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A note on the sum of sets of m-tuples

by

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For m a natural number, let J^m be the set of m-tuples of nonnegative integers. The element of J^m having each coordinate equal to zero is represented by 0. For $z = (z_1, z_2, \ldots, z_m)$ in J^m, define

$$I_z = \{(x_1, x_2, \ldots, x_m) \mid 0 \leqslant x_i \leqslant z_i \text{ for } i = 1, 2, \ldots, m\}.$$

Addition of elements of J^m, as well as subtraction of elements of I_z from z, is done coordinatewise. When $A_1, A_2, \ldots, A_k$ are subsets of J^m, let $\{\sum_{i=1}^{k} a_i \mid a_i \in A_i\}$ be their sum and denote it by $\sum_{i=1}^{k} A_i$. When A is a subset of J^m and $z \in J^m$, let $A(z)$ be the cardinality of the set $(A \cap I_z) \sim \{0\}$. With the Cartesian product of k copies of the power set of J^m denoted by P_m^k, for each natural number $k \geqslant 2$ define the function f_k on the nonzero elements of z of J^m as follows:

$$f_k(z) = \max\Big\{\sum_{i=1}^{k} A_i(z) \mid (A_1, A_2, \ldots, A_k) \in P_m^k,\ 0 \in \bigcap_{i=1}^{k} A_i,\ z \notin \sum_{i=1}^{k} A_i\Big\}.$$

When n is a natural number Erdös and Scherk [1] have shown that $f_k(n) = kn/2 - k/2$ if n is odd and

$$kn/2 - k + 1 \leqslant f_k(n) \leqslant kn/2 - k/2$$

if n is even. We evaluate $f_k(n)$ when n is even and extend the result to the m-dimensional space J^m.

THEOREM. *Let $z \in J^m$, $z \neq 0$, and let n be the cardinality of $I_z \sim \{0\}$. For each natural number $k \geqslant 2$,*

$$f_k(z) = \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $z = (z_1, z_2, \ldots, z_m)$, $z \neq 0$, be an element of J^m for which n is the cardinality of $I_z \sim \{0\}$. Let $k \geqslant 2$.

We first determine an upper bound for $f_k(z)$. Let $A_1, A_2, \ldots, A_k$ be subsets of J^m satisfying $0 \in \bigcap_{i=1}^{k} A_i$ and $z \notin \sum_{i=1}^{k} A_i$. If $A_i(z) < n/2$ for $i = 1, 2, \ldots, k$, then we immediately obtain

$$\sum_{i=1}^{k} A_i(z) \leqslant \begin{cases} k(n-1)/2 & \text{if } n \text{ is odd,} \\ k(n/2-1) & \text{if } n \text{ is even.} \end{cases}$$

Next consider the case when $A_i(z) \geqslant n/2$ for some i, $1 \leqslant i \leqslant k$, and let $t = \max\{A_i(z) | \ i = 1, 2, \ldots, k\}$. Let j be such that $A_j(z) = t$. If $x \in A_j \cap I_z$, $x \neq 0$, and $x \neq z$ then $z - x \notin A_i$, $z - x \in I_z \sim A_i$, $z - x \neq 0$, and $z - x \neq z$ $(i = 1, 2, \ldots, k;\ i \neq j)$. Also, $z \notin A_j$ but $z \in I_z \sim A_i$ $(i = 1, 2, \ldots, k)$. It follows that

$$(I_z \sim A_i)(z) \geqslant A_j(z) + 1 = t + 1 \quad (i = 1, 2, \ldots, k;\ i \neq j).$$

Hence,

$$A_i(z) = n - (I_z \sim A_i)(z) \leqslant n - (t+1) \quad (i = 1, 2, \ldots, k;\ i \neq j),$$

and

$$\sum_{i=1}^{k} A_i(z) \leqslant t + (k-1)(n-t-1) = (2-k)t + (k-1)(n-1).$$

However, the function a defined on the set of real numbers by

$$a(y) = (2-k)y + (k-1)(n-1)$$

is decreasing since $a'(y) = 2 - k \leqslant 0$. Since $t \geqslant n/2$, then

$$\sum_{i=1}^{k} A_i(z) \leqslant a(t) \leqslant a(n/2) = kn/2 - k + 1.$$

From the preceding considerations we have

$$f_k(z) \leqslant \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

That equality holds will next be established by exhibiting subsets $B_1, B_2, \ldots, B_k$ of J^m for which $0 \in \bigcap_{i=1}^{k} B_i$, $z \notin \sum_{i=1}^{k} B_i$, and $\sum_{i=1}^{k} B_i(z)$ is equal to the upper bound of $f_k(z)$ just obtained.

If n is odd, then $n + 1 = \prod_{i=1}^{m} (z_i + 1)$ is even; consequently, z_v is odd for some v, $1 \leqslant v \leqslant m$. For $i = 1, 2, \ldots, k$, define

$$B_i = \{(x_1, x_2, \ldots, x_m) | \ 0 \leqslant x_j \leqslant z_j \text{ for } 1 \leqslant j \leqslant m,\ j \neq v, \text{ and } (z_v+1)/2 \leqslant x_v < z_v \text{ or } x_v = 0\}.$$

Then $z \notin \sum_{i=1}^{k} B_i$, and

$$\sum_{i=1}^{k} B_i(z) = k\Big(\tfrac{1}{2} \prod_{i=1}^{m} (z_i+1) - 1\Big) = kn/2 - k/2.$$

Next assume n is even; hence, z_i is even for $i = 1, 2, \ldots, m$. Let $\Delta = \{i | \ z_i > 0 \text{ and } 1 \leqslant i \leqslant m\}$ and $u = \min\{i | \ i \in \Delta\}$. For $i \in \Delta$, define

$$D_i = \{(x_1, x_2, \ldots, x_m) | \ x_j = z_j/2 \text{ for } j < i, \ z_i/2 < x_i \leqslant z_i, \text{ and } 0 \leqslant x_j \leqslant z_j \text{ for } j > i\}.$$

Set $C_u = D_u \sim \{z\}$ and $C_i = D_i$ for $i \in \Delta$ and $i \neq u$.

If $x = (x_1, x_2, \ldots, x_m)$ and $x \in C_i$, then $x_i > z_i/2$ and so $x \notin C_j$ for $j > i$. Hence,

$$\Big(\bigcup_{i \in \Delta} C_i\Big)(z) = \sum_{i \in \Delta} C_i(z) = \sum_{i=1}^{m-1} z_i/2 \Big(\prod_{t=i+1}^{m} (z_t + 1)\Big) + z_m/2 - 1 = n/2 - 1.$$

Define

$$B_1 = \Big(\bigcup_{j \in \Delta} C_j\Big) \cup \{0, (z_1/2, \ldots, z_m/2)\}$$

and

$$B_i = \Big(\bigcup_{j \in \Delta} C_j\Big) \cup \{0\} \quad \text{for} \quad i = 2, 3, \ldots, k.$$

If $i \leqslant j$, $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$, $x \in C_i$, and $y \in C_j$, then $x_i > z_i/2$ and $y_i \geqslant z_i/2$. Thus $x_i + y_i > z_i$. If follows that $z \notin \sum_{i=1}^{k} B_i$. Furthermore, $\sum_{i=1}^{k} B_i(z) = k(n/2-1) + 1$.

References

[1] P. Erdös and P. Scherk, *On a question of additive number theory*, Acta Arith. 5 (1958), pp. 45–55.

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