ *is geometric. Furthermore* $d_n = a_{n+1}-2a_n+a_{n-1} > (2a_n)^{1/2}$ *for all* $n$, *and*

$$|(\theta+\theta^{-1})-(a_{n+1}+a_{n-1})/a_n| \leqslant 1/\left(d_n+(d_n^2-2a_n)^{1/2}\right). \tag{20}$$



Proof. By the assumptions, there is a $c > 0$ satisfying

$$d_n = a_{n+1}-2a_n+a_{n-1} \geqslant a_n/(2c)+c, \tag{21}$$

for $n = 1$; namely $c = a_1/(d_1+(d_1^2-2a_1)^{1/2})$. We shall prove by induction that (21) and $a_n \geqslant a_{n-1}$ hold for all $n$. For, by (2)

$$d_{n+1} \geqslant (a_{n+1}/a_n)d_n-1/2 \geqslant (a_{n+1}/a_n)(a_n/(2c)+c)-1/2 \tag{22}$$
$$\geqslant a_{n+1}/(2c)+(a_{n+1}/a_n)c-1/2.$$

However, $a_n \geqslant a_{n-1}$ so that (21) implies $a_{n+1} \geqslant (1+(2c)^{-1})a_n+c$, and this combines with (22) to show (21) for $d_{n+1}$. It also shows that $a_{n+1} \geqslant a_n$ so the induction is complete.

By Lemma 1, $a_{n+1}/a_n \to \theta > 1$. We obtain (20) by using (14) and $a_k \geqslant (1+(2c)^{-1})^{k-n}a_n$ for $k \geqslant n$, where now we can use $c = a_n/(d_n+(d_n^2-2a_n)^{1/2})$.

## 3. The characteristic inequality

THEOREM 1. (a) *Suppose* $a_n = \lambda\theta^n+\varepsilon_n$ *is a geometric F-sequence. Then*

$$\limsup|(E-\theta)^2(E-\theta^{-1})\varepsilon_n| \leqslant 1/2, \tag{23}$$

*where* $E\xi_n = \xi_{n+1}$ *denotes the (backward) shift operator.*

(b) *Conversely, if* $a_n = \lambda\theta^n+\varepsilon_n$ *with* $\lambda > 0$ *and* $\theta > 1$ *and* $\varepsilon_n$ *bounded, and if*

$$\limsup|(E-\theta)^2(E-\theta^{-1})\varepsilon_n| < 1/2, \tag{24}$$

*then* $a_n$ *satisfies* (2) *for all sufficiently large* $n$.

(c) *The inequality* (23) *implies*

$$\limsup|\varepsilon_n| \leqslant \theta/(2(\theta-1)^3), \tag{25}$$

*and* (24) *is implied by*

$$\limsup|\varepsilon_n| < \theta/(2(\theta+1)^3). \tag{26}$$

Proof. Write $D_n(a) = a_n(a_{n+2}+a_n)-a_{n+1}(a_{n+1}+a_{n-1})$. Then (2) implies $|D_n(a)| \leqslant a_n/2$. But

$$D_n(a) = \lambda\theta^n\{(E-\theta)^2(E-\theta^{-1})\varepsilon_{n-1}\}+D_n(\varepsilon),$$

which immediately implies (a) and (b).

To prove (c), we write $(E-\theta)^2(E-\theta^{-1})\varepsilon_n = \zeta_n$, so that $\limsup|\zeta_n| \leqslant 1/2$. Since $E$ is an operator of norm 1 on bounded sequences, we can immediately conclude that $E-\theta = -\theta(1-\theta^{-1}E)$ is invertible and

$$(E-\theta)(E-\theta^{-1})\varepsilon_n = -(\theta^{-1}+\theta^{-2}E+\ldots)\zeta_n,$$

so that

$$\limsup|(E-\theta)(E-\theta^{-1})\varepsilon_n| \leqslant (\theta-1)^{-1}\limsup|\zeta_n| \leqslant (\theta-1)^{-1}/2.$$

A repetition of this gives $\limsup |(E-\theta^{-1})\varepsilon_n| \leqslant 1/(2(\theta-1)^2)$. The operator $E-\theta^{-1}$ is not invertible but has a one-dimensional null space. Thus if $\varepsilon_0 = \mu$ is prescribed, we may solve $\varepsilon_{n+1} - \theta^{-1}\varepsilon_n = \eta_n$, say by successive substitution to obtain

$$\varepsilon_{n+1} = \eta_n + \theta^{-1}\eta_{n-1} + \ldots + \theta^{-n}\eta_0 + \mu\theta^{-n-1},$$

from which $\limsup|\varepsilon_n| \leqslant \theta/(\theta-1)\limsup|\eta_n|$ follows.

The proof that (26) implies (24) is straightforward.

Remarks. The inequality for $E$-sequences corresponding to (23) is

$$\limsup |(E-\theta)^2\varepsilon_n| \leqslant 1/2 \tag{27}$$

which implies Pisot's result that $\limsup|\varepsilon_n| \leqslant 1/(2(\theta-1)^2)$, as above.

**4. Linear recurrence relations satisfied by $F$-sequences.** This section contains generalizations of some of Flor's [5] results to $F$-sequences. The proof of Theorem 2 is familiar but is included for completeness. In Theorem 3, we use Dirichlet's theorem rather than Minkowski's lemma to obtain a somewhat more constructive result.

THEOREM 2. *Suppose $a_n = \lambda\theta^n + \varepsilon_n$ is a geometric $F$-sequence which satisfies a linear recurrence relation. Then $\theta$ is a Pisot or Salem number.*

*Conversely, if $\theta$ is a Pisot or Salem number then there is an $F$-sequence with $\theta = \lim(a_{n+1}/a_n)$.*

Proof. If $a_n$ has generating function $f(z) = A(z)/Q(z)$, with $Q(0) = 1$, then Fatou's lemma implies that $Q$ has integer coefficients. Thus if $P(z) = z^kQ(z^{-1})$, $k = \deg Q$, then the roots of $P$ are algebraic integers. Clearly $f$ has a simple pole at $z = \theta^{-1}$, since $a_n = \lambda\theta^n + \varepsilon_n$. Since $\varepsilon_n$ is bounded the remaining poles are outside or on the unit circle with those on the circle being simple. Thus $P$ has exactly one root $\theta$ outside the unit circle which thus must be in $S$ or $T$. The roots inside the circle must be conjugates of $\theta$ hence are also simple roots. The roots on the circle are either roots of unity or conjugates of $\theta$, if $\theta$ is in $T$.

Conversely if $\theta$ is in $S$ or $T$, then one can find $\lambda$ in $Q(\theta)$ so that $\|\lambda\theta^n\| < \theta(\theta+1)^{-3}/2$ for all $n \geqslant 0$, and hence by Theorem 1, $a_n = \mathrm{Tr}(\lambda\theta^n)$ eventually satisfies (2).

THEOREM 3. *Let $\theta$ be a Salem number or a reciprocal quadratic Pisot number. Then there is a special $F$-sequence $F(0, a_1, a_2)$ with $\lim(a_{n+1}/a_n) = \theta$.*

Proof. Let $T(z)$, of degree $2m$, be the minimal polynomial of $\theta$. Then $T(z)$ is reciprocal, and we may write $T(z) = z^mR(\zeta)$, with $\zeta = z+z^{-1}$. We will construct a polynomial $B(\zeta)$ of degree $m-1$ so that if $A(z) = z^{m-1}B(\zeta)$, then $zA(z)/T(z)$ will be the generating function of a special $F$-sequence, i.e.

$$zA(z)/T(z) = a_1z + a_2z^2 + \ldots \tag{28}$$



Let $\theta, \theta^{-1}, \alpha_2, \bar{\alpha}_2, \ldots, \bar{\alpha}_m$ be the zeros of $T(z)$ so that the real numbers $\varrho = \theta+\theta^{-1}$, $\varrho_2 = \alpha_2+\bar{\alpha}_2, \ldots, \varrho_m = \alpha_m+\bar{\alpha}_m$ are the zeros of $R(\zeta)$. The relation (28) implies

$$a_n = \lambda\theta^n + \mu\theta^{-n} + \sum_{k=2}^{m}(\beta_k\alpha_k^n + \bar{\beta}_k\bar{\alpha}_k^n), \tag{29}$$

where

$$\lambda = -\theta A(\theta^{-1})/T'(\theta^{-1}) = (\theta-\theta^{-1})^{-1}B(\varrho)/R'(\varrho) = -\mu, \tag{30}$$

while

$$\beta_k = (\alpha_k-\bar{\alpha}_k)^{-1}B(\varrho_k)/R'(\varrho_k) = -\bar{\beta}_k, \tag{31}$$

so $\beta_k$ is pure imaginary.

Writing (29) as $a_n = \lambda\theta^n - \lambda\theta^{-n} + \delta_n$, we see from (2) that the condition to be satisfied is

$$\begin{gathered}|(E-\theta)^2(E-\theta^{-1})\delta_{n-1} - \theta^{-2n}(E-\theta)(E-\theta^{-1})^2\delta_{n-1} + D_n(\delta)\lambda^{-1}\theta^{-n}| \\ < 2^{-1}(1-\theta^{-2n}+\lambda^{-1}\theta^{-n}\delta_n), \quad \text{for} \quad n \geqslant 1,\end{gathered} \tag{32}$$

using the notation of Theorem 1. This can clearly be satisfied by making $\beta_2, \ldots, \beta_m$ sufficiently small (thus forcing $\lambda$ to be large). To be more explicit, let

$$R_\varrho(\zeta) = R(\zeta)/(\zeta-\varrho) = \zeta^{m-1} + c_1\zeta^{m-2} + \ldots + c_{m-1},$$

where $c_1, \ldots, c_{m-1}$ are certain real numbers. By Dirichlet's theorem, there are arbitrarily large integers $a_1$ and $n_1, \ldots, n_{m-1}$ so that

$$|a_1c_k - n_k| < a_1^{-1/(m-1)} \quad \text{for} \quad k = 1, 2, \ldots, m-1.$$

Let

$$B(\zeta) = a_1\zeta^{m-1} + n_1\zeta^{m-2} + \ldots + n_{m-1},$$

and then determine $\lambda, \ldots, \beta_m$ by (30) and (31). Then $R_\varrho(\varrho_k) = 0$ so

$$|B(\varrho_k)/R'(\varrho_k)| = |(B(\varrho_k) - a_1R_\varrho(\varrho_k))/R'(\varrho_k)| < a_1^{-1/(m-1)}2^m/R'(\varrho_k). \tag{33}$$

By taking $a_1$ sufficiently large, $\beta_2, \ldots, \beta_m$ may be made as small as desired. Note that $\lambda \sim a_1/(\theta-\theta^{-1})$.

EXAMPLE. The $F$-sequence $F(0, 5, 11)$ can be shown to be of the type discussed in Theorem 3, with $A(z) = 5+6z+8z^2+6z^3+5z^4$, and $T(z) = 1-z-z^3-z^5+z^6$, so $a_{n+1}/a_n \to \theta = 1.5061356795$, a Salem number. In this case $B(\zeta) = 5\zeta^2+6\zeta-2$, while $5R_\varrho(\zeta) \simeq 5\zeta^2+5.85\,\zeta-2.30$. Using (31), one verifies that $|\beta_2| = .053260$ and $|\beta_3| = .009382$. To check (32), one needs

$$(E-\theta)^2(E-\theta^{-1})\delta_n = 2\,\mathrm{Re}\Big\{\sum_{k=2}^{m}\beta_k(\alpha_k-\theta)(\varrho_k-\varrho)\alpha_k^{n-1}\Big\},$$

so that

$$\gamma = \sup_n |(E-\theta)^2(E-\theta^{-1})\,\delta_n| = 2\sum_{k=2}^{m}|\beta_k|\,|\alpha_k-\theta|(\varrho-\varrho_k).$$

We can estimate

$$\sup_n |(E-\theta)(E-\theta^{-1})^2\,\delta_n| \leqslant \theta(\theta+1)(\theta-1)^{-1}\gamma.$$

In this case, $\gamma = .492015$ so that (32) is readily established for $n \geqslant 9$, and (2) can be verified directly for $n < 9$.

Remark. We have not attempted a complete investigation of non-geometric $F$-sequences. For symmetry, if $a_n$ is allowed to be negative, the rule (2) should be modified to "round down" to the nearest integer if $a_{n+2} < 0$. The proof of Lemma 2 shows that if $a_n$ exhibits a rate of growth greater than $n^2$, then $a_n$ is geometric. This is sharp since $a_n = \binom{n+k}{2}$ is an $F$-sequence for $k \geqslant 5$, since $a_{n+1}-2a_n+a_{n-1} = 1$.

For special $F$-sequences, it may be true that a superlinear rate of growth implies $a_n$ is geometric. By Lemma 2, $F(0, a, 2a+b)$ is geometric if $b > 0$, and clearly $F(0, a, 2a)$ is linear. But, for example, $F(0, 20, 39)$ is periodic (with period 26), while $F(0, 21, 41)$ tends to $-\infty$ linearly, after a single positive excursion.

**5. A criterion for pure $T$-recurrence.** The following result is proved using the techniques of [1]. The only basic change is that we now have (23) rather than (27), so we avoid the estimate

$$\limsup |(E-\theta)^2(E-\theta^{-1})\,\varepsilon_n| \leqslant (1+\theta^{-1})\limsup |(E-\theta)^2\varepsilon_n|.$$

As in [1], we say that $a_n$ satisfies a *pure recurrence* if its generating function $A/Q$ has $\deg A < \deg Q$. We say it is *$T$-recurrent* if $Q(z) = T(z)K(z)$, where $T$ is the minimal polynomial of a Salem number and $K$ is cyclotomic with simple roots.

Using the estimates obtainable from Lemma 1, and the techniques of Theorem 1 of [1], it is possible to effectively deal with $T$-recurrences which are not pure, and thus show that there are $F$-sequences satisfying no linear recurrence whatsoever. However, because of Theorem 3, the following result is sufficient for the application suggested in the introduction. In fact, the convergence results of Section 2 are not really needed for this purpose, except to suggest that the method will be fairly efficient.

As in [1], we note that if $F(a_0, a_1, a_2)$ satisfies a pure $T$-recurrence then

$$a_n = \lambda\theta^n + \mu\theta^{-n} + \delta_n, \tag{34}$$



where $\delta_n$ is a linear combination of powers of numbers of modulus 1, and hence is almost periodic in $n$ ([1], p. 92). The equation (34) may be used to define $a_n$ for $n < 0$, and since the leading coefficient of $Q$ is $\pm 1$, it follows that $a_n$ is an integer for all $n$.

THEOREM 4. *Suppose that $F(a_0, a_1, a_2)$ satisfies a pure $T$-recurrence corresponding to the Salem number $\theta$. Then $\theta$ satisfies*

$$|\theta+\theta^{-1}-(a_{n+1}+a_{n-1})/a_n| \leqslant 1/(2|a_n|(\theta-1)) \quad \textit{for all } n. \tag{35}$$

*Furthermore, if $\Delta_\theta(x, y, z) = \{x\theta(2+\theta^2)-y(1+2\theta^2)+z\theta\}/\theta^2$, then for all $n$, positive or negative,*

$$|a_n - \Delta_\theta(a_{n+1}, a_{n+2}, a_{n+3})| \leqslant 1/(2\theta). \tag{36}$$

Proof. Since $\varepsilon_n = \mu\theta^{-n}+\delta_n$, we have $(E-\theta^{-1})\varepsilon_n = (E-\theta^{-1})\delta_n$, so (23) implies that $\limsup |(E-\theta)^2(E-\theta^{-1})\,\delta_n| \leqslant 1/2$, and since $\delta_n$ is almost periodic we have

$$|(E-\theta)^2(E-\theta^{-1})\,\delta_n| \leqslant 1/2 \quad \text{for all } n. \tag{37}$$

This implies, as in Theorem 1, that

$$|(E-\theta)(E-\theta^{-1})\,\delta_n| \leqslant 1/(2(\theta-1)) \quad \text{for all } n. \tag{38}$$

But $(E-\theta)(E-\theta^{-1})\,\delta_n = (E-\theta)(E-\theta^{-1})\,a_n$, so (38) implies (35). Similarly, (37) implies (36) since $(E-\theta)^2(E-\theta^{-1})\,a_n = \theta(\Delta_\theta(a_{n+1}, a_{n+2}, a_{n+3})-a_n)$.

APPLICATION. To test $F(a_0, a_1, a_2)$ for pure $T$-recurrence, one generates $a_1, \ldots, a_N$ for large $N$, then uses (35) to estimate $\theta$. For (35) to be effective, one needs a crude lower bound $\theta \geqslant \alpha > 1$, since $\theta = 1$ satisfies (35). This bound may be obtained from the results of Lemma 2 or Lemma 3, or from the estimate $|(E-\theta)^2(E-\theta^{-1})\,a_n| \leqslant 1/2$, regarded as a polynomial in $\theta$. Given $\theta \geqslant \alpha$, (35) allows $\theta$ to be computed with arbitrary (and known) accuracy. One uses this approximation in $\Delta_\theta$ and then uses (36) to compute $a_{-1}, a_{-2}, \ldots, a_{-M}$, which can be done accurately for an $M$ which tends to infinity with $N$, since $1/(2\theta) < 1/2$. If (36) fails (to within the known accuracy) for any $n$, then $F(a_0, a_1, a_2)$ does not satisfy a pure $T$-recurrence.

For example $F(0, 30, 61)$ has $\theta = 1.184214547464$, and is not pure $T$-recurrent since (36) fails for $n = -8$. In fact, since $\theta < \min S$, it is not hard to show that $F(0, 30, 61)$ satisfies no recurrence whatsoever.

We used a combination of Theorem 4 and the method of Cantor [3] to test a large number of special $F$-sequences for pure $T$-recurrence. If a sequence passes the test (36) for a reasonably large interval of $n$, we use the algorithm of [3] to determine the possible recurrence relation. The ideas discussed for the example $F(0, 5, 11)$ can then be used to test the validity of the recurrence so determined.

For example $F(0, 185, 375)$ is recurrent with $\theta = \sigma_1$, the smallest known Salem number [2]. An interesting example is $F(0, 46, 95)$ which satisfies (36) for $-80 \leqslant n \leqslant -1$ and appears to have the generating function $zA(z)/T(z) = B(\zeta)/R(\zeta)$ where $T(z) = z^8 - z^5 - z^4 - z^3 + 1$, $B(\zeta) = 46\zeta^3 + 95\zeta^2 + 12\zeta - 22$, $R(\zeta) = \zeta^4 - 4\zeta^2 - \zeta - 1$, and $A(z) = z^3 B(z + z^{-1})$. Here $\theta = \sigma_{22}$ in the notation of [2]. However, computations show that $\sup|(E-\theta)^2(E-\theta^{-1})\delta_n| = .525458$ so that $zA(z)/T(z)$ does not generate an $F$-sequence. On the other hand, if we write $a_n = \mathrm{Tr}(\lambda\theta^n)$ then it may be shown that $\mathrm{Tr}(\lambda\varrho\theta^n)$ and $\mathrm{Tr}(\lambda\varrho^2\theta^n)$ are the $F$-sequences $F(0, 95, 196)$ and $F(0, 196, 404)$. It is perhaps worth recording that $196R_\varrho(\zeta) = 196\zeta^3 + 404.05\zeta^2 + 48.96\zeta - 95.08$.

**6. Some other sequences.** Let $N(x) = [x + 1/2]$ denote the "nearest" integer to $x$. The following are some interesting integer sequences, their "typical" behaviour, and their characteristic inequalities:

(a) $D(a_m, m)$:
$$a_{n+1} = N(a_n^{(n+1)/n}), \quad n \geqslant m,$$
$$a_n = \theta^n + \varepsilon_n,$$
$$\limsup|(E-\theta)\varepsilon_n| \leqslant 1/2.$$

(b) $E(a_0, a_1)$:
$$a_{n+1} = N(a_n^2/a_{n-1}), \quad n \geqslant 2,$$
$$a_n = \lambda\theta^n + \varepsilon_n,$$
$$\limsup|(E-\theta)^2\varepsilon_n| \leqslant 1/2.$$

(c) $D'(a_{m-1}, a_m, m)$:
$$a_{n+1} = N(a_n^{(n+1)/n} + a_n^{(n-1)/n} - a_{n-1}), \quad n \geqslant m,$$
$$a_n = \theta^n + \theta^{-n} + \varepsilon_n,$$
$$\limsup|(E-\theta)(E-\theta^{-1})\varepsilon_n| \leqslant 1/2.$$

(d) $F(a_0, a_1, a_2)$:
$$a_{n+1} = N\big(a_n(a_n + a_{n-2})/a_{n-1} - a_{n-1}\big), \quad n \geqslant 2,$$
$$a_n = \lambda\theta^n + \mu\theta^{-n} + \varepsilon_n,$$
$$\limsup|(E-\theta)^2(E-\theta^{-1})\varepsilon_n| \leqslant 1/2.$$

(e) $G(a_0, a_1, a_2, a_3)$:
$$a_{n+2} = N\Big(\big\{a_{n+1}^2 + 2(a_{n+1}a_{n-1} - a_n^2) - (a_n a_{n-2} - a_{n-1}^2)\big\}/a_n\Big), \quad n \geqslant 2,$$
$$a_n = \lambda\theta^n + \mu\theta^{-n} + \varepsilon_n,$$
$$\limsup|(E-\theta)^2(E-\theta^{-1})^2\varepsilon_n| \leqslant 1/2.$$

Of these, (a) and (b) were considered by Pisot [6]. In (a), it can be shown that $a_n$ grows geometrically if and only if $a_m^{m+1} > (a_m + 1/2)^m$. The relation between the $E$-, $F$- and $G$-sequences is that in each case a certain finite difference of $a_{n+1}a_{n-1} - a_n^2$ is made as small as possible. An analogue of Theorem 2 applies to all these sequences, but for $D$ and $D'$ sufficiently large Salem numbers are ruled out by the fact that $\theta^n$ is dense modulo 1 if $\theta$ is a Salem number. An analogue of Theorem 4 can be proved for $G$-sequences but the analogue of (36) now gives no effective test for $n < 0$, since the final coefficient of the polynomial $(E-\theta)^2(E-\theta^{-1})^2$ is 1.



A sequence of a somewhat different sort is obtained by analogy with the fact that the $F$-sequence recursion is
$$(a_{n+2} - 2a_{n+1} + a_n) = N\big((a_{n+1}/a_n)(a_{n+1} - 2a_n + a_{n-1})\big).$$
Namely

(f)
$$a_{n+2} = 2a_{n+1} + N\big((a_n/a_{n-1})(a_{n+1} - 2a_n)\big),$$
$$a_n = \lambda\theta^n + \mu(2-\theta)^n + \varepsilon_n,$$
$$\limsup|(E-\theta)^2(E+\theta-2)\varepsilon_n| \leqslant 1/2.$$

In contrast to (c)–(e), the second term $\mu(2-\theta)^n$ has a real meaning if $\theta > 2$. The set of limits $\theta$ includes, in addition to $S$ and $T$, such quadratics as $1 + 5^{1/2}$ which is obtained from the sequence 4, 13, 42, 136, ...

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*Received on 11. 11. 1977* (999)