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On power series with only finitely many coefficients (mod 1): Solution of a problem of Pisot and Salem

by

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1. Introduction. Denote by S the set of algebraic integers θ, all of whose conjugates (except for θ itself), have absolute value < 1. Let T be the set of algebraic integers θ, all of whose conjugates (except for θ itself), have absolute value $\leqslant 1$ with at least one conjugate of absolute value $= 1$. Pisot [3] has shown that if $\lambda > 1$, $\theta > 1$ are real numbers for which

$$\sum_{n=0}^{\infty} \|\lambda\theta^n\|^2 < \infty, \tag{1.1}$$

($\|x\|$ denotes the distance from x to the nearest integer) then $\theta \in S$ and λ is in the field $\boldsymbol{Q}(\theta)$. He has also shown [4] that if

$$\|\lambda\theta^n\| \leqslant \frac{1}{2e\theta(\theta+1)(1+\log\lambda)} \tag{1.2}$$

for all integers $n \geqslant 0$ then $\theta \in S \cup T$ and $\lambda \in \boldsymbol{Q}(\theta)$. In [1] I give an extension of Pisot's result (1.2). A major open question is whether (1.1) can be replaced by $\lim_{n\to\infty} \|\lambda\theta^n\| = 0$.

In [6], Pisot and Salem ask for a theorem which includes both cited results of Pisot. Theorem 1 (below) answers this question and shows moreover that the term $1/(1+\log\lambda)$ in (1.2) can, in essence, be replaced by the less restrictive term $1/(2+\sqrt{\log\lambda})$.

As usual, we say $x, y \in \boldsymbol{R}$ are *distinct* (mod 1) if $x-y \notin \boldsymbol{Z}$.

In [4] Pisot proved that if there exists a sequence of real numbers $a_0, a_1, a_2, \ldots$ which assume at most f distinct values (mod 1) and $\lambda > 1$, $\theta > 1$ which satisfy

$$\lambda\theta^n = a_n + o(1/n^f), \quad n = 0, 1, 2, \ldots \tag{1.3}$$

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then again $\theta \in S \cup T$. We shall show that the rate of decrease $o(1/n^f)$ can be replaced by the much slower rate $o(1/n^{1-1/(2f)})$. Furthermore we shall replace the term $\lambda\theta^n$ in (1.3) by the more general expression $\sum_{i=1}^{m} \lambda_i(n)\theta_i^n$ where the $\lambda_i(n)$ are polynomials and $|\theta_i| \geqslant 1$. Finally, in [5], Pisot showed that if $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series defining a function $a(z)$ meromorphic in the disk $|z| < R$, where $R > 1$ and if the a_n assume only finitely many distinct values (mod 1), then $a(z)$ is a rational function. We shall extend this result to functions $a(z)$ which need only be meromorphic in $|z| < 1$ and which can be expressed as the ratio of analytic functions whose Taylor series satisfy suitable growth conditions (these conditions are trivially satisfied in the case considered by Pisot).

The key idea in proving these results is contained in Lemma 3.1, which gives a new criterion for a sequence of integers to satisfy a linear recurrence relationship with constant coefficients.

If x is a real number, then $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$ and $\lceil x \rceil$ denotes the least integer $\geqslant x$. Note that $\|x\| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$.

2. Main theorems.

2.1. THEOREM. *Suppose $a_n = \lambda\theta^n + \varepsilon_n$, $n = 0, 1, 2, \ldots$ is a sequence of positive integers with $\lambda > 0$, $\theta > 1$. Suppose that there exist μ and σ satisfying $0 < \mu \leqslant 1$, $0 < \sigma \leqslant 1$ such that*

$$(1+\theta)^2 \sum_{i=m}^{m+n-1} (\varepsilon_{i+1} - \theta\varepsilon_i)^2 < \mu n^\sigma \tag{2.2}$$

for all integers $m \geqslant 0$, $n \geqslant 1$. If

$$\mu \leqslant [\log(a_0^2 + 1/8) + 2\sigma]^{-\sigma} (\sigma/e)^\sigma, \tag{2.3}$$

then $\sum_{n=0}^{\infty} a_n z^n$ is a rational function, $\theta \in S \cup T$, and θ has degree $< 1/(e\mu^{1/\sigma})$.

2.4. Remark. Condition (2.2) will be satisfied if

$$(1+\theta)^4 \sum_{i=m}^{m+n-1} \varepsilon_i^2 < \mu n^\sigma \tag{2.5}$$

for all integers $m \geqslant 0$, $n \geqslant 1$ and condition (2.3) will be satisfied if

$$(e\mu)^{1/\sigma} \leqslant 1/\big(\log(a_0^2 + 1/8) + 2\big). \tag{2.6}$$

If (2.3) is satisfied we may increase μ until $\mu = 1$ or there is equality in (2.3) and this increase will not violate condition (2.2). If this is done we can conclude that θ has degree $< 2 + \big(\log(a_0^2 + 1/8)\big)/\sigma$.

2.7. THEOREM. *Suppose $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series, where the a_n assume $\leqslant f$ distinct values* (mod 1), *which can be written in the form $s(z)/t(z)$ where $s(z) = \sum_{n=0}^{\infty} s_n z^n$ and $t(z) = \sum_{n=0}^{\infty} t_n z^n$ are analytic in $|z| < 1$ and satisfy*

$$\sum_{n=0}^{\infty} |t_n| < \infty, \tag{2.8}$$

$$\sum_{m=n}^{2n} |s_m|^2 = o(1/n^\alpha), \tag{2.9}$$

$$\sum_{m=n}^{2n} |t_m| = o(1/n^\beta), \tag{2.10}$$

where $0 \leqslant \alpha \leqslant \beta \leqslant 1$ and $\alpha + \beta \geqslant 2 - 1/f$. Then $a(z)$ and $\sum_{n=0}^{\infty} \lfloor a_n \rfloor z^n$ are rational functions.

2.11. COROLLARY. *Suppose $a_0, a_1, a_2, \ldots$ is a sequence of real numbers taking $\leqslant f$ distinct values* (mod 1). *Suppose $\lambda_1(n), \lambda_2(n), \ldots, \lambda_m(n)$ are polynomials with complex coefficients and $\theta_1, \theta_2, \ldots, \theta_m$ are complex numbers with $|\theta_i| \geqslant 1$. Suppose finally that*

$$\sum_{i=1}^{m} \lambda_i(n)\theta_i^n = a_n + o(1/n^{1-1/(2f)}).$$

Then $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} \lfloor a_n \rfloor z^n$ are rational functions.

2.12. COROLLARY. *If $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a rational function and the a_n assume only finitely many values* (mod 1), *then $\sum_{n=0}^{\infty} \lfloor a_n \rfloor z^n$ is a rational function.*

Before we give the proofs we give some comments. When $\sigma = 1$ and all $|\varepsilon_i| \leqslant \varepsilon$, then conditions (2.2) and (2.3) are satisfied if

$$(1+\theta)^2 \varepsilon < \big(e\log(a_0^2 + 1/8) + 2e\big)^{-1/2}. \tag{2.13}$$

Thus to show that Theorem 2.1 includes the cited result (1.2) of Pisot, it suffices to show that if ε satisfies (2.13) and if $\lambda > 1$ then

$$(1+\theta)^2 \big(e\log(a_0^2 + 1/8) + 2e\big)^{1/2} < 2e\theta(\theta + 1)(1 + \log\lambda).$$

We shall omit the elementary verification. To show that Theorem 2.1 implies the other cited result (1.1) of Pisot, suppose that $\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$. By omitting a finite number of the a_n and renumbering the a_n, if necessary, we may assume that $(1+\theta)^4\sum_{n=0}^{\infty}\varepsilon_n^2 < 1$. The right hand side of (2.3) approaches 1 as $\sigma\to 0$. Thus if we put $\mu = (1+\theta)^4\sum_{n=0}^{\infty}\varepsilon_n^2$ and choose $\sigma > 0$ sufficiently small, conditions (2.2), (2.3) will be satisfied and hence $\theta\in S\cup T$.

It is interesting to note that the special case of Theorem 2.7 when $t_0 = 1$, $t_1 = -\theta$, $t_2 = t_3 = \ldots = 0$, $a_n = \lambda\theta^n+\varepsilon_n$, $f = 1$, $\alpha = 0$, $\beta = 1$ shows that Pisot's condition $\sum_{n=0}^{\infty}\varepsilon_n^2 < \infty$ can be replaced by the weaker condition $\sum_{m=n}^{2n}\varepsilon_m^2\to 0$ as $n\to 0$.

We note that throughout this paper the word "integer" could be replaced by "Gaussian integer" or, more generally, "integer of L" where L is a quadratic imaginary algebraic number field. All results would remain valid (*mutatis mutandis*) and a slightly different definition of S and T would be necessary.

3. Proofs.

3.1. LEMMA. *Suppose $c_0\neq 0, c_1, c_2, \ldots$ is a sequence of integers and $t_0 = 1, t_1, t_2, \ldots$ is a sequence of complex numbers. Put*

$$x_{mn} = \sum_{h=0}^{m}\sum_{i=0}^{n} t_h t_i c_{m+n-h-i}.$$

If $p_0 < p_1 < \ldots < p_n$ is a finite, increasing sequence of non-negative integers define $A_n(p_0, p_1, \ldots, p_n)$ to be the $(n+1)\times(n+1)$ matrix whose (h, i) entry is $x_{p_h,i}$ for $0\leqslant h, i\leqslant n$. If there exists $r\geqslant 0$ such that

$$|\det(A_r(p_0, p_1, \ldots, p_r))| < 1 \qquad \textit{for all } p_0 < p_1 < \ldots < p_r$$

then there exist q satisfying $1\leqslant q\leqslant r$ and $d_0 = 1, d_1, \ldots, d_q$ such that

$$\sum_{i=0}^{q} d_i c_{n-i} = 0 \qquad \textit{for all } n\geqslant q.$$

Proof. For $n\leqslant r$ define the matrix $H_n(p_0, p_1, \ldots, p_n)$ to be the $(n+1)\times(n+1)$ matrix whose (h, i) entry is c_{p_h+i} for $0\leqslant h, i\leqslant n$. Put $J_n = H_n(0, 1, \ldots, n)$ and let U_r be the upper triangular $(r+1)\times(r+1)$ matrix whose (h, i) entry is t_{i-h} when $i\geqslant h$ and 0 when $i < h$ for $0\leqslant h, i\leqslant r$. Since $t_0 = 1$, $\det(U_r) = 1$. The matrix $U_r^t J_r U_r$ has (m, n) entry $x_{m,n}$. Hence $U_r^t J_r U_r = A_r(0, 1, \ldots, r)$ and by hypothesis $|\det(J_r)| < 1$. Since J_r has integral entries, $\det(J_r) = 0$. Let q be the least integer for which $\det(J_n) = 0$ for $q\leqslant n\leqslant r$. Since $\det(J_0) = c_0$ is not zero, q is $\geqslant 1$ and $\det(J_{q-1})\neq 0$. Thus there exists a non-trivial linear combination of the columns of J_q which vanish. Hence there exist $d_0, d_1, \ldots, d_q$, not all 0, such that $\sum_{h=0}^{q} d_h c_{n-h} = 0$ for $q\leqslant n\leqslant 2q$. Since $\det(J_{q-1})\neq 0$, we see that $d_0\neq 0$ and we may assume, with no loss in generality, that $d_0 = 1$. We now show, by induction, that $\sum_{h=0}^{q} d_h c_{n-h} = 0$ for $q\leqslant n\leqslant q+r$. Suppose we have shown that $\sum_{h=0}^{q} d_h c_{m-h} = 0$ for $q\leqslant m\leqslant q+n-1$, where $q+1\leqslant n\leqslant r$. The matrix J_n has determinant 0. Call its columns $\gamma_0, \gamma_1, \ldots, \gamma_n$. Successively, for $j = n, j = n-1, \ldots, j = q$ replace column γ_j by $\sum_{h=0}^{q} d_h\gamma_{j-h}$. The determinant of this new matrix is still 0. If $q\leqslant j\leqslant n$, the elements in the $(n+q-j, j)$ positions $= \sum_{h=0}^{q} d_h c_{n+q-h}$ and the elements above these elements are of the form $\sum_{h=0}^{q} d_h c_{k+q-h}$ where $q\leqslant k < n$, hence are 0. It follows that the determinant of the new matrix is $(\sum_{h=0}^{q} d_h c_{n+q-h})^{n-q+1}\times\det(J_{q-1}) = 0$ and hence $\sum_{h=0}^{q} d_h c_{n+q-h} = 0$. This shows that $\sum_{h=0}^{q} d_h c_{n-h} = 0$ for $q\leqslant n\leqslant q+r$. Put $K_n = H_r(0, 1, \ldots, q-1, n-r, n-r+1, \ldots, n-q)$. We will show, by induction, for all $n\geqslant q+r$, that $\det(K_n) = 0$ and that $\sum_{h=0}^{q} d_h c_{n-h} = 0$. We have already shown this when $n = q+r$. Suppose we have proven it for all m satisfying $q+r\leqslant m < n$. Let δ_i denote the row vector $(c_i, c_{i+1}, \ldots, c_{i+r})$. By induction, each of the rows $\delta_q, \delta_{q+1}, \ldots$ $\ldots, \delta_{n-r-1}$, which are not present in K_n, is a linear combination of the first q rows of K_n: $\delta_0, \delta_1, \ldots, \delta_{q-1}$. By adding appropriate linear combinations of the preceding rows to the last row δ_{n-q} of K_n we can replace the last row by $\sum_{h=0}^{n-q} t_h\delta_{n-q-h}$, then we can replace the next-to-last row by $\sum_{h=0}^{n-q-1} t_h\delta_{n-q-1-h}$, etc. Thus, at the completion of the row operations, the row δ_j is replaced by $\sum_{h=0}^{j} t_h\delta_{j-h}$. Let γ_j denote the jth column of this new matrix; we can replace γ_j by $\sum_{h=0}^{j} t_h\gamma_{j-h}$, successively for $j = r$, $r-1, \ldots, 0$. The matrix we obtain in this way will be $A_r(0, 1, \ldots, q-1$, $n-r, n-r+1, \ldots, n-q)$ and will have the same determinant as K_n. Thus $|\det(K_n)| < 1$ and since K_n has integral entries, $\det(K_n) = 0$. Denote the jth column of K_n by η_j. Now, successively for $j = r, r-1, \ldots, q$ replace η_j by $\sum_{h=0}^{q} d_h\eta_{j-h}$. The new matrix still has determinant 0. If $q\leqslant j\leqslant r$ the elements in the $(r+q-j, j)$ positions $= \sum_{h=0}^{q} d_h c_{n-h}$. The elements above these elements are of the form $\sum_{h=0}^{q} d_h c_{k-h}$ where $q\leqslant k < n$, and are

0 by the inductive hypothesis. It follows that $\det(K_n) = \left(\sum_{h=0}^{q} d_h c_{n-h}\right)^{r-q+1} \times$ $\times \det(J_{q-1}) = 0$. Hence $\sum_{h=0}^{q} d_h c_{n-h} = 0$. ∎

3.2. Lemma. *Suppose* $f(x) = \mu^{x-1} x^{\sigma(x-1)}$ *where* $0 < \mu \leqslant 1$, $0 < \sigma \leqslant 1$. *Then there exists a positive integer* $r \leqslant 1 + 1/(e\mu^{1/\sigma})$ *such that*

$$\log f(r) \leqslant \sigma\left(1 + e\mu^{1/\sigma} - 1/(e\mu^{1/\sigma})\right).$$

Proof. Put $g(x) = \mu^x x^{\sigma x}$ and choose $x_0 \geqslant 0$ so that $g(x_0)$ is the minimum of $g(x)$ for $x \geqslant 0$. Now

$$\log g(x) = x \log \mu + \sigma x \log x,$$

$$g'(x)/g(x) = \log \mu + \sigma + \sigma \log x.$$

Since $g'(x_0) = 0$,

$$\log \mu + \sigma + \sigma \log x_0 = 0,$$

$$\log x_0 = -1 - (\log \mu)/\sigma.$$

Thus $x_0^\sigma = 1/(\mu e^\sigma)$ and

$$\log g(x_0) = -\sigma x_0 = -\sigma/(e\mu^{1/\sigma}).$$

Put $r = \lceil x_0 \rceil$. Clearly $1 \leqslant r \leqslant 1 + 1/(e\mu^{1/\sigma})$. Since $g'(x) \geqslant 0$ for $x \geqslant x_0$, we see that $g(r) \leqslant g(x_0+1)$. Next

$$g(x_0+1)/g(x_0) = \mu(1+1/x_0)^{x_0\sigma}(x_0+1)^\sigma \leqslant \mu e^\sigma (x_0+1)^\sigma = (1+e\mu^{1/\sigma})^\sigma.$$

Then,

$$\log g(r) \leqslant \log g(x_0) + \log\left(g(x_0+1)/g(x_0)\right)$$
$$\leqslant -\sigma/(e\mu^{1/\sigma}) + \sigma \log(1+e\mu^{1/\sigma}) \leqslant -\sigma/(e\mu^{1/\sigma}) + \sigma e\mu^{1/\sigma}.$$

Now, $f(x) = g(x)/(\mu x^\sigma)$, hence

$$f(r) = g(r)/(\mu r^\sigma) \leqslant g(r)/(\mu x_0^\sigma) = e^\sigma g(r).$$

Thus

$$\log f(r) \leqslant \sigma + \log g(r) \leqslant \sigma\left(1 + e\mu^{1/\sigma} - 1/(e\mu^{1/\sigma})\right). \ ∎$$

Proof of Theorem 2.1. Write $u = 1 + \left(\log(a_0^2 + 1/8)\right)/\sigma$ and $v = e\mu^{1/\sigma}$. Then (2.3) becomes

$$\mu \leqslant \left(\sigma(1+u)\right)^{-\sigma}(\sigma/e)^\sigma \quad \text{or} \quad \mu e^\sigma \leqslant (1+u)^{-\sigma}.$$

Hence

$$v \leqslant \frac{1}{1+u} = \frac{2}{u+(u+2)} < \frac{2}{u+\sqrt{u^2+4}} = (-u+\sqrt{u^2+4})/2.$$

Now the polynomial $X^2 + uX - 1$ has roots $(-u \mp \sqrt{u^2+4})/2$ and v lies between these two roots. Hence $v^2 + uv - 1 < 0$ or $u + v - 1/v < 0$. Thus

$$\left(\log(a_0^2 + 1/8)\right)/\sigma + 1 + e\mu^{1/\sigma} - 1/(e\mu^{1/\sigma}) < 0.$$

By Lemma 3.2 there exists a positive integer $r \leqslant 1 + 1/(e\mu^{1/\sigma})$ such that

$$\log(\mu^{r-1} r^{\sigma(r-1)}) \leqslant \sigma\left(1 + e\mu^{1/\sigma} - 1/(e\mu^{1/\sigma})\right).$$

Then

$$\log(a_0^2 + 1/8) + \log(\mu^{r-1} r^{\sigma(r-1)}) < 0. \tag{3.3}$$

Put $\delta_n = \varepsilon_n - \theta\varepsilon_{n-1}$ and set $t_0 = 1$, $t_1 = -\theta$, and $t_i = 0$ when $i \geqslant 2$. Define $x_{mn} = \sum_{h=0}^{m} \sum_{i=0}^{n} t_h t_i a_{m+n-h-i}$. Then $x_{00} = a_0$, $x_{0n} = x_{n0} = \delta_n$ when $n \geqslant 1$, and $x_{mn} = \delta_{m+n} - \theta\delta_{m+n-1}$ when $m, n \geqslant 1$. Next, if $m, n \geqslant 1$,

$$x_{mn}^2 = \delta_{m+n}^2 - 2\theta\delta_{m+n}\delta_{m+n-1} + \theta^2\delta_{m+n-1}^2$$
$$\leqslant \delta_{m+n}^2 + \theta(\delta_{m+n}^2 + \delta_{m+n-1}^2) + \theta^2\delta_{m+n-1}^2 = (1+\theta)(\delta_{m+n}^2 + \theta\delta_{m+n-1}^2).$$

Now let $p_0, p_1, \ldots, p_{r-1}$ be an increasing sequence of positive integers and define the matrix $A_{r-1} = A_{r-1}(p_0, p_1, \ldots, p_{r-1})$ to be the $r \times r$ matrix whose (h, i) entry is $x_{p_h, i}$ for $0 \leqslant h, i \leqslant r-1$. Let $B_{r-1} = B_{r-1}(p_0, p_1, \ldots$ $\ldots, p_{r-1})$ be the $r \times r$ matrix obtained from A_{r-1} by multiplying the first column by $(1+\theta)^{1/2}$ and dividing the first row by $(1+\theta)^{1/2}$. Call the elements of the ith row of B_{r-1}, $y_{i0}, y_{i1}, \ldots, y_{i,r-1}$. Then if $p_i > 0$

$$\sum_{j=0}^{r-1} y_{ij}^2 \leqslant (1+\theta)\left(\delta_{p_i}^2 + \sum_{j=1}^{r-1} (\delta_{p_i+j}^2 + \theta\delta_{p_i+j+1}^2)\right)$$
$$\leqslant (1+\theta)\left(\sum_{j=0}^{r-1} \delta_{p_i+j}^2 + \theta \sum_{j=1}^{r} \delta_{p_i+j}^2\right) \leqslant \mu r^\sigma.$$

Similarly, if $p_0 = 0$,

$$\sum_{j=0}^{r-1} y_{0j}^2 \leqslant a_0^2 + \sum_{j=1}^{r-1} \delta_j^2/(1+\theta) \leqslant a_0^2 + \mu r^\sigma/(1+\theta)^3.$$

Since $\log(a_0^2 + 1/8) > 0$, we see from (3.3) that $\mu r^\sigma < 1$, hence

$$\mu r^\sigma/(1+\theta)^3 < 1/8 \quad \text{and} \quad \sum_{j=0}^{r-1} y_{0j}^2 < a_0^2 + 1/8.$$

By Hadamard's inequality, if $p_0 = 0$, then

$$|\det(A_{r-1})|^2 = |\det(B_{r-1})|^2 < (a_0^2 + 1/8)(\mu r^\sigma)^{r-1},$$

while if $p_0 > 0$, then

$$|\det(A_{r-1})|^2 < (\mu r^\sigma)^r.$$

By (3.3) $|\det(A_{r-1})|<1$ and by Lemma 3.1, $a(z)=\sum_{n=0}^{\infty}a_nz^n$ is a rational function whose denominator has degree $\leqslant r-1\leqslant 1/(e\mu^{1/\sigma})$ (since r as used in Lemma 3.1 is 1 less than r as used in this proof). By the Fatou-Hurwitz Lemma [7], we may write $a(z)=P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are relatively prime polynomials with integral coefficients satisfying $Q(0)=1$ and $\deg Q(z)\leqslant 1/(e\mu^{1/\sigma})$. Now $P(z)/Q(z)-\lambda(1-\theta z)^{-1}=\sum_{n=0}^{\infty}\varepsilon_nz^n$ is analytic in $|z|<1$. Since $\lambda\neq 0$, $1/\theta$ is a pole of $P(z)/Q(z)$ and $Q(1/\theta)=0$. Since $Q(0)=1$, θ is an algebraic integer. As all other roots of $Q(z)$ lie in $|z|\geqslant 1$, all conjugates of θ (except θ itself) lie in $|z|\leqslant 1$ and $\theta\in S\cup T$. ■

3.4. Lemma. *Suppose $y_1, y_2, y_3, \ldots$ is a sequence of positive numbers satisfying*

$$\sum_{i=m}^{2m-1}y_i\to 0 \quad as \quad m\to\infty.$$

Put $\delta_n=\sup_{m\geqslant n}\sum_{i=m}^{2m-1}y_i$. Then if $i_1<i_2<\ldots<i_r$ is a strictly increasing sequence of positive integers, we have

$$\sum_{j=1}^{r}\sum_{k=1}^{r}y_{i_j+k}\leqslant 4\sum_{i=1}^{r}\delta_i.$$

Proof. Choose $j\leqslant r$ and let h be the largest power of 2 which is $\leqslant r/i_j$; let l be the largest power of 2 which is $\leqslant j$; let g be the largest power of 2 which is $\leqslant r$. Since $i_j\geqslant j\geqslant l$,

$$\sum_{k=1}^{r}y_{i_j+k}\leqslant \delta_{i_j}+\delta_{2i_j}+\delta_{4i_j}+\ldots+\delta_{hi_j}\leqslant \delta_j+\delta_{2j}+\delta_{4j}+\ldots+\delta_{hj}$$
$$\leqslant \delta_l+\delta_{2l}+\delta_{4l}+\ldots+\delta_g.$$

Hence

$$\begin{aligned}\sum_{j=1}^{r}\sum_{k=1}^{r}y_{i_j+k}\leqslant{} & \delta_1+\delta_2+\delta_4+\ldots+\delta_g+\\ & +2(\delta_2+\delta_4+\delta_8+\ldots+\delta_g)+\\ & +4(\delta_4+\delta_8+\delta_{16}+\ldots+\delta_g)+\\ & \vdots\\ & +g\delta_g\\ \leqslant{} & \delta_1+3\delta_2+7\delta_4+\ldots+(2g-1)\delta_g\\ \leqslant{} & \delta_1+3\delta_2+4\big(2\delta_4+4\delta_8+\ldots+(g/2)\delta_g\big)\\ \leqslant{} & \delta_1+(\delta_1+2\delta_2)+4\big((\delta_3+\delta_4)+(\delta_5+\delta_6+\delta_7+\delta_8)+\\ & \qquad +\ldots+(\delta_{g/2+1}+\ldots+\delta_g)\big)\\ \leqslant{} & 4(\delta_1+\delta_2+\ldots+\delta_g). \ ■\end{aligned}$$

3.5. Lemma. *If $X=(x_{ij})$ is an $r\times r$ matrix and $\sum_{i=1}^{r}\sum_{j=1}^{r}|x_{ij}|^2<r$, then $|\det(X)|<1$.*

Proof. By Hadamard's inequality and the arithmetic-geometric mean inequality,

$$|\det(X)|^2\leqslant\prod_{i=1}^{r}\sum_{j=1}^{r}|x_{ij}|^2\leqslant\Big((1/r)\sum_{i=1}^{r}\sum_{j=1}^{r}|x_{ij}|^2\Big)^r<1. \ ■$$

Proof of Theorem 2.7. By subtracting a constant multiple of $1/(1-z)$ from $a(z)$, if necessary, we may assume that some of the a_n are $0 \pmod 1$. This will not change the number of distinct values (mod 1) assumed by the a_n. It will replace $s(z)$ by $(1-z)s(z)-\tau t(z)$ and $t(z)$ by $(1-z)t(z)$, where τ is a constant. Since $\alpha\leqslant\beta$, (2.8), (2.9), and (2.10) will remain satisfied. We may assume that $a(z)$ is not constant. Since $a(z)$ is regular in a neighborhood of 0 we may divide $s(z)$ and $t(z)$ by the same integral power of z and assume that $t_0\neq 0$. By further dividing $s(z)$ by an integral power of z, if necessary, we may assume that $s_0\neq 0$. We may multiply $s(z)$ and $t(z)$ by the same non-zero constant and assume that $t_0=1$. Thus, after appropriate renaming, we may assume the hypotheses of Theorem 2.7 and, in addition, that $s_0\neq 0$, $t_0=1$, $a(z)$ is not constant, and that one of the $\leqslant f$ distinct values (mod 1) assumed by the a_n is 0. If N is a positive integer we may write

$$Na_n=c_n+\varepsilon_n \quad \text{where} \quad c_n\in\mathbf{Z} \text{ and } -1/2\leqslant\varepsilon_n<1/2.$$

By Dirichlet's theorem ([2], p. 14, Th. VII) there are infinitely many choices for N for which

$$(3.6)\qquad |\varepsilon_n|\leqslant 1/N^{1+1/(f-1)},\quad n=0,1,2,\ldots$$

where the right hand side of (3.6) is interpreted as 0 if $f=1$. We may decrease α and β, if necessary, and assume that $0\leqslant\alpha\leqslant\beta\leqslant 1$ and $\alpha+\beta=2-1/f$.

In the succeeding lemmas we use the following notation:

$T=\sum_{h=0}^{\infty}|t_h|$;

N is a positive integer satisfying (3.6);

$$u_{m,n}=N\sum_{h=0}^{n}t_hs_{m+n-h};$$

$$v_{m,n}=-N\sum_{h=0}^{n-1}s_ht_{m+n-h};$$

$$w_{m,n}=-\sum_{h=0}^{m}\sum_{i=0}^{n}t_ht_i\varepsilon_{m+n-h-i};$$

$$x_{m,n}=\sum_{h=0}^{m}\sum_{i=0}^{n}t_ht_ic_{m+n-h-i}.$$

We shall let $p_0 < p_1 < \ldots < p_r$ be a strictly increasing, finite sequence of non-negative integers. Define $A_r = A_r(p_0, p_1, \ldots, p_r)$ to be the $(r+1)\times(r+1)$ matrix whose (m, n) entry is $x_{p_m, n}$, $0 \leqslant m, n \leqslant r$.

3.7. Lemma. *We have* $x_{m,n} = u_{m,n} + v_{m,n} + w_{m,n}$.

Proof.

$$x_{m,n} = \sum_{i=0}^{n}\sum_{h=0}^{m} t_h t_i c_{m+n-h-i} = \sum_{i=0}^{n}\sum_{h=0}^{m} t_h t_i (N a_{m+n-h-i} - \varepsilon_{m+n-h-i})$$

$$= w_{m,n} + N\sum_{i=0}^{n} t_i \sum_{h=0}^{m} t_h a_{m+n-h-i}$$

$$= w_{m,n} + N\sum_{i=0}^{n} t_i \Big(s_{m+n-i} - \sum_{h=m+1}^{m+n-i} t_h a_{m+n-h-i}\Big),$$

since $\sum_{h=0}^{m+n-i} t_h a_{m+n-h-i} = s_{m+n-i}$. Continuing,

$$x_{m,n} = w_{m,n} + u_{m,n} - N\sum_{h=m+1}^{m+n} t_h \sum_{i=0}^{m+n-h} t_i a_{m+n-h-i}$$

$$= w_{m,n} + u_{m,n} - N\sum_{h=m+1}^{m+n} t_h s_{m+n-h} = w_{m,n} + u_{m,n} + v_{m,n}. \ \blacksquare$$

3.8. Lemma. *We have* $\sum_{h=0}^{r} |s_h| = o(r^{(1-\alpha)/2})$.

Proof. By the Cauchy–Schwarz inequality,

$$\sum_{h=j}^{2j-1} |s_h| \leqslant j^{1/2}\Big(\sum_{h=j}^{2j-1} |s_h|^2\Big)^{1/2} \leqslant j^{1/2}\big(o(1/j^{\alpha})\big)^{1/2} = o(j^{(1-\alpha)/2}).$$

Let 2^l be the largest power of 2 which is $\leqslant r$. Then

$$\sum_{h=0}^{r} |s_h| \leqslant |s_0| + \sum_{i=0}^{l}\sum_{j=2^i}^{2^{i+1}-1} |s_j| \leqslant |s_0| + \sum_{i=0}^{l} o(2^{i(1-\alpha)/2}) = o(2^{l(1-\alpha)/2}) = o(r^{(1-\alpha)/2}),$$

since $\alpha < 1$. $\blacksquare$

3.9. Lemma. *We have*

$$\sum_{m=0}^{r}\sum_{n=0}^{r} |w_{p_m,n}|^2 \leqslant (r+1)^2 T^4/N^{2+2/(f-1)};$$

$$\sum_{m=0}^{r}\sum_{n=0}^{r} |u_{p_m,n}|^2 = N^2 o(r^{1/f});$$

$$\sum_{m=0}^{r}\sum_{n=0}^{r} |v_{p_m,n}|^2 = N^2 o(r^{1/f}).$$

Proof. Since

$$|w_{p_m,n}| \leqslant \sum_{h=0}^{p_m}\sum_{i=0}^{n} |t_h|\cdot|t_i|\cdot|\varepsilon_{p_m+n-h-i}| \leqslant T^2 N^{1+1/(f-1)},$$

the first inequality is clear. Next, since $n \leqslant r$, $u_{p_m,n}$ is the coefficient of z^{p_m+n} in

$$N\sum_{h=0}^{r} t_h z^h \sum_{i=p_m}^{r+p_m} s_i z^i.$$

By Parseval's equality with $z = e^{\theta\sqrt{-1}}$ in the following integrals

$$\sum_{n=0}^{r} |u_{p_m,n}|^2 \leqslant \frac{N^2}{2\pi}\int_0^{2\pi} \Big|\sum_{h=0}^{r} t_h z^h \sum_{i=p_m}^{r+p_m} s_i z^i\Big|^2 d\theta$$

$$\leqslant \frac{N^2}{2\pi}\int_0^{2\pi} \Big(\sum_{h=0}^{r} |t_h|\cdot\Big|\sum_{i=p_m}^{r+p_m} s_i z^i\Big|\Big)^2 d\theta$$

$$\leqslant \frac{N^2}{2\pi}\Big(\sum_{h=0}^{r} |t_h|\Big)^2 \int_0^{2\pi} \Big|\sum_{i=p_m}^{r+p_m} s_i z^i\Big|^2 d\theta$$

$$\leqslant N^2 T^2 \sum_{i=p_m}^{r+p_m} |s_i|^2.$$

Put $\sigma_i = \sup_{j\geqslant i} \sum_{h=j}^{2j} |s_h|^2$. Since $0 \leqslant \alpha \leqslant 1/2$, and $\sigma_i = o(1/i^{\alpha})$, we have

$$\sum_{=0}^{r} \sigma_i = o(r^{1-\alpha}).$$

Next, using Lemma 3.4 with $y_k = |s_{k-1}|^2$

$$\sum_{m=0}^{r}\sum_{n=0}^{r} |u_{p_m,n}|^2 \leqslant N^2 T^2 \sum_{m=0}^{r}\sum_{i=p_m}^{r+p_m} |s_{p_m+i}|^2 = N^2 T^2 \sum_{m=0}^{r}\sum_{i=0}^{r} |s_{p_m+i}|^2$$

$$\leqslant 4N^2 T^2 \sum_{i=0}^{r} \sigma_i = N^2 o(r^{1-\alpha}) = N^2 o(r^{1/f}),$$

since $\alpha + \beta = 2 - 1/f$ and $\beta \leqslant 1$ implies $\alpha \geqslant 1 - 1/f$ or $1 - \alpha \leqslant 1/f$. This proves the second inequality. Similarly, $v_{p_m,n}$ is the coefficient of z^{p_m+n} in

$$-N\sum_{h=0}^{r} s_h z^h \sum_{i=p_m+1}^{r+p_m} t_i z^i$$

and proceeding as for $u_{p_m,n}$, using Lemmas 3.4 and 3.8, we obtain

$$\sum_{n=0}^{r} |v_{p_m,n}|^2 \leqslant N^2\Big(\sum_{h=0}^{r} |s_h|\Big)^2 \sum_{i=p_m+1}^{r+p_m} |s_i|^2 \leqslant N^2\big(o(r^{(1-\alpha)/2})\big)^2 o(r^{1-\beta})$$

$$= N^2 o(r^{2-\alpha-\beta}) = N^2 o(r^{1/f}). \ \blacksquare$$

Proof of Theorem 2.7. (Completed.) If $f \geqslant 2$, put

$$X_r = \sum_{m=0}^{r} \sum_{n=0}^{r} x^2_{p_m,n}, \quad \delta = 1/(36T^4) \quad \text{and} \quad r = \lfloor \delta N^{2f/(f-1)} \rfloor$$

where N will be chosen later and chosen so large that $r \geqslant 2$. Since

$$x^2_{m,n} \leqslant 3(u^2_{m,n} + v^2_{m,n} + w^2_{m,n}),$$

we have by Lemma 3.9

$$X_r \leqslant 6r^2T^4/N^{2+2/(f-1)} + N^2 o(r^{1/f}) \leqslant 6(\delta N^{2f/(f-1)})/N^{2+2(f-1)} + o(N^{2f/(f-1)})$$
$$\leqslant 12\delta T^4 r + o(r) \leqslant r/3 + o(r).$$

Thus X_r, which is the sum of the squares of A_r, is $\leqslant r/2$ for all sufficiently large r. Choose N satisfying (3.6) so large that $X_r \leqslant r/2$ and that $r \geqslant 2$. Then by Lemma 3.5,

$$|\det(A_r(p_0, p_1, \ldots, p_r))| < 1,$$

and this is true for all $p_0 < p_1 < \ldots < p_r$. By Lemma 3.1, $c(z) = \sum_{n=0}^{\infty} c_n z^n$ is rational. If $f = 1$, we choose $N = 1$ and then by Lemma 3.9, $X_r = o(r)$. Then exactly as above, using Lemmas 3.5 and 3.1, $c(z) = \sum_{n=0}^{\infty} c_n z^n$ is rational. Put $\varepsilon(z) = \sum_{n=0}^{\infty} \varepsilon_n z^n$ so that $Na(z) = c(z) + \varepsilon(z)$. If $c(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials with $Q(0) = 1$, then

$$\varepsilon(z) = d(z)/e(z)$$

where

$$d(z) = Na(z)Q(z) - t(z)P(z) \quad \text{and} \quad e(z) = t(z)Q(z).$$

If $d(z) = \sum_{n=0}^{\infty} d_n z^n$ and $e(z) = \sum_{n=0}^{\infty} e_n z^n$, then

$$\lim_{n\to\infty} d_n = 0, \quad e_0 = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} |e_n| < \infty.$$

There are $\leqslant f$ distinct ε_n. Put $\sigma = \min_{\varepsilon_i \neq \varepsilon_j} |\varepsilon_i - \varepsilon_j|$ and $\tau = \max_n |\varepsilon_n|$. Choose n_0 so large that $\tau \sum_{n=n_0+1}^{\infty} |e_n| < \sigma/4$ and $|d_n| < \sigma/4$ for all $n \geqslant n_0$. Now $d_n = \sum_{m=0}^{\infty} e_m \varepsilon_{n-m}$ and if $n \geqslant n_0$ $\left|\sum_{m=0}^{n_0} e_m \varepsilon_{n-m} - d_n\right| < \sigma/4$, hence

$$\left|\varepsilon_n - \sum_{m=0}^{n_0} (-e_m)\varepsilon_{n-m}\right| < \sigma/2.$$

Thus ε_n is determined by the previous $(n_0+1)\varepsilon_i$ and hence the sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ is eventually periodic with period $\leqslant f^{n_0+1}$. Thus there exists ϱ

so that $\varepsilon_n = \varepsilon_{n-\varrho}$ for all sufficiently large n and $\varepsilon(z) = q(z)/(1-z^\varrho)$ where $q(z)$ is a polynomial. Hence $\varepsilon(z)$ and $a(z) = (c(z)+\varepsilon(z))/N$ are rational functions. By the Fatou–Hurwitz lemma [7] we may assume that $Q(z)$ has integral coefficients, hence that $a(z)$ may be written in the form $a(z) = u(z)/v(z)$ where $v(z) = (1-z^\varrho)Q(z)$ has integral coefficients and $v(0) = 1$. Write $v(z) = \sum_{i=0}^{l} v_i z^i$ with $v_0 = 1$. Then

$$\sum_{i=0}^{l} v_i a_{n-i} = 0 \quad \text{for all } n > n_1 = \deg(u(z)).$$

Write $a_n = \lfloor a_n \rfloor + \gamma_n$. There are $\leqslant f$ distinct γ_n and

$$\sum_{i=0}^{l} v_i \gamma_{n-i} \equiv 0 \pmod 1 \quad \text{for} \quad n > n_1.$$

Thus $\gamma_n = -\sum_{i=1}^{l} v_i \gamma_{n-i} \pmod 1$ if $n > n_1$. Hence γ_n is determined by the previous l γ_i's and the sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ is ultimately periodic. Hence $\sum_{n=0}^{\infty} \lfloor a_n \rfloor z^n = a(z) - \sum_{n=0}^{\infty} \gamma_n z^n$ is a rational function. ∎

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