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MATHEMATISCHES INSTI UT DER UNIVERSITÄT
Münster

Eingegangen am 22. 4. 1976

ACTA ARITHMETICA
XXXIV (1977)

On permutations containing no long arithmetic progressions

by

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Introduction. It has often been noted (e.g., see [1], [4], [5]) that it is possible to arrange n consecutive integers into a sequence $a_1a_2\ldots a_n$ which contains no subsequence forming an increasing or decreasing 3-term arithmetic progression (A.P.). In other words, if $a_i = c$, $a_j = c+d$, $a_k = c+2d$ for some positive d, then either $j = \max\{i, j, k\}$ or $j = \min\{i, j, k\}$. In this note we investigate several questions related to this idea. For example, we show that any doubly-infinite permutation $\ldots a_{-2}a_{-1}a_0a_1a_2\ldots$ of *all* the positive integers must contain an increasing or decreasing (i.e., monotone) 3-term A.P. as a subsequence. On the other hand, we construct a doubly-infinite permutation of the positive integers which contains no monotone 4-term A.P.

Permutations of finite intervals. Let us denote by $M(n)$ the number of permutations $a_1a_2\ldots a_n$ of $\{1, 2, \ldots, n\} \equiv [1, n]$ containing no monotone 3-term A.P. To see that $M(n) > 0$ for all n simply note if $A = a_1a_2\ldots a_m$ has no monotone 3-term A.P. then

$$A' = (2A)(2A-1) \equiv (2a_1)(2a_2)\ldots(2a_m)(2a_1-1)\ldots(2a_m-1)$$

also has no monotone 3-term A.P. (since the first and last terms of a 3-term A. P. must have the same parity!) Of course, if A is a permutation of $[1, m]$ then A' is a permutation of $[1, 2m]$. Finally, since no monotone A.P.'s are created by *deleting* entries of A, the assertion $M(n) > 0$ for all n follows immediately. In fact, much more is true.

Fact 1.

(1) $$M(n) \geqslant 2^{n-1} \quad \textit{for} \quad n \geqslant 1.$$

Proof. As we have already noted, if A has no monotone 3-term A.P., then neither do $2A$ and $2A-1$. Thus, if A and A' are 3-term A.P.-

free permutations of $[1, m]$, then $(2A)(2A'-1)$ and $(2A'-1)(2A)$ are 3-term A.P.-free permutations of $[1, 2m]$. Hence,

$$M(2n) \geqslant 2M(n)^2.$$

Similarly, we have

$$M(2n+1) \geqslant 2M(n+1)M(n).$$

Since $M(2) = 2$, $M(3) = 4$ then (1) follows. ■

H. E. Thomas [6] has independently proved (1) by a somewhat more complicated construction.

In Table 1, we give a list of values of $M(n)$ for $n \leqslant 20$.

Table 1

n	$M(n)$	n	$M(n)$
1	1	11	2460
2	2	12	6128
3	4	13	12840
4	10	14	29380
5	20	15	74904
6	48	16	212728
7	104	17	368016
8	282	18	659296
9	496	19	1371056
10	1066	20	2937136

By using the fact that $M(16) = 212728$, it follows from the preceding argument that

$$M(2^t) > \tfrac{1}{2}(2.248)^{2^t}, \quad t \geqslant 4.$$

In the other direction, we have the following result:

FACT 2.

$$(2) \qquad M(2n-1) \leqslant (n!)^2, \quad M(2n) \leqslant (n+1)(n!)^2.$$

Proof. Let $\mathscr{M}(t)$ denote the set of permutations of $[1, t]$ containing no monotone 3-term A.P.'s. Any permutation $X \in \mathscr{M}(n+1)$ generates a permutation $X' \in \mathscr{M}(n)$ by just deleting $n+1$. Consider an element $A = a_1 a_2 \ldots a_n \in \mathscr{M}(n)$ to which $n+1$ can be added *somewhere* to form an $A' \in \mathscr{M}(n+1)$. If a_i satisfies

$$(3) \qquad \left[\frac{n+3}{2}\right] \leqslant a_i \leqslant n,$$

then the three values

$$n+1, a_i, 2a_i-n-1$$

form an arithmetic progression which is not allowed to occur monotonely in A'. Hence, for each a_i satisfying (3), $n+1$ is prohibited from being placed just to the right (left) of a_i if $2a_i-n-1$ occurs to the left (right) of a_i. Also, if $n+1$ were prohibited from going to the right of a_i *and* to the left of a_{i+1} then A could not be extended to an element of $\mathscr{M}(n+1)$. Hence, each of the $n-\left[\frac{n+3}{2}\right]+1$ values a_i satisfying (3) rules out at least one of the $n+1$ possible locations in A for $n+1$, leaving at most $\left[\frac{n+3}{2}\right]$ places where $n+1$ might go. This implies

$$M(n+1) \leqslant \left[\frac{n+3}{2}\right] M(n)$$

which, in turn, implies (2). ■

Permutations of the positive integers. Let $A = a_1 a_2 a_3 \ldots$ be a permutation of the set Z^+ of positive integers. Denote by $\mathscr{S}_k$ the set of those A which contain no monotone k-term A.P.

FACT 3.

$$\mathscr{S}_3 = \emptyset.$$

Proof. Let $A = a_1 a_2 a_3 \ldots$ be a permutation of Z^+. If i denotes the least index for which $a_i > a_1$ then for some $j > i$,

$$a_j = 2a_i - a_1$$

and so we always have, in fact, an increasing 3-term A.P. in A. ■

FACT 4.

$$\mathscr{S}_5 \neq \emptyset.$$

Proof. For $k \geqslant 0$, define the intervals A_k and B_k as follows:

$$A_k = [a_k+1, a_k+10^k], \quad B_k = [b_k+1, b_k+10^k]$$

where $a_0 = 0$, $b_0 = 1$, and in general,

$$a_k = 2\sum_{i=0}^{k-1} 10^i, \quad b_k = a_k + 10^k.$$

Thus, Z^+ is partitioned into disjoint intervals A_k, B_k, $k \geqslant 0$. Note that $A_0 = \{1\}$ and

$$|A_k| = |B_k| = 10^k.$$

Let A_k^* and B_k^* denote arbitrary fixed permutations of A_k and B_k, respectively, which contain no monotone 3-term A.P.'s. Finally, let P be the permutation of Z^+ given by

$$P = B_0^* A_0^* B_1^* A_1^* B_2^* A_2^* \ldots B_k^* A_k^* \ldots$$

We claim that P contains no monotone 5-term A.P. Suppose the contrary, i.e., suppose $X = \{x_1, x_2, x_3, x_4, x_5\}$ with $x_{k+1} - x_k = d > 0$ is a 5-term A.P. occurring monotonely in P. There are several possibilities:

(i) X is a *decreasing* subsequence of P. Thus, for some k, $X \subseteq A_k \cup B_k$. But this implies that either x_5, x_4, x_3 is a decreasing A.P. in B_k^* or x_3, x_2, x_1 is a decreasing A.P. in A_k^*. Since neither of these possibilities can occur, this case is impossible.

(ii) X is an *increasing* subsequence of P.

(a) Suppose $|X \cap (A_k \cup B_k)| \leqslant 1$ for all k. Let $x_k \in A_{i_k} \cup B_{i_k}$, $1 \leqslant k \leqslant 5$. Thus, $i_1 < i_2 < i_3 < i_4 < i_5$. Since

$$x_5 - x_3 > a_{i_5} - a_{i_4} \geqslant 2 \cdot 10^{i_5}$$

then

$$d = \tfrac{1}{2}(x_5 - x_3) > 10^{i_5}.$$

Thus,

$$x_2 = x_3 - d < a_{i_4} - 10^{i_5} \leqslant 2(1 + 10 + \ldots + 10^{i_4}) - 10^{i_4+1} < 0$$

which is impossible. Hence, in this case we cannot even have a 4-term A.P.

(b) Suppose for some k, $|X \cap (A_k \cup B_k)| \geqslant 2$. Of course, since X is increasing and B_k precedes A_k in P, then X cannot intersect both A_k and B_k. Therefore, by the construction of P (which uses A_k^* and B_k^*), we must have $|X \cap (A_k \cup B_k)| = 2$. There are two possibilities.

(α) Suppose $|X \cap B_k| = 2$. If $x_2, x_3 \in B_k$ then $d = x_3 - x_2 < 10^k$ and

$$x_1 = x_2 - d > b_k - 10^k = a_k,$$

i.e., $x_1 \in A_k$ which, as we have just noted, is impossible. A similar argument applies if $x_3, x_4 \in B_k$ or $x_4, x_5 \in B_k$. Thus,

$$x_5 = x_2 + 3d < a_{k+1} + 3 \cdot 10^k$$

which implies $x_5 \in A_{k+1}$ and consequently, $x_3, x_4 \in A_{k+1}$ as well, which is impossible.

(β) Suppose $|X \cap A_k| = 2$. If $x_3, x_4 \in A_k$ then $d = x_4 - x_3 < 10^k$ and

$$x_5 = x_4 + d < a_k + 10^k + 10^k = a_{k+1},$$

i.e., $x_5 \in B_k$ which is impossible. The same argument applies if $x_1, x_2 \in A_k$ or $x_2, x_3 \in A_k$. Thus, the only possibility remaining is $x_4, x_5 \in A_k$.

Now, if $x_2 \in B_{k-1}$ then we also must have $x_3 \in B_{k-1}$ and this is impossible from Case (i). On the other hand, if $x_2 \in A_{k-1}$ then $x_3 \in A_{k-1}$ and $d = x_3 - x_2 < 10^{k-1}$ which implies

$$x_4 = x_3 + d < x_3 + 10^{k-1},$$

i.e., $x_4 \in B_{k-1}$, a contradiction. Thus, $x_2 \leqslant a_{k-1}$ and so,

$$d = \tfrac{1}{2}(x_4 - x_2) > \tfrac{1}{2}(a_k - a_{k-1}) = 10^{k-1}.$$

Therefore,

$$x_1 = x_2 - d < a_{k-1} - 10^{k-1} < 0$$

which is a contradiction.

This completes the proof that P contains no monotone 5-term A.P. and Fact 4 is proved. ■

One of the most tantalizing questions still open is whether or not $\mathcal{S}_4$ is empty; i.e., whether every permutation of $\mathbf{Z}^+$ must contain monotone 4-term A.P.'s. Current opinions are about evenly divided.

Doubly-infinite permutations of the positive integers. If we are allowed to arrange the positive integers into a *doubly-infinite* sequence $A = \ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ then, in principle, we have more opportunity to prevent the occurrence of monotone A.P.'s. Denote by $\mathcal{D}_k$ the set of those A which contain no monotone k-term A.P. As in the case of $\mathcal{S}_3$, $\mathcal{D}_3$ is also empty. This time however, a little more work is required to prove it.

FACT 5.

$$\mathcal{D}_3 = \emptyset.$$

Proof #1 (J. H. Folkman [2]). Let $A = \ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots$ be a doubly-infinite permutation of $\mathbf{Z}^+$. For $n \in \mathbf{Z}^+$, let $A(n)$ denote the *index* of n in A, i.e., $A(n)$ is defined by

$$a_{A(n)} = n.$$

Suppose A contains no monotone 3-term A. P. Thus, for all $a, d > 0$,

$$A(a) < A(a+d) \quad \text{iff} \quad A(a+d) > A(a+2d)$$

and

$$A(a) > A(a+d) \quad \text{iff} \quad A(a+d) < A(a+2d).$$

Iterating these relations we obtain

$$(4) \qquad A(a) < A(a+d) \quad \text{iff} \quad \begin{cases} A(a+2md) < A(a+d+2md) \text{ and} \\ A\big(a+(2m+1)d\big) > A\big(a+d+(2m+1)d\big), \\ \qquad m = 0, 1, 2, \ldots \end{cases}$$

$$(4') \qquad A(a) > A(a+d) \quad \text{iff} \quad \begin{cases} A(a+2md) > A(a+d+2md) \text{ and} \\ A\big(a+(2m+1)d\big) < A\big(a+d+(2m+1)d\big), \\ \qquad m = 0, 1, 2, \ldots \end{cases}$$

We may assume without loss of generality that $A(1) < A(2)$ (otherwise, reverse the sequence). By (4), we have

$$(5) \qquad A(2m-1) < A(2m), \quad m = 1, 2, \ldots$$

We claim that for *any* odd a and d,

$$A(a) < A(a+d). \tag{6}$$

For $d = 1$, this is just (5). Assume (6) holds for a fixed odd $d \geqslant 1$. Let a be odd and let $b = a+2d+4$. By assumption

$$A(b) < A(b+d).$$

(i) Suppose $A(b+d) < A(b+d+2)$. Then $A(b) < A(b+d+2)$ and so

$$A(a) = A\big(b-2(d+2)\big) < A\big(b+d+2-2(d+2)\big) = A(a+d+2)$$

by (4).

(ii) Suppose $A(b+d) > A(b+d+2)$. Then by (5)

$$A(a+d) = A\big(b+d-(d+2)\cdot 2\big) < A\big(b+d+2-(d+2)\cdot 2\big) = A(a+d+2).$$

Since $A(a) < A(a+d)$ then $A(a) < A(a+d+2)$.

Thus, in either case, we have $A(a) < A(a+d+2)$. This completes the induction step and (6) is proved. We are now finished, since by (6)

$$A(1) < A(2m) \quad \text{for all } m > 0.$$

Thus, as in the argument that $\mathscr{S}_3 = \emptyset$, if $2r$ is the first even number to the right of 1 and $2r+2d$ is the first even number to the right of $2r$ which is larger than $2r$, then $2r+4d$ is to the right of $2r+2d$ and $2r$, $2r+2d$, $2r+4d$ forms an increasing 3-term A.P. in A. This completes Proof #1 of Fact 5.

We sketch another proof of Fact 5 which is conceptually somewhat simpler although it involves some computation.

Proof #2. We form a directed tree T as follows. The vertices of T will be certain permutations $A \in \mathscr{M}(n)$ for various n. T will have 4 root vertices 132, 213, 231 and 312. Suppose A is a vertex of T in which the subblock $B = a_i a_{i+1} \ldots a_{i+r}$ spanned by $\{1, 2, 3\}$ contains some *other* 3-term A.P. (necessarily non-monotone). We call such a vertex *special*. If $A \in \mathscr{M}(n)$ is a non-special vertex of T and A is a subsequence of $A' \in \mathscr{M}(n+1)$ then A' is also a vertex of T and (A, A') is a directed edge of T. If no such A' exists for A then A is called a *terminal* vertex of T. We show a portion of T in Fig. 1. The basic fact concerning T is that it is *finite*. In fact, straightforward computation shows that T contains no vertices $A \in \mathscr{M}(n)$ with $n > 17$.

To complete the proof, we make the following observation. As we adjoin consecutive integers, starting with $A^* \in \mathscr{M}(3)$, to form a permutation P of Z^+, we move in the obvious way along a directed path in the tree. Suppose we reach a special vertex $A = a_1 \ldots a_n$. By definition, the block of A spanned by $\{1, 2, 3\}$ contains a subsequence $a_{i_1} a_{i_2} a_{i_3}$ which

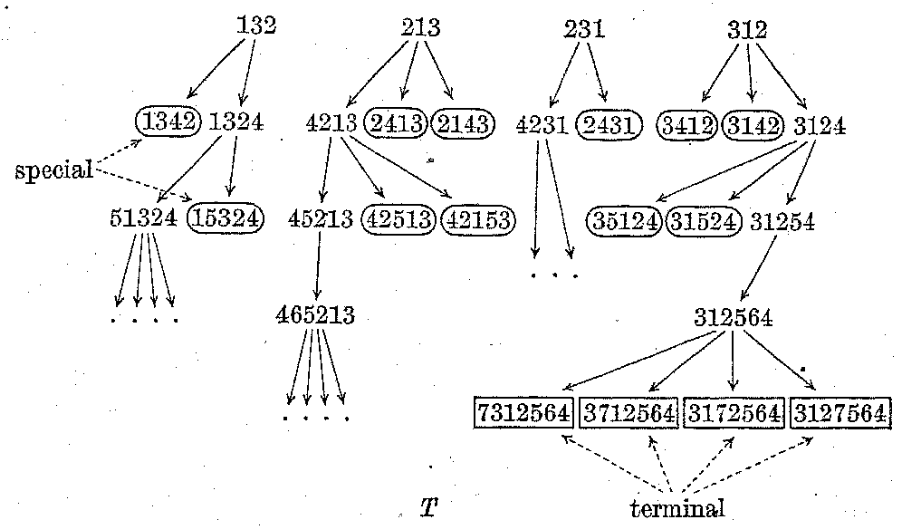

Fig. 1

is a permutation of $\{a, a+d, a+2d\} \neq \{1, 2, 3\}$. If we restrict our attention from now on to just those integers of the form $a+md$, $m \geqslant 0$, then we can move back to the appropriate *root* of T, i.e., the permutation of $\{1, 2, 3\}$ having the same relative order as $a_{i_1} a_{i_2} a_{i_3}$. Since T is finite then as we form P, we must pass through the roots of T an unbounded number of times. However, this implies that in P some pair of integers in $\{1, 2, 3\}$ must have an unbounded number of integers separating them. This, however, contradicts the definition of a permutation of Z^+, and the proof is completed. ∎

The additional freedom allowed by doubly-infinite permutations can be used to prevent the occurrence of monotone 4-term A.P.'s.

FACT 6. $\mathscr{D}_4 \neq \emptyset$.

Proof. Define the blocks B_i, $i > 0$, as follows:

$$B_0 = 1, \quad B_{2i+1} = (2B_{2i})'(2B_{2i}+1)', \quad B_{2i+2} = (2B_{2i+1}+1)'(2B_{2i+1})', \quad i \geqslant 0,$$

where B' denotes the block B written in reverse order. Define the doubly-infinite permutation P of Z^+ by

$$P = \ldots B_4 B_2 B_0 B_1 B_3 \ldots$$
$$= \ldots 28, 20, 24, 16, 7, 5, 6, 4, 1, 2, 3, 8, 12, 10, 14, 9, 13, 11, 5, \ldots$$

We claim that $P \in \mathscr{D}_4$.

We first note that for all $i \geqslant 0$, B_i is a permutation of $[2^i, 2^{i+1}-1]$ containing no monotone 3-term A.P. Suppose now that P contains a monotone 4-term A.P. $X = \{x, y, z, w\}$ with either $x > y > z > w$ or

$x < y < z < w$, where we have chosen X so that $d = |x-y|$ is minimal. There are several possibilities:

(i) The smallest two elements of X belong to the same block B_i. Then $d < 2^i$ so that the largest two elements of X are in B_{i+1}. Consequently, x, y, z and w all have the same parity. If $2j+1$ and $2k+1$ are in B_i then $2j$ and $2k$ are also in B_i with the same relative order. Hence, we may assume x, y, z, and w are all even. But then

$$\frac{1}{2}X = \left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right\}$$

is a monotone 4-term A.P. in P since the smallest two elements of $\frac{1}{2}X$ appear in B_{i-1} in reverse order of their appearance in B_i, the largest two appear in B_i in reverse order of the appearance in B_{i+1}, and the order of B_i and B_{i-1} in P is the reverse of that of B_{i+1} and B_i. However, this contradicts the minimality of d.

(ii) Suppose y and z occur in the same block B_i. Then the largest element of X occurs in B_{i+1} and the smallest occurs in B_j for some $j < i$. But this requires B_i to appear between B_{i+1} and B_j in P which is impossible.

(iii) Suppose the largest two elements of X occur in the same block B_i. The third largest element of X must be at least as large as 2^{i-1} since otherwise, we would have $d < 2^{i-1}$ and consequently, the second largest element of X would be less than 2^i and therefore, not in B_i. Thus, the third largest element of X is in B_{i-1}. Hence, by (i), the smallest element of X is in B_j for some $j < i-1$. As before, this requires B_{i-1} to appear between B_j and B_i in P which is impossible.

(iv) Suppose each element of X belongs to a different block B_i of P. Let B_i denote the block containing the largest element of X. Then we may argue as in (ii) and (iii) that the second largest element of X is not contained in B_{i-1}. Consequently $d > 2^{i-1}$ so that the third largest element of X must be negative, a contradiction.

Since the construction of the B_i prohibits the occurrence of 3 elements of X in a single block then we have proved that P has no monotone 4-term A.P. ∎

Concluding remarks. There are a number of questions which we were either unable to resolve or did not have a chance to look at. We mention a few of these.

1. The most natural question remaining is whether or not $\mathscr{S}_4 = \emptyset$, i.e., whether or not every singly-infinite permutation of Z^+ contains a monotone 4-term A.P. It is not clear at present in which direction the truth lies.

2. The following modular analogue to the finite problem has been studied by M. Nathanson [3]. A subsequence $a_{i_0}, \ldots, a_{i_{t-1}}$ of a permutation $a_1 a_2 \ldots a_n$ of $[1, n]$ is called a *monotone A. P. modulo n* if for some a and $d \neq 0$,

$$a_{i_k} \equiv a + kd \pmod{n}, \quad 0 \leqslant k < t.$$

Nathanson has shown (see [3]) that:

(i) If $n \neq 2^r$ then any permutation of $[1, n]$ contains a monotone 3-term A.P. modulo n.

(ii) If $n = 2^r$ then there is a permutation of $[1, n]$ which contains no monotone 3-term A.P.

On the other hand, it is easily seen that a permutation of $[1, n]$ which contains no monotone 3-term A. P. also contains no monotone 5-term A. P. modulo n. As in the preceding question, the situation for 4-term A. P.'s modulo n is unclear.

3. It is possible to partition $\boldsymbol{Z}^+$ into three sets, each of which can be permuted so as to have no monotone 3-term A. P. For example, define the partition of $\boldsymbol{Z}^+$ into consecutive intervals A_k by:

$$A_1 = [1, 100], \quad |A_{k+1}| = [\tfrac{3}{2}|A_k|], \quad k \geqslant 1.$$

Now, rearrange each A_k into A_k^* containing no monotone 3-term A.P. and define

$$\mathscr{A} = A_1^* A_4^* A_7^* A_{10}^* \ldots,$$
$$\mathscr{B} = A_2^* A_5^* A_8^* A_{11}^* \ldots,$$
$$\mathscr{C} = A_3^* A_6^* A_9^* A_{12}^* \ldots$$

It is easily checked that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ form the desired partition. Whether this can be done for some partition of $\boldsymbol{Z}^+$ into *two* sets is not known.

4. Let $\mathscr{A}$ denote the set of all infinite subsets A of $\boldsymbol{Z}^+$ for which there exists a (singly-infinite) permutation of A having no monotone 3-term A. P. What is

$$\sup_{A \in \mathscr{A}} \liminf_{n} \frac{|A \cap [1, n]|}{n} ?$$

What is

$$\sup_{A \in \mathscr{A}} \limsup_{n} \frac{|A \cap [1, n]|}{n} ?$$

5. The preceeding questions could also be asked for $\boldsymbol{Z}$, the set of all the integers, as well. Only preliminary results are known for this case. For example, using Fact 4, it is easy to construct permutations of $\boldsymbol{Z}$ which have no monotone 7-term A. P.

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Received on 29. 6. 1976 (