A note on the sum of sets of \( m \)-tuples

by

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For \( m \) a natural number, let \( J^m \) be the set of \( m \)-tuples of nonnegative integers. The element of \( J^m \) having each coordinate equal to zero is represented by 0. For \( z = (x_1, x_2, \ldots, x_m) \) in \( J^m \), define

\[
I_z = \{ (x_1, x_2, \ldots, x_m) | 0 \leq x_i \leq z_i \text{ for } i = 1, 2, \ldots, m \}.
\]

Addition of elements of \( J^m \), as well as subtraction of elements of \( I_z \) from \( z \), is done coordinatewise. When \( A_1, A_2, \ldots, A_k \) are subsets of \( J^m \), let

\[
\sum_{i=1}^k A_i
\]

be their sum and denote it by \( \sum A_i \). When \( A \) is a subset of \( J^m \) and \( z \in J^m \), let \( A(z) \) be the cardinality of the set \( (A \cap I_z) \sim \{0\} \).

With the Cartesian product of \( k \) copies of the power set of \( J^m \) denoted by \( P_{\infty}^k \), for each natural number \( k \geq 2 \) define the function \( f_k \) on the nonzero elements of \( z \) of \( J^m \) as follows:

\[
f_k(z) = \max \left\{ \sum_{i=1}^k A_i(z) | (A_1, A_2, \ldots, A_k) \in P_{\infty}^k, 0 \not\in \bigcup_{i=1}^k A_i, z \not\in \bigcup_{i=1}^k A_i \right\}.
\]

When \( n \) is a natural number Erdős and Sechert [1] have shown that

\[
f_k(n) = kn/2 - k/2 \text{ if } n \text{ is odd and } kn/2 - k + 1 \leq f_k(n) \leq kn/2 - k/2
\]

if \( n \) is even. We evaluate \( f_k(n) \) when \( n \) is even and extend the result to the \( m \)-dimensional space \( J^m \).

**Theorem.** Let \( z \in J^m, z \neq 0 \), and let \( n \) be the cardinality of \( I_z \sim \{0\} \). For each natural number \( k \geq 2 \),

\[
f_k(z) = \begin{cases} 
kn/2 - k/2 & \text{if } n \text{ is odd,} \\
kn/2 - k + 1 & \text{if } n \text{ is even.}
\end{cases}
\]

**Proof.** Let \( z = (x_1, x_2, \ldots, x_m), z \neq 0 \), be an element of \( J^m \) for which \( n \) is the cardinality of \( I_z \sim \{0\} \). Let \( k \geq 2 \).
We first determine an upper bound for $f_k(z)$. Let $A_1, A_2, \ldots, A_k$ be subsets of $J^m$ satisfying $0 \in \bigcap_{i=1}^k A_i$ and $z \notin \sum_{i=1}^k A_i$. If $A_i(z) < n/2$ for $i = 1, 2, \ldots, k$, then we immediately obtain

$$\sum_{i=1}^k A_i(z) \leq \begin{cases} \frac{k(n-1)^2}{2} & \text{if } n \text{ is odd}, \\ k(n-1) & \text{if } n \text{ is even}. \end{cases}$$

Next consider the case when $A_i(z) \geq n/2$ for some $i$, $1 \leq i \leq k$, and let $t = \max(A_i(z))$, $i = 1, 2, \ldots, k$. Let $j$ be such that $A_j(z) = t$. If $x \in A_j \cap I_{x+1}$, $x \neq 0$, and $x \notin A_i$ then $z-x \in I_{x+1} \setminus A_i$, $z-x \neq 0$, and $z-x \neq i$ ($i = 1, 2, \ldots, k; i \neq j$). Also, $x \notin A_j$ but $x \in I_{x+1} \setminus A_i$ ($i = 1, 2, \ldots, k$). It follows that

$$(I_{x+1} \setminus A_i)(z) \geq A_j(z) + 1 = t + 1 \quad (i = 1, 2, \ldots, k; i \neq j).$$

Hence,

$$A_i(z) = n - (I_{x+1} \setminus A_i)(z) \leq n - (t+1) \quad (i = 1, 2, \ldots, k; i \neq j),$$

and

$$\sum_{i=1}^k A_i(z) \leq t + (k-1) (n-t-1) = (2-k) t + (k-1) (n-1).$$

However, the function $a$ defined on the set of real numbers by

$$a(y) = 2 - k \cdot y + (k-1)(n-1)$$

is decreasing since $a'(y) = 2-k \leq 0$. Since $t \geq n/2$, then

$$\sum_{i=1}^k A_i(z) \leq a(t) \leq a(n/2) = kn/2 - k + 1.$$

From the preceding considerations we have

$$f_k(z) \leq \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd}, \\ kn/2 - k + 1 & \text{if } n \text{ is even}. \end{cases}$$

That equality holds will next be established by exhibiting subsets $B_1, B_2, \ldots, B_k$ of $J^m$ for which $0 \in \bigcap_{i=1}^k B_i$, $z \notin \sum_{i=1}^k B_i$, and $\sum_{i=1}^k B_i(z)$ is equal to the upper bound of $f_k(z)$ just obtained.

If $n$ is odd, then $n+1 = \prod_{i=1}^m (x_i+1)$ is even; consequently, $x_i$ is odd for some $i$, $1 \leq i \leq m$. For $i = 1, 2, \ldots, k$, define

$$B_i = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_j \leq x_i \text{ for } 1 \leq j \leq m, j \neq i, \text{ and } (x_1+1)/2 \leq x_v < x_u \text{ or } x_v = 0\}.$$