ACTA ARITHMETICA
XXXIV (1979)

The Lagrange spectrum of a set

by

T. W. Cusick (Buffalo, N. Y.) and M. Mendès France (Talence)

à Ch. Pisot

1. The Lagrange spectrum of a set. As usual, let Q denote the rational numbers, Z the integers and N the positive integers; also let Q^* denote the nonzero rationals. Let ξ be an irrational number and let

$$\xi = [c_0, c_1, c_2, \ldots]$$

be its continued fraction expansion. The Lagrange constant $L(\xi)$ of ξ is the least upper bound of the η's for which the inequality

$$\left|\xi - \frac{a}{q}\right| < \frac{1}{\eta q^2}$$

has infinitely many rational solutions a/q. It is well known that $L(\xi) \geqslant \sqrt{5}$ and that $L(\xi)$ can also be defined by either one of the two formulas:

$$L(\xi) = \limsup_{q\to\infty} (q\|q\xi\|)^{-1} \tag{1}$$

where $\|x\| = \min_{n\in Z} |x-n|$, and

$$L(\xi) = \limsup_{j\to\infty} ([c_j, c_{j+1}, \ldots] + [0, c_{j-1}, c_{j-2}, \ldots, c_1]). \tag{2}$$

For rational ξ we define $L(\xi) = +\infty$.

Let S be a set of real numbers. The Lagrange spectrum $L(S)$ of S is the set

$$L(S) = \{L(\xi) \mid \xi \in S\}.$$

The purpose of this article is to establish a simple property of the spectrum and to apply it to the case of real quadratic number fields.

Theorem 1. *Let ξ be a real number and let $n \geqslant 1$ be an integer. Then*

$$\max_{d_1, d_2, a} L\left(\frac{d_1\xi + a}{d_2}\right) = nL(\xi)$$

where the maximum is taken over all divisors d_1 and $d_2 > 0$ of n such that $d_1 d_2 = n$, and over all integers a.

As a consequence, we have the following result.

COROLLARY. *Let S be a set of real numbers such that $Q^* \cdot S + Q \subset S$. Then*

$$L(S) = \bigcup_{x \in L(S)} xN.$$

When S is a real quadratic number field, say $Q(\sqrt{d})$ ($d \geqslant 2$, squarefree), we can specify the structure of $L(S)$ to some extent.

THEOREM 2. *Let $d \geqslant 2$ be a squarefree integer.*

(i) *There exists a set $A = A(d)$ of positive rational numbers such that*

$$L(Q(\sqrt{d})) = \sqrt{d} AN.$$

(ii) *If $I(\sqrt{d})$ is the ring of integers of the field $Q(\sqrt{d})$, then*

$$L(I(\sqrt{d})) = \begin{cases} \sqrt{d}N & \text{if} \quad d \equiv 1 \pmod 4, \\ 2\sqrt{d}N & \text{if} \quad d \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

To give a complete characterization of $L(Q(\sqrt{d}))$ seems to be rather difficult. We do not even know if the set $A(d)$ can be finite, which would imply that $L(Q(\sqrt{d}))$ is a discrete set. We conjecture that $L(Q(\sqrt{d}))$ is not discrete, so $A(d)$ is always an infinite set. [1] It is possible to determine some members of $A(d)$ for a given d by computing the chains of reduced indefinite binary quadratic forms of discriminant $k^2 d$ ($k = 1, 2, \ldots$) and then looking for those chains of forms $a_i x^2 + b_i xy + c_i y^2$ for which $\min |a_i|$ does not divide k. (See Dickson [2], Chapter VII, for the relevant theory of quadratic forms.) For example, we find that $A(5) \supset \{1, 7/3, 13/5\}$ in this way; the latter two numbers arise from $(-13+7\sqrt{5})/6 = [0, \overline{2, 3, 1, 4}]$ ($k = 21$) and $(-25+13\sqrt{5})/10 = [0, \overline{2, 2, 5}]$ ($k = 13$), respectively.

For arbitrary squarefree d, even the determination of the smallest member of $L(Q(\sqrt{d}))$ seems difficult. It would also be interesting to know whether there exists a constant B such that for every real quadratic field, $L(Q(\sqrt{d}))$ contains a number $\leqslant B$. We conjecture that such a constant does exist.

2. Proof of Theorem 1.

We first establish some lemmas.

LEMMA 1. (i) *Let a, b, c, d be four integers such that $ad - bc = \pm 1$.*

[1] $A(d)$ is infinite. See Appendix.

Then for all ξ

$$L\left(\frac{a\xi + b}{c\xi + d}\right) = L(\xi).$$

(ii) *For any integer $h \neq 0$,*

$$L(h\xi) \leqslant |h| L(\xi).$$

Proof. (i) The numbers ξ and $(a\xi + b)(c\xi + d)^{-1}$ are equivalent in the sense of continued fraction theory ([3], p. 141). Equality (2) then implies (i). (ii) Obvious from formula (1). ∎

LEMMA 2. *Let d_1 and d_2 be two positive integers. Then for all ξ and all integers a*

$$L\left(\frac{d_1 \xi + a}{d_2}\right) \leqslant d_1 d_2 L(\xi).$$

Proof. Using Lemma 1,

$$L\left(\frac{d_1 \xi + a}{d_2}\right) = L\left(\frac{d_2}{d_1 \xi + a}\right) \leqslant d_2 L\left(\frac{1}{d_1 \xi + a}\right) = d_2 L(d_1 \xi + a) \leqslant d_1 d_2 L(\xi). \quad ∎$$

In order to establish Theorem 1, it is enough to prove the following result.

LEMMA 3. *For all real ξ and for all integers $n \geqslant 1$, there exist divisors of n, d_1 and $d_2 = n/d_1$, and an integer $a \in \{0, 1, \ldots, n-1\}$ such that*

$$L\left(\frac{d_1 \xi + a}{d_2}\right) \geqslant nL(\xi).$$

Proof. We prove the lemma by induction on the number of divisors of n. We suppose ξ is irrational.

Step 1. Assume $n = p$ is prime. Using an idea of Davenport [1], let $\eta \in (0, L(\xi))$. By definition of $L(\xi)$, there exist infinitely many c_n/q_n such that

$$\left|\xi - \frac{c_n}{q_n}\right| < \frac{1}{\eta q_n^2}.$$

Suppose p divides infinitely many q_n's. For those q_n's, $q_n = p q_n'$, hence

$$\left|p\xi - \frac{c_n}{q_n'}\right| < \frac{1}{p\eta q_n'^2},$$

and

$$L(p\xi) \geqslant p\eta.$$

As η is arbitrarily close to $L(\xi)$,

$$L(p\xi) \geqslant pL(\xi) \tag{3}$$

as was to be shown.

Suppose, on the contrary that p is coprime to q_n for all sufficiently large n. For all those n's, there exist $a_n \in \{0, 1, \ldots, p-1\}$ and $r_n \in \mathbf{Z}$ such that

$$c_n = -a_n q_n + r_n p.$$

Let $a \in \{0, 1, \ldots, p-1\}$ be a cluster point of the infinite sequence (a_n). Then for infinitely many n's

$$c_n = -aq_n + r_n p$$

so that

$$\left|\frac{\xi+a}{p} - \frac{r_n}{q_n}\right| < \frac{1}{\eta p q_n^2}.$$

Therefore

$$L\left(\frac{\xi+a}{p}\right) \geqslant \eta p$$

and, as above

$$L\left(\frac{\xi+a}{p}\right) \geqslant pL(\xi). \tag{4}$$

Combining inequalities (3) and (4) we have then proved that among the numbers

$$\left\{p\xi, \frac{\xi}{p}, \frac{\xi+1}{p}, \ldots, \frac{\xi+p-1}{p}\right\}$$

one at least, say ξ', satisfies

$$L(\xi') \geqslant pL(\xi).$$

Lemma 3 is thus established when n is a prime.

Step 2. Let n be a composite number. Assume the truth of the lemma for all $m < n$. Let m and $m' = n/m$ be two divisors of n such that $1 < m < n$. By step 1, there exist δ_1, $\delta_2 = m/\delta_1$ and a such that

$$L\left(\frac{\delta_1\xi+a}{\delta_2}\right) \geqslant m\,L(\xi).$$

Again, using step 1, there exist δ_1', $\delta_2' = m'/\delta_1'$ and a' such that

$$L\left(\frac{\delta_1'\left(\frac{\delta_1\xi+a}{\delta_2}\right)+a'}{\delta_2'}\right) \geqslant m' L\left(\frac{\delta_1\xi+a}{\delta_2}\right).$$

Define $d_1 = \delta_1\delta_1'$, $d_2 = \delta_2\delta_2'$, $b = a'\delta_2 + a\delta_1'$. Then

$$L\left(\frac{d_1\xi+b}{d_2}\right) \geqslant m' L\left(\frac{\delta_1\xi+a}{\delta_2}\right) \geqslant nL(\xi). \ \blacksquare$$

3. Proof of Theorem 2. To prove Theorem 2 (i), let $\xi \in \mathbf{Q}(\sqrt{d})$. The continued fraction expansion of ξ is periodic from some point on, say

$$\xi = [c_0, c_1, \ldots, c_k, \overline{a_1, a_2, \ldots, a_s}].$$

By formula (2),

$$L(\xi) = \max_{1\leqslant j\leqslant s}([\overline{a_j, a_{j+1}, \ldots, a_{j+s-1}}] + [0, \overline{a_{j+s-1}, a_{j+s-2}, \ldots, a_j}]).$$

We may assume without loss of generality that

$$L(\xi) = [\overline{a_1, a_2, \ldots, a_s}] + [0, \overline{a_s, a_{s-1}, \ldots, a_1}].$$

Now $\alpha = [\overline{a_1, \ldots, a_s}] \in \mathbf{Q}(\sqrt{d})$, say $\alpha = (a+b\sqrt{d})/c$ for $a, b, c \in \mathbf{Z}$. By the theorem of Galois about continued fractions with reversed period, we have for the algebraic conjugate α^* of α:

$$\alpha^* = -[0, \overline{a_s, a_{s-1}, \ldots, a_1}] = (a-b\sqrt{d})/c.$$

Hence $L(\xi) = \alpha - \alpha^* = 2b\sqrt{d}/c \in \sqrt{d}\mathbf{Q}$. Combining this result with the Corollary of Theorem 1, we obtain Theorem 2(i).

To prove the second part of Theorem 2, we need the following lemma.

LEMMA 4. *Let $d \geqslant 2$ be a squarefree number.*

(i) *Let a and b be integers. Any number equivalent to $a+b\sqrt{d}$ can be represented as*

$$\frac{a'+b\sqrt{d}}{c'}, \quad a', c' \in \mathbf{Z}.$$

(ii) *If $d \equiv 1 \pmod 4$, any number equivalent to*

$$\frac{a+b\sqrt{d}}{2}, \quad a \equiv 1 \pmod 2,\ b \equiv 1 \pmod 2$$

can be represented as

$$\frac{a'+b\sqrt{d}}{c'}, \quad a', c' \in \mathbf{Z},$$

where c' is even.

Proof. We consider (ii) first. Suppose $(a''+b''\sqrt{d})/c''$ is equivalent to $(a+b\sqrt{d})/2$, where a and b are odd and $d \equiv 1 \pmod 4$. Then there exist integers A, B, C, D such that $AD-BC = \pm 1$ and

$$\frac{a''+b''\sqrt{d}}{c''} = \frac{A(a+b\sqrt{d})+2B}{C(a+b\sqrt{d})+2D} = \frac{a_1 \pm 2b\sqrt{d}}{c_1}$$

where

$$a_1 = (Aa+2B)(Ca+2D)-ACb^2d, \quad c_1 = (Ca+2D)^2-C^2b^2d.$$

We have $a_1 \equiv 0 \pmod 2$ and $c_1 \equiv 0 \pmod 4$, so if we define $a' = \pm a_1/2$ and $c' = \pm c_1/2$, the conditions of (ii) are met. A similar but simpler calculation proves part (i) of the lemma. ∎

We now proceed to establish Theorem 2(ii).

First case: $d \equiv 2$ or $3 \pmod 4$. Let $a \in I(\sqrt{d})$. Then $a = a+b\sqrt{d}$ where a and b are rational integers. We shall assume $b > 0$ disregarding the trivial case $b = 0$. The continued fraction of a is periodic from the beginning provided the integer a is suitably chosen, say $a = [\overline{a_1, a_2, \ldots, a_s}]$.

Define

$$a_i = [\overline{a_i, a_{i+1}, \ldots, a_{i+s-1}}] \quad (1 \leqslant i \leqslant s)$$

and let a_i^* $(1 \leqslant i \leqslant s)$ denote the algebraic conjugate of a_i. By formula (2),

$$L(a) = \max_{1\leqslant i\leqslant s} (a_i - a_i^*).$$

Since the a_i are equivalent, Lemma 4 implies $a_i = (a_i + b\sqrt{d})/c_i$ for some integers a_i, c_i. Hence

(5) $$L(a) = 2b\sqrt{d}/\min_{1\leqslant i\leqslant s} |c_i|.$$

Obviously $c_1 = 1$ is the minimum of the $|c_i|$, so $L(a) = 2b\sqrt{d}$. Hence $L(I(\sqrt{d})) = 2\sqrt{d}\,N$.

Second case: $d \equiv 1 \bmod 4$. Let $a \in I(\sqrt{d})$. Then either $a = a+b\sqrt{d}$ or $a = \frac{1}{2}(a+b\sqrt{d})$ with a and b odd. The former situation can be treated as above, so we only consider the latter one. As above, we find that (5) holds. We have $c_1 = 2$ and by Lemma 4, c_i is always even. Thus $L(a) = b\sqrt{d}$, so we have $L(I(\sqrt{d})) = \sqrt{d}\,N$. ∎

Appendix

H. Cohen and S. M. J. Wilson have independently answered the question whether $A(d)$ is infinite: $A(d)$ *is* indeed *necessarily infinite*. With his kind permission we reproduce Wilson's proof.

Let d be a squarefree integer. Let θ be the fundamental unit in $\boldsymbol{Q}(\sqrt{d})$, let $r_n = (\theta^n - \theta^{-n})^2$ and

$$a = \frac{r_n + \sqrt{r_n(r_n+4)}}{2b}$$

where b divides r_n and where $b^2 < r_n$ to ensure (iii) below. One easily checks that

(i) $a = (\theta^{2n}-1)/b \in \boldsymbol{Q}(\sqrt{d})$,

(ii) $a = [\overline{r_n/b, b}]$ and

(iii) $L(a) = \dfrac{\sqrt{r_n(r_n+4)}}{b}$.

Now let p and q be two primes such that $p < q$ and $(p, 4d) = (q, 4d) = 1$. Choose $n = (p^2-1)(q^2-1)$ and now p and q divide r_n. Define k by $p^k \| (\theta^n - \theta^{-n})/\sqrt{d}$ and take $b = p^{k+1}$ (a possible choice). Then

$$L(a) = \frac{c}{p}\sqrt{d},$$

where c is some integer not divisible by p. Hence $A(d)$ is infinite.

Added in proof (March 1979): In a recent paper of A. C. Woods (*The Markoff spectrum of an algebraic number field*, J. Austral. Math. Soc. (A) 25 (1978), pp. 486–488), it was proved (by looking at the corresponding problem for the Markoff spectrum) that $L(\boldsymbol{Q}(\sqrt{5}))$ is not a discrete set. After seeing this paper S.M.J. Wilson was able to generalize the work and prove that $L(\boldsymbol{Q}(\sqrt{d}))$ is not a discrete set. This settles the conjecture stated after Theorem 2 above.

References

[1] H. Davenport, *A remark on continued fractions*, Michigan Math. J. 11 (1964), pp. 343–344.

[2] L. E. Dickson, *Introduction to the theory of numbers*, Dover Publications, New York 1957.

[3] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, Oxford 1960.

DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK AT BUFFALO
Buffalo, New York 14214, U.S.A.
U.E.R. DE MATHÉMATIQUES ET D'INFORMATIQUE
UNIVERSITÉ DE BORDEAUX I
33405 Talence, France

Received on 8. 11. 1977 (996)