4. Proof of Lemmas A and B. We shall use the following formulae:

$$\sum_{m\leqslant x} \frac{G(m)}{\log m} = \int_{2}^{x} \frac{\Psi(t, K)}{t\log^2 t}\,dt + \frac{\Psi(x, K)}{\log x}, \tag{4.1}$$

$$\sum_{m\leqslant x} \frac{G(m)}{\log x} = \Pi(x, K) + \tfrac{1}{2}\Pi(x^{1/2}, K) + \tfrac{1}{3}\Pi(x^{1/3}, K) + \ldots \tag{4.2}$$

$$= \Pi(x, K) + O\left(\frac{n}{\log x}\, x^{1/2}\right).$$

From Lemma 9 and Lemma 8 for $\varepsilon = 1/4C_4$ it follows that

$$\Psi(x, K) = x + O\left(x\exp\left(-C_{25}\frac{\log^{1/2}x}{n^{1/2}}\right)\right) \tag{4.3}$$

for $1 \leqslant |\Delta| \leqslant \log^{C_4}x$, $1 \leqslant n \leqslant C_5\dfrac{\log_2 x}{\log_3 x}$.

Hence owing to (4.1)

$$\sum_{m\leqslant x} \frac{G(m)}{\log m} = \mathrm{li}\,x + O\left(x\exp\left(-C_{25}\frac{\log^{1/2}x}{n^{1/2}}\right)\right). \tag{4.4}$$

Combining (4.2) and (4.4), we get Lemma A. Similarly, the proof of Lemma B follows.

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On the zeros of the Riemann zeta-function and L-series

by

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1. Introduction. The object of this paper is to prove some theorems of which two typical ones are as follows.

THEOREM 1. *Let $\beta_0+i\gamma_0$ be any zero of $\zeta(s)$ with $\beta_0 \geqslant \frac{1}{2}$ and $\gamma_0 \geqslant 100$. With every positive constant λ not exceeding $\frac{1}{2}$ and every complex number $1+i\mu$ (μ real) let $D_\lambda(1+i\mu)$ denote the disc $|1+i\mu-s| \leqslant \lambda$. Then there exist effective positive absolute constants C_1, C_2, C_3, C_4 (depending only on λ) such that for all Y satisfying $C_1\log\log\gamma_0 \leqslant Y \leqslant C_2(1-\beta_0)^{-1}$ there holds*

$$\sum_{\varrho} e^{-Y(1-\beta)} > C_3\big(Y(1-\beta_0)\big)^{-1} - C_4,$$

where $\varrho = \beta+i\gamma$ runs over all the zeros of $\zeta(s)$ which lie in $D_\lambda(1+i\gamma_0)$ and $D_\lambda(1+2i\gamma_0)$.

Remark. Since we let $C_1, C_2, \ldots, C_9$ depend on λ, in Theorems 1, 2 the upper bounds like $Y \leqslant C_2(1-\beta_0)^{-1}$ are plainly unnecessary. But we have retained them only for trivial reasons. However, we have good reasons to retain it in Theorem 3 which we state at the end of the paper.

THEOREM 2. *Let $\beta_1+i\gamma_1$ and $\beta_2+i\gamma_2$ be two zeros of $\zeta(s)$ with $\beta_1 \geqslant \frac{1}{2}$, $\beta_2 \geqslant \frac{1}{2}$, $100 \leqslant \gamma_1 < \gamma_2 \leqslant 2\gamma_1$, $10^{-8} \leqslant \gamma_2-\gamma_1$ (10^{-8} is unimportant and can be replaced by smaller positive constant as well). Then there exist effective positive absolute constants C_5, C_6, C_7, C_8 and C_9 (depending only on λ) such that for all Y satisfying*

$$C_5\log\log\gamma_1 \leqslant Y \leqslant C_6\min\big((1-\beta_1)^{-1}, (1-\beta_2)^{-1}\big)$$

and all γ_1, γ_2 satisfying

$$\gamma_2-\gamma_1 < \exp\big(C_7(\log\log\gamma_1)^{3/2}(\log\log\log\gamma_1)^{-2}\big)$$

there holds

$$\sum_{\varrho} e^{-Y(1-\beta)} > C_8Y^{-1}\min\big((1-\beta_1)^{-1}, (1-\beta_2)^{-1}\big) - C_9,$$

where $\varrho = \beta+i\gamma$ runs over all the zeros of $\zeta(s)$ which lie in $D_\lambda(1+i\gamma_1)$ and $D_\lambda(1+i\gamma_2)$.

The earlier works in this direction is due to N. Levinson and later to H. L. Montgomery and for the history of these works we refer the reader to the thesis [1] (the chapter relevant to our paper is 11) of Montgomery, which is perhaps the most valuable work on these and related topics. We make some brief comments. The methods of Levinson and Montgomery were completely different and their results also appeared to be slightly different from each other. While Levinson roughly proved that $D_\lambda(1+i\gamma_0)$ and $D_\lambda(1+2i\gamma_0)$ together contain at least two zeros, Montgomery proved that 'in an average sense' the same region contains 'many more zeros'. This work of Montgomery provided a novel alternative approach to establishing the zeros free regions of I. M. Vinogradov still depending (like other approaches) on the deep estimates of Vinogradov for $|\zeta(s)|$. In his thesis Montgomery proposes the problem of proving 'a good lower bound for the zeros in the union of $D_\lambda(1+i\gamma_0)$ and $D_\lambda(1+2i\gamma_0)$' instead of his lower bound 'in the average sense'. This problem of Montgomery is solved in the present work (Theorem 1 solves the problem and like Montgomery's result gives the zero free region [1] of Vinogradov) which may be looked upon as a continuation of the ideas of Montgomery. However, in this paper there are some other important ideas like

$$-2\frac{\zeta'(s)}{\zeta(s)}-\frac{\zeta'(s+ia)}{\zeta(s+ia)}-\frac{\zeta'(s-ia)}{\zeta(s-ia)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}|1+n^{ia}|^2$$

which coupled with inequalities like

$$10^8|1+n^{ia}|^2+|1+n^{2ia}|^2\geqslant 3$$

(in fact 10^8 and 3 can both be replaced by 4) and

$$10^8(|1+n^{ia_1}|^2+|1+n^{ia_2}|^2)+|1+n^{i(a_1-a_2)}|^2\geqslant 3$$

are responsible for proving Theorems 1 and 2 respectively. These last inequalities may be generalized as follows:

$$C_{10}\sum_{k=1}^{R}|1+n^{ia_k}|^2+|1+n^{iA}|^2\geqslant 3$$

where A is a linear combination $\sum_{k=1}^{R}a_k\alpha_k$ with integer coefficients a_k such that $\sum|a_k|$ is a bounded even integer and C_{10} depends only on this bound. (This inequality can also be generalized in a useful way when a_j are rational and satisfy some conditions.) It is possible that these ideas may play some unexpected role in the future development of this subject. (In all these statements $a, a_1, a_2, \ldots, a_R$ are real.) It must be mentioned that

(1) See the Appendix at the end.

Theorem 2 depends on the zero free region of Vinogradov. We end this paragraph with the remarks that these ideas lead to a quick and easy proof that $L(1+it,\chi)$ ($t\neq 0$ if χ is real) is different from zero and that the investigations of this paper go through for L-series considerably. We have only to note that

$$-2\frac{\zeta'(s)}{\zeta(s)}-\frac{L'(s+ia,\chi)}{L(s+ia,\chi)}-\frac{L'(s-ia,\bar\chi)}{L(s-ia,\bar\chi)}$$
$$=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}|1+\bar\chi(n)n^{ia}|^2+\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}(1-|\chi(n)|).$$

It is a pleasure to record here my warmest thanks to my colleague Shri R. Balasubramanian who shared much of my thoughts on the subject and checked this manuscript carefully.

2. Proof of Theorem 1. We give a sketch of the proof of Theorem 1 and leave the proof of Theorem 2, which is analogous, to the reader. Also in this section s will denote a real number exceeding 1 and a a real number exceeding 100 and W will denote a complex variable. The rest of the notation is the same as already explained with these differences. We will denote by β_0+ia a fixed zero of $\zeta(W)$ with $\beta_0\geqslant 1/2$, $a>100$ and $C_1, C_2, C_3, \ldots$ and the O-constants which are effective positive constants (not to be confused with the constants of the introduction). These constants are absolute numerical constants with the exception of C_8 and C_{10} which depend on λ and only on λ.

The following lemmas lead to the proof.

LEMMA 1. *Let*

$$\varphi(s)=\left(\zeta^2(s)\zeta(s-ia)\zeta(s+ia)\right)^{10^8}\left(\zeta^2(s)\zeta(s-2ia)\zeta(s+2ia)\right)\left(\zeta(s)\right)^{-3}.$$

Then

$$-\frac{\varphi'(s)}{\varphi(s)}=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$$

where b_n are non-negative real numbers.

LEMMA 2. *For $X\geqslant 1$, we have, by writing $F(s)=\varphi'(s)/\varphi(s)$,*

$$\frac{1}{2\pi i}\int F(s+W)X^W\Gamma\left(\frac{W}{2}\right)dW=-2\sum_{n=1}^{\infty}\frac{b_n}{n^s}\exp\left(-\left(\frac{n}{X}\right)^2\right),$$

where the line of integration is $\operatorname{Re}W=2$.

Moving the line of integration to $\operatorname{Re}W=-2+1/100$ and assuming that $1<s\leqslant 1+1/100$ we have

LEMMA 3. *If $1 < s \leqslant 1+1/100$, then*

$$2\sum_{n=1}^{\infty}\frac{b_n}{n^s}\left(1-\exp\left(-\left(\frac{n}{X}\right)^2\right)\right) = 2\cdot 10^8\Sigma_1+2\Sigma_2-3\Sigma_3+O\left(X^{1-s}+\frac{\log a}{X}\right)$$

where

$$\Sigma_1 = \mathrm{Re}\left(-X^{1-s}\Gamma\left(\frac{1-s}{2}\right)+\sum_{\varrho}X^{\varrho-s-ia}\Gamma\left(\frac{\varrho-s-ia}{2}\right)\right),$$

$$\Sigma_3 = -X^{1-s}\Gamma\left(\frac{1-s}{2}\right),$$

and Σ_2 is the same as Σ_1 with a replaced by $2a$. Here $\varrho = \beta+i\gamma$ runs over the zeros of $\zeta(W)$ with positive real part. (In future the same convention will be in force. Also the restriction of the sums to zeros which satisfy some conditions will be indicated below the summation symbol or explained at appropriate places.)

We now write $\varrho_0 = \beta_0+ia$ and

$$\Sigma_4 = \mathrm{Re}\sum_{\varrho\neq\varrho_0}X^{\varrho-s-ia}\Gamma\left(\frac{\varrho-s-ia}{2}\right),\qquad \Sigma_5 = \mathrm{Re}\sum_{\varrho}X^{\varrho-s-2ia}\Gamma\left(\frac{\varrho-s-2ia}{2}\right)$$

and we see that Lemma 3 gives

LEMMA 4. *Let $1 < s \leqslant 1+1/100$, $X \geqslant (\log a)^{80}$. Then*

$$0 \leqslant 2\cdot 10^8\left(-X^{1-s}\Gamma\left(\frac{1-s}{2}\right)+X^{\beta_0-s}\Gamma\left(\frac{\beta_0-s}{2}\right)\right)+X^{1-s}\Gamma\left(\frac{1-s}{2}\right)+ +2\cdot 10^8\Sigma_4+2\Sigma_5+O(X^{1-s}).$$

We now impose some further conditions on s and X so as to satisfy

$$2\cdot 10^8\left(-X^{1-s}\Gamma\left(\frac{1-s}{2}\right)+X^{\beta_0-s}\Gamma\left(\frac{\beta_0-s}{2}\right)\right)+X^{1-s}\Gamma\left(\frac{1-s}{2}\right)\leqslant \tfrac{1}{2}X^{1-s}\Gamma\left(\frac{1-s}{2}\right).$$

This requires

$$-(4\cdot 10^8-1)\Gamma\left(\frac{1-s}{2}\right)+4\cdot 10^8X^{\beta_0-1}\Gamma\left(\frac{\beta_0-s}{2}\right)\leqslant 0,$$

i.e.

$$X^{1-\beta_0}\Gamma\left(\frac{1-s}{2}\right)\left(\Gamma\left(\frac{\beta_0-s}{2}\right)\right)^{-1}\leqslant\frac{4\cdot 10^8}{4\cdot 10^8-1}.$$

To satisfy this it suffices to impose the conditions

$$\log X\leqslant C_1(1-\beta_0)^{-1},\quad s\leqslant\beta_0+C_2\quad\text{and}\quad (s-\beta_0)(s-1)^{-1}\leqslant C_3$$

where C_1 and C_2 are small positive constants and C_3 is a constant which

is greater than 1 but close enough to it. The third condition can be satisfied by requiring that $s\geqslant 1+C_4(1-\beta_0)$ where C_4 is a sufficiently large constant. To satisfy conditions already imposed on s we have to require that $1-\beta_0$ shall be small. Collecting these observations we state

LEMMA 5. *Let $1+C_4(1-\beta_0)\leqslant s\leqslant\beta_0+C_2$ where C_4 is a certain large constant and C_2 is a certain small positive constant. Let $80\log\log a\leqslant\log X \leqslant C_1(1-\beta_0)^{-1}$ where C_1 is a certain small constant. Then*

$$0\leqslant -\frac{X^{1-s}}{100(s-1)}+2\cdot 10^8\Sigma_4+2\Sigma_5+O(X^{1-s}).$$

Our next lemma is essentially due to Montgomery.

LEMMA 6. *We have*

$$\mathrm{Re}\,X^{\varrho-s-ia}\Gamma\left(\frac{\varrho-s-ia}{2}\right)\leqslant C_5X^{\beta-s}\log X$$

where C_5 is a certain positive constant.

Proof. Let a denote the complex number $\varrho-s-ia$. Observe that the real part of a lies between $-3/2$ and zero and write

$$X^a\Gamma\left(\frac{a}{2}\right) = X^a\left(\Gamma\left(\frac{a}{2}\right)-\frac{2}{a}\right)+\frac{2}{a}(X^a-X^{\beta-s})+\frac{2}{a}X^{\beta-s}.$$

The first two terms on the right are $O(X^{\beta-s}\log X)$ and the real part of the last term is negative. These remarks complete the proof of Lemma 6.

We next write

$$\Sigma_4 = \mathrm{Re}\sum_{\varrho\neq\varrho_0}X^{\varrho-s-ia}\Gamma\left(\frac{\varrho-s-ia}{2}\right) = \Sigma_6+\Sigma_7$$

where Σ_6 is the same as Σ_4 with the sum restricted to zeros in $D_\lambda(1+ia)$ and Σ_7 is the remaining portion of the sum. We also split up

$$\Sigma_5 = \mathrm{Re}\sum_{\varrho}X^{\varrho-s-2ia}\Gamma\left(\frac{\varrho-s-2ia}{2}\right) = \Sigma_8+\Sigma_9$$

where Σ_8 is the same as Σ_5 with the sum restricted to zeros in $D_\lambda(1+2ia)$ and Σ_9 is the remaining portion of the sum. By Lemmas 5 and 6 we have

LEMMA 7. *Let D denote the union of $D_\lambda(1+ia)$ and $D_\lambda(1+2ia)$. Write*

$$\Sigma_{10} = \Sigma_{10}(X) = \sum X^{\beta-1}$$

where the sum is over all the zeros in D. Then for some large constants C_6 and C_7, we have,

$$0\leqslant X^{1-s}\left(-\frac{1}{100(s-1)}+C_6\log X\,\Sigma_{10}(X)+C_7\right)+2\cdot 10^8\Sigma_7+2\Sigma_9$$

provided $1+C_4(1-\beta_0)\leqslant s\leqslant \beta_0+C_2$ *and*

$$80\log\log a\leqslant \log X\leqslant C_1(1-\beta_0)^{-1}.$$

We next write

$$X=e^{Y+u_1+u_2+\ldots+u_m}$$

where Y is fixed to be large, m is fixed to be the integral part of Y and finally $u_1,\ldots,u_m$ are variables subject to $0\leqslant u_k\leqslant C_8$ $(k=1,2,\ldots,m)$ where C_8 is a constant to be chosen later. Writing $X_0=e^Y$ and $X_1=e^{(C_8+1)Y}$, we have to see that all the numbers X in between these satisfy our requirements, i.e.

$$80\log\log a\leqslant Y\leqslant C_1(C_8+1)^{-1}(1-\beta_0)^{-1}.$$

We note that $(\Sigma_{10}(X))\log X$ does not exceed $(\Sigma_{10}(X_0))(C_8+1)Y$. Trivially

$$\int\int\ldots\int(2\cdot 10^8\Sigma_7+2\Sigma_9)\,du_1\ldots du_m$$
$$=O\Big(\sum_{|\varrho-1-ia|\geqslant\lambda}\frac{2^m X_0^{\beta-s}}{|\varrho-s-ia|^{m+1}}+\sum_{|\varrho-1-2ia|\geqslant\lambda}\frac{2^m X_0^{\beta-s}}{|\varrho-s-2ia|^{m+1}}\Big)$$
$$=O\Big(\Big(\frac{2}{\lambda}\Big)^{m+1}(\log a)\,X_0^{1-s}\Big).$$

Now we are going to show that

$$C_6(\Sigma_{10}(X_0))\,(C_8+1)\,Y+C_7\geqslant\frac{1}{200(s-1)}.\tag{1}$$

If this is false then we have

$$0\leqslant X_1^{1-s}\Big(-\frac{1}{200(s-1)}\Big)+2\cdot 10^8\Sigma_7+2\Sigma_9.$$

Integrating this with respect to $u_1,\ldots,u_m$ we have

$$0\leqslant-\frac{X_1^{1-s}C_8^m}{200(s-1)}+C_9(\log a)\Big(\frac{2}{\lambda}\Big)^{m+1}X_0^{1-s},\tag{2}$$

i.e.

$$\Big(\frac{X_0}{X_1}\Big)^{s-1}\big(200(s-1)\big)^{-1}\leqslant C_9(\log a)\Big(\tfrac{1}{2}\lambda C_8^{\frac{m}{m+1}}\Big)^{-(m+1)}.$$

It is plain that the right-hand side is less than $C_9(\log a)^{-1}$ if we set $C_8=400\lambda^{-2}$. Now

$$\Big(\frac{X_0}{X_1}\Big)^{s-1}=e^{-C_8Y(s-1)}\geqslant e^{-C_8C_{10}}$$

provided the maximum value of Y does not exceed $C_{10}(s-1)^{-1}$. We have not yet fixed s. So far we had to satisfy only

$$1+C_4(1-\beta_0)\leqslant s\leqslant\beta_0+C_2,$$

but we now fix $s=1+C_4(1-\beta_0)$ and insist that $Y\leqslant C_{10}C_4^{-1}(1-\beta_0)^{-1}$. Putting $C_{10}=C_1C_4(C_8+1)^{-1}$ we see that this condition on Y is automatically satisfied if our earlier inequality for Y is satisfied.

These remarks prove that for all zeros $\varrho_0=\beta_0+ia$ with

$$C_9(\log a)^{-1}\leqslant\frac{1}{400}e^{-C_8C_{10}},\quad\text{ i.e. }\quad \log a\geqslant 400\,C_9^{-1}\exp(C_1C_4),$$

(note that the last inequality implies the last but one), the inequality (2) can not hold. Hence under this condition on a we must have (1) and so

$$\sum_{\varrho\text{ in }D}e^{-Y(1-\beta)}\geqslant\Big(\frac{C_{11}}{Y(1-\beta_0)}-C_{12}\Big)(C_8+1)^{-1}$$

provided $80\log\log a\leqslant Y\leqslant(C_8+1)^{-1}C_1(1-\beta_0)^{-1}$ with $C_8=400\lambda^{-2}$. Here C_{11} and C_{12} are positive numerical constants. We end with the remark that it is possible to prove slightly more viz.

THEOREM 3. *Let β_0+ia be a zero of $\zeta(W)$ with $\beta_0\geqslant\frac{1}{2}$ and $a>100$. Then there exist effective positive numerical constants C_{13}, C_{14}, C_{15} and C_{16} such that if λ is a variable satisfying*

$$C_{13}(\log\log a)\,(1-\beta_0)\leqslant\lambda\leqslant\tfrac{1}{2}$$

then with the condition $80\log\log a\leqslant Y\leqslant C_{14}\lambda(1-\beta_0)^{-1}$ on Y, there holds

$$\sum e^{-Y(1-\beta)}\geqslant\Big(\frac{C_{15}}{Y(1-\beta_0)}-C_{16}\Big)\lambda,$$

where the sum on the left extends over the zeros $\varrho=\beta+i\gamma$ in the union of $D_\lambda(1+ia)$ and $D_\lambda(1+2ia)$.

Appendix (added 15. 10. 1976)

Instead of altering the main contents of this paper we now amplify a certain remark (concerning the deduction of zero free regions) mentioned in the introduction. This is desirable because Theorem 1 assumes a slightly improved nice form. With each $D_\lambda(1+i\mu)$ associate the sum

$$S_{\lambda,\mu}=\sum_{\varrho}\delta(\gamma,\mu,\lambda)e^{-Y(1-\beta)}$$

where $\varrho=\beta+i\gamma$ runs over all the zeros of $\zeta(s)$ in $D_\lambda(1+i\mu)$ and $\delta(\gamma,\mu,\lambda)$ is equal to 1 if $|\gamma-\mu|\leqslant 1/Y$ and $Y^{-2}|\gamma-\mu|^{-2}$ otherwise. Then in the deduction of Theorem 1 from Lemma 7 we write

$$X=e^{Y+u_1+\ldots+u_m+u_{m+1}}$$

where $u_1, \ldots, u_m$ are as before and $0 \leqslant u_{m+1} \leqslant Y$. Arguing as in the deduction of Theorem 1 from Lemma 7 we can prove

THEOREM 1-A. *Let $\beta_0+i\gamma_0$ be a zero of $\zeta(s)$. Then there exist effective positive constants E_1, E_2, E_3 and E_4 depending only on λ such that if $\gamma_0 \geqslant E_1$, $Y \geqslant E_2 \log\log\gamma_0$, then*

$$S_{\lambda,\gamma_0}+S_{\lambda,2\gamma_0} > \frac{E_3}{Y(1-\beta_0)} - E_4.$$

Next we can break up the sum over ϱ in S_{λ,γ_0} into subsums consisting of zeros ϱ with $n/Y \leqslant |\gamma-\gamma_0| \leqslant (n+1)/Y$ where $n = 0, 1, 2, 3, \ldots$ and estimate the subsums for any fixed n and add these subsums together. We treat $S_{\lambda,2\gamma_0}$ similarly. Appraising these subsums a little we can restate Theorem 1-A in a nicer form as follows. Let $N_v(r)$ denote the number (counted with multiplicity) of zeros of $\zeta(s)$ in $D_r(1+iv)$. Next with $0<\lambda\leqslant\frac{1}{2}$, $Y\geqslant 2$ and $V \geqslant 10$ let us write

$$f(V) = f(\lambda, Y, V) = \max_{V-\lambda\leqslant v\leqslant V+\lambda}\left\{N_v(\lambda)e^{-Y\lambda}+\int_0^{Y\lambda} N_v\left(\frac{u}{Y}\right)e^{-u}\,du\right\}.$$

Then we have

THEOREM 1-B. *Let $\beta_0+i\gamma_0$ be a zero of $\zeta(s)$. Then there exist effective positive constants E_5, E_6, E_7 and E_8 depending only on λ such that if $\gamma_0 \geqslant E_5$, $Y \geqslant E_6 \log\log\gamma_0$, then*

$$f(\gamma_0)+f(2\gamma_0) > \frac{E_7}{Y(1-\beta_0)} - E_8.$$

This (with Jensen's theorem for example) leads at once to the zero free regions.

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Über Hasses Verallgemeinerung des Syracuse-Algorithmus (Kakutanis Problem)

von

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1. Einleitung. Ist $N_2' := \{n \in \mathbf{N};\ 2\nmid n\}$ und $S\colon N_2' \to N_2'$, $n \mapsto (3n+1)/2^a$ mit eindeutig bestimmtem $a = a(n) \in \mathbf{N}$, so heißt die Folge der iterierten Funktionen $(S^k)_{k\in \mathbf{N}_0}$ *Syracuse-Algorithmus*, weil ein Mitglied der Universität Syracuse (USA) die folgende merkwürdige Eigenschaft dieses Algorithmus (wieder-) entdeckte: Welche $n \in N_2'$ er auch probierte, die Folge $(S^k(n))_{k\in\mathbf{N}_0}$ war immer periodisch mit der Periode $\bar{1}$. Die Vermutung, daß dieses für alle $n \in N_2'$ gilt, ist bis heute unbewiesen, aber mindestens für $n < 2^{50}$ richtig (Fraenkel).

Obwohl dieses Problem, das eine Reihe von verschiedenen Namen trägt (z.B. Kakutani-Problem, Collatz-Problem), von vielen Mathematikern untersucht wurde, ist bis heute nur sehr wenig darüber bekannt (1).

Die folgende weitgehende Verallgemeinerung des Syracuse-Algorithmus stammt von H. Hasse, der auch erkannte, daß die entsprechenden Periodizitätsprobleme mit gewissen Reihenentwicklungen in den (Henselschen) d-adischen Vervollständigungen des Ringes der ganzrationalen Zahlen zusammenhängen.

Sei $(m, d) \in \mathbf{N}^2$ mit $d \geqslant 2$ und $\mathrm{ggT}(m, d) = 1$, $N_d := \{n \in \mathbf{Z};\ d\nmid n\}$ sowie R_d ein vollständiges Restsystem modulo d ohne Vielfaches von d. Dann gibt es zu jedem $x \in N_d$ genau ein Paar $(r, a) \in R_d \times \mathbf{N}$, so daß $(mx-r)/d^a \in N_d$ gilt. Damit ist eine Abbildung $H = H(m, d, R_d)\colon N_d \to N_d$, $x \mapsto (mx-r)/d^a$ mit $r \in R_d$, $a \in \mathbf{N}$ definiert. Die Folge der iterierten Funktionen $(H^k)_{k\in\mathbf{N}_0}$ mit $H^0 := \mathrm{id}$ und $H^{k+1} = H \circ H^k$ $(k \in \mathbf{N}_0)$ bezeichnen wir als *Hasse-Algorithmus*.

Wie bei dem Kakutani-Collatz-Problem stellt sich die Frage, für welche $n \in N_d$ die Folge $(H^k(n))_{k\in\mathbf{N}_0}$ periodisch ist, wenn (m, d, R_d) vor-

(1) Herrn Professor Dr. H. Hasse verdanke ich die folgenden Hinweise: J. H. Conway stellte fest, daß Probleme dieser Art unentscheidbar sein können; Riho Terras bewies die Existenz einer „Verteilungsfunktion", die im Unendlichen den Grenzwert Null besitzt.