On power series with only finitely many coefficients (mod 1):
Solution of a problem of Pisot and Salem
by

DAVID G. CANTOR* (Los Angeles, Calif.)

1. Introduction. Denote by $S$ the set of algebraic integers $\theta$, all of whose conjugates (except for $\theta$ itself), have absolute value $< 1$. Let $T$ be the set of algebraic integers $\theta$, all of whose conjugates (except for $\theta$ itself), have absolute value $\leq 1$ with at least one conjugate of absolute value $= 1$. Pisot [3] has shown that if $\lambda > 1$, $\theta > 1$ are real numbers for which

\[ \sum_{n=0}^{\infty} ||\lambda^n\theta^n||^2 < \infty, \]  

($||w||$ denotes the distance from $w$ to the nearest integer) then $\theta \in S$ and $\lambda$ is in the field $\mathbb{Q}(\theta)$. He has also shown [4] that if

\[ ||\lambda^n\theta^n|| \leq \frac{1}{2e\theta(\theta+1)(1+\log \lambda)} \]  

for all integers $n \geq 0$ then $\theta \in S \cup T$ and $\lambda \in \mathbb{Q}(\theta)$. In [1] I give an extension of Pisot's result (1.2). A major open question is whether (1.1) can be replaced by $\lim_{n \to \infty} ||\lambda^n\theta^n|| = 0$.

In [6], Pisot and Salem ask for a theorem which includes both cited results of Pisot. Theorem 1 (below) answers this question and shows moreover that the term $1/(1+\log \lambda)$ in (1.2) can, in essence, be replaced by the less restrictive term $1/(2+V\log \lambda)$.

As usual, we say $x, y \in \mathbb{R}$ are distinct (mod 1) if $x - y \notin \mathbb{Z}$.

In [4] Pisot proved that if there exists a sequence of real numbers $a_0, a_1, a_2, \ldots$ which assumes at most $f$ distinct values (mod 1) and $\lambda > 1$, $\theta > 1$ which satisfy

\[ \lambda^n\theta^n = a_n + o(1/n^2), \quad n = 0, 1, 2, \ldots \]  

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then again \( \theta \in S \cup T \). We shall show that the rate of decrease \( o(1/n^k) \) can be replaced by the much slower rate \( o(1/n^{k-\beta/(2\mu)}) \). Furthermore we shall replace the term \( \lambda \theta^\beta \) in (1.3) by the more general expression \( \sum \lambda_i(n) \theta_i^\beta \) where the \( \lambda_i(n) \) are polynomials and \( |\theta_i| \geq 1 \). Finally, in [5], Pisot showed that if \( a(z) = \sum a_n z^n \) is a power series defining a function \( a(z) \) meromorphic in the disk \( |z| < R \), where \( R > 1 \) and if the \( a_n \) assume only finitely many distinct values \( \pmod{1} \), then \( a(z) \) is a rational function. We shall extend this result to functions \( a(z) \) which need only be meromorphic in \( |z| < 1 \) and which can be expressed as the ratio of analytic functions whose Taylor series satisfy suitable growth conditions (these conditions are trivially satisfied in the case considered by Pisot).

The key idea in proving these results is contained in Lemma 3.1, which gives a new criterion for a sequence of integers to satisfy a linear recurrence relationship with constant coefficients.

If \( x \) is a real number, then \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \) and \( \lceil x \rceil \) denotes the least integer \( \geq x \). Note that \( |x| = \min(\lfloor x \rfloor, \lceil x \rceil - x) \).

2. Main theorems.

2.1. Theorem. Suppose \( a_n = \lambda \theta^n + e_n \), \( n = 0, 1, 2, \ldots \) is a sequence of positive integers with \( \lambda > 0 \), \( \theta > 1 \). Suppose that there exist \( \mu \) and \( \sigma \) satisfying \( 0 < \mu < 1 \), \( 0 < \sigma < 1 \) such that

\[
(1 + \theta^\beta) \sum_{m=0}^{m+n-1} (e_{m+1} - e_m)^2 < \mu n^2
\]

for all integers \( m \geq 0 \), \( n \geq 1 \). If

\[
\mu < \left[ \log(a_0^2 + 1/8) + 2\sigma \right]^{-1/2} |\theta|^\sigma,
\]

then \( \sum a_n z^n \) is a rational function, \( \theta \in S \cup T \), and \( \theta \) has degree \( < 1/(e \mu^{1/2}) \).

2.4. Remark. Condition (2.2) will be satisfied if

\[
(1 + \theta^\beta) \sum_{m=0}^{m+n-1} e_m^2 < \mu n^2
\]

for all integers \( m \geq 0 \), \( n \geq 1 \) and condition (2.3) will be satisfied if

\[
(e \mu)^n \log(a_0^2 + 1/8 + 2\theta) < 1/(\log(a_0^2 + 1/8) + 2\theta).
\]

If (2.3) is satisfied we may increase \( \mu \) until \( \mu = 1 \) or there is equality in (2.3) and this increase will not violate condition (2.2). If this is done we can conclude that \( \theta \) has degree \( < 2 + \log(a_0^2 + 1/8)/\sigma \).

2.7. Theorem. Suppose \( a(z) = \sum a_n z^n \) is a power series, where the \( a_n \) assume \( \leq f \) distinct values \( \pmod{1} \), which can be written in the form

\[
s(z)/t(z) \quad \text{where} \quad s(z) = \sum a_n z^n \quad \text{and} \quad t(z) = \sum b_n z^n \quad \text{are analytic in} \ |z| < 1
\]

and satisfy

\[
\sum_{n=0}^{\infty} |t_n| < \infty,
\]

\[
\sum_{n=0}^{m} \theta^n = o(1/n^\alpha),
\]

\[
\sum_{n=-m}^{\infty} \theta^n = o(1/n^\beta),
\]

where \( 0 < a < \beta < 1 \) and \( a + \beta < 2 - 1/f \). Then \( a(z) \) and \( \sum a_n z^n \) are rational functions.

2.11. Corollary. Suppose \( a_0, a_1, a_2, \ldots \) is a sequence of real numbers taking \( \leq f \) distinct values \( \pmod{1} \). Suppose \( \lambda_1(n), \lambda_2(n), \ldots, \lambda_n(n) \) are polynomials with complex coefficients and \( \theta_1, \theta_2, \ldots, \theta_n \) are complex numbers with \( |\theta_i| \geq 1 \). Suppose finally that

\[
\sum_{n=0}^{\infty} \lambda_i(n) \theta_i^\beta = a_n + o(1/n^{1-\beta/(2\mu)}).
\]

Then \( \sum a_n z^n \) and \( \sum |a_n| z^n \) are rational functions.

2.12. Corollary. If \( a(z) = \sum a_n z^n \) is a rational function and the \( a_n \) assume only finitely many values \( \pmod{1} \), then \( \sum a_n z^n \) is a rational function.

Before we give the proofs we give some comments. When \( \sigma = 1 \) and all \( |e_i| \leq e \), then conditions (2.2) and (2.3) are satisfied if

\[
(1 + \theta^\beta) e < (\log(a_0^2 + 1/8) + 2\theta)^{-1/2}.
\]

Thus to show that Theorem 2.1 includes the cited result (1.2) of Pisot, it suffices to show that if \( e \) satisfies (2.13) and if \( \lambda > 1 \) then

\[
(1 + \theta^\beta)(\log(a_0^2 + 1/8) + 2\theta)^{1/2} < 2e\theta|\theta + 1|(1 + \log \lambda).
\]

We shall omit the elementary verification. To show that Theorem 2.1 implies the other cited result (1.1) of Pisot, suppose that \( \sum_{n=0}^{\infty} e_n^2 < \infty \). By
omitting a finite number of the \( a_n \) and renumbering the \( a_n \), if necessary, we may assume that \((1+\rho)^{\sum_{n=0}^{\infty} e_n} < 1\). The right hand side of (2.3) approaches 1 as \( \sigma \to 0 \). Thus if we put \( \mu = (1+\rho)^{\sum_{n=0}^{\infty} e_n} \) and choose \( \sigma > 0 \) sufficiently small, conditions (2.2), (2.3) will be satisfied and hence \( \theta \in S \cup T \).

It is interesting to note that the special case of Theorem 2.7 when \( t_0 = 1, t_1 = -2, t_2 = t_3 = \ldots = 0, a_n = \lambda^m + e_n, f = 1, \alpha = 0, \beta = 1 \) shows that Picard's condition \( \sum_{n=0}^{\infty} a_n^m < \infty \) can be replaced by the weaker condition \( \sum_{n=0}^{\infty} a_n^m \to 0 \) as \( n \to 0 \).

We note that throughout this paper the word "integer" could be replaced by "Gaussian integer" or, more generally, "integer of \( L \)" where \( L \) is a quadratic imaginary algebraic number field. All results would remain valid (mutatis mutandis) and a slightly different definition of \( T \) and \( \beta \) would be necessary.

3. Proofs.

3.1. Lemma. Suppose \( c_n \neq 0, c_1, c_2, \ldots \), is a sequence of integers and \( t_0 = 1, t_1, t_2, \ldots \), is a sequence of complex numbers. Put

\[
x_{mn} = \sum_{h=0}^{m} \sum_{i=0}^{n} t_h k_m c_{m+i-h+i}.
\]

If \( p_0 < p_1 < \ldots < p_r \) is a finite, increasing sequence of non-negative integers, define \( A_{p_i}(p_0, p_1, \ldots, p_r) \) to be the \((n+1) \times (m+1)\) matrix whose \((h, i)\) entry is \( x_{p_{h+i}} \) for \( 0 < h, i < n \). If \( \det A_{p_i}(p_0, p_1, \ldots, p_r) \) then there exist \( g \) satisfying \( 1 < g < r, \alpha_1 = 1, \alpha_2, \ldots, \alpha_g \) such that

\[
\sum_{l=1}^{g} \delta_l c_{p_{g-l}} = 0 \quad \text{for all} \quad n \neq g.
\]

Proof. For \( n \leq r \) define the matrix \( H_{n}(p_0, p_1, \ldots, p_r) \) to be the \((n+1) \times (m+1)\) matrix whose \((h, i)\) entry is \( c_{h+i} \) for \( 0 \leq h, i \leq n \). Put \( J_{h} = H_{n}(0, 1, \ldots, n) \) and let \( U_{s} \) be the upper triangular \((r+1) \times (r+1)\) matrix whose \((h, i)\) entry is \( s_{h+i} \) when \( h > i \) and 0 when \( h < i \) or \( h = i \). Since \( t_0 = 1, \det(U_{0}) = 1 \). The matrix \( U_{1}J_{h}U_{s} \) has \((m, n)\) entry \( a_{m+n} \). Hence \( U_{1}J_{h}U_{s} = A_{0}(0, 1, \ldots, r) \) and by hypothesis \( \det(J_{h}) < 1 \). Since \( J_{h} \) has integral entries, \( \det(J_{h}) = 0 \). Let \( \delta_{l} \) be the least integer for which \( \det(J_{h}) = 0 \) for \( g < n \leq 1 \). Since \( \det(J_{h}) = c_{p_{g-l}} \) is not zero, \( g \) is \( > 1 \) and \( \det(J_{h}) \neq 0 \). Thus there exists a non-trivial linear combination of the columns of \( J_{h} \) which vanish. Hence there exist \( \delta_{l}, \delta_{2}, \ldots, \delta_{g}, \) not all 0, such that \( \sum_{h=0}^{g} \delta_{h} a_{p_{h}} = 0 \) for \( g < n \leq 2g \). Since \( \det(J_{h-g}) \neq 0 \), we see that \( \delta_{g} \neq 0 \) and we may assume, with no loss in generality, that \( \delta_{g} = 1 \). We now show, by induction, that \( \sum_{h=0}^{g} \delta_{h} a_{p_{h}} = 0 \) for \( g < m < q + r \). Suppose we have shown that \( \sum_{h=0}^{q} \delta_{h} a_{p_{h}} = 0 \) for \( g < m < q + r \). The matrix \( J_{h} \) has determinant 0. Call its columns \( y_{0}, y_{1}, \ldots, y_{n-1} \). Successively, for \( j = n, j = n-1, \ldots, j = q \), replace column \( y_{j} \) by \( \sum_{h=0}^{q} \delta_{h} y_{j-h} \).

The determinant of this new matrix is still 0. If \( q < j < n \), the elements in \((n+q-j, j)\) positions \( = \sum_{h=0}^{q} \delta_{h} y_{j-h} \). The elements above these elements are of the form \( \sum_{h=0}^{q} \delta_{h} c_{p_{g-h}} \) where \( q < k < n \), hence are 0. It follows that the determinant of the new matrix is \( \sum_{h=0}^{q} \delta_{h} a_{p_{h}+g-n} \), and the elements above these elements are of the form \( \sum_{h=0}^{q} \delta_{h} c_{p_{g-h}} \) where \( q < k < n \), hence are 0. It follows that the determinant of the new matrix is \( \sum_{h=0}^{q} \delta_{h} a_{p_{h}+g-n} \), and the elements above these elements are of the form \( \sum_{h=0}^{q} \delta_{h} c_{p_{g-h}} \) where \( q < k < n \), hence are 0. It follows that the determinant of the new matrix is \( \sum_{h=0}^{q} \delta_{h} a_{p_{h}+g-n} \), and the elements above these elements are of the form \( \sum_{h=0}^{q} \delta_{h} c_{p_{g-h}} \) where \( q < k < n \), hence are 0. It follows that the determinant of the new matrix is \( \sum_{h=0}^{q} \delta_{h} a_{p_{h}+g-n} \), and the elements above these elements are of the form \( \sum_{h=0}^{q} \delta_{h} c_{p_{g-h}} \)
0 by the inductive hypothesis. It follows that \( \det(K_n) = \left( \sum_{k=0}^{n} a_k c_{n-k} \right)^{r-2+1} \times \det(J_{n-1}) = 0 \). Hence \( \sum_{k=0}^{n} a_k c_{n-k} = 0 \). \( \blacksquare \)

3.2. **Lemma.** Suppose \( f(x) = x^{r-1} \psi (x - 1) \) where \( 0 < \mu \leq 1, 0 < \sigma \leq 1 \). Then there exists a positive integer \( r \leq 1 + 1/(\sigma \mu^{1/3}) \) such that

\[
\log f(r) \leq \sigma \left( 1 + \epsilon \mu^{1/3} - 1/(\sigma \mu^{1/3}) \right).
\]

**Proof.** Put \( g(x) = x^{r-1} \psi (x - 1) \) and choose \( x_0 \geq 0 \) so that \( g(x_0) \) is the minimum of \( g(x) \) for \( x \geq 0 \). Now

\[
\log g(x) = x \log x + \sigma \log x,
\]

\[
g'(x)g(x) = \log x + x + \sigma \log x.
\]

Since \( g'(x_0) = 0 \),

\[
\log x_0 + x_0 + \sigma \log x_0 = 0,
\]

\[
\log x_0 = -1/(\sigma \mu^{1/3}).
\]

Thus \( x_0 = 1/(\mu^{1/3}) \) and

\[
\log g(x_0) = -x_0 - \sigma = -\sigma/(\sigma \mu^{1/3}).
\]

Let \( v = [x_0] \). Clearly \( 1 \leq v \leq 1 + 1/(\sigma \mu^{1/3}) \). Since \( g'(x) > 0 \) for \( x \geq x_0 \), we see that \( g(r) \leq g(x_0 + 1) \). Next

\[
g(x_0 + 1)g(x_0) = (x_0 + 1)^{2r} x_0 = (1 + x_0)^r \leq \mu^{1/3} \mu^{1/3} = (1 + \epsilon \mu^{1/3})^r.
\]

Then,

\[
\log g(r) \leq \log g(x_0) + \log g(x_0 + 1)/g(x_0) \leq \sigma/(\sigma \mu^{1/3}) + 2 \log(1 + x_0) \leq \sigma/(\sigma \mu^{1/3}) + 2 \epsilon \mu^{1/3}.
\]

Now, \( f(x) = g(x)/(\mu^{1/3}) \), hence

\[
f(x) = g(x)/(\mu^{1/3}) \leq g(r)/(\mu^{1/3}) = \mu^{1/3} g(r).
\]

Thus

\[
\log f(r) \leq \sigma + \log g(r) \leq \sigma \left( 1 + \epsilon \mu^{1/3} - 1/(\sigma \mu^{1/3}) \right).
\]

\( \blacksquare \)

**Proof of Theorem 2.1.** Write \( u = 1 + \log(x_0 + 1/8)/\sigma \) and \( v = \epsilon \mu^{1/3} \).

Then (2.3) becomes

\[
\mu \leq \left( \sigma (1 + u) \right)^{-v} (\sigma / \epsilon) \quad \text{or} \quad \mu \leq u^{-v}.
\]

Hence

\[
v \leq \frac{1}{1 + u} = \frac{2}{u + (u + 2)} < \frac{2}{u + V u^2 + 4} = (-u + V u^2 + 4)/2.
\]

Now the polynomial \( x^2 + u x - 1 \) has roots \(-u \pm \sqrt{u^2 + 4}\)/2 and \( v \) lies between these two roots. Hence \( u^2 + v^2 - 1 < 0 \) or \( u + v - 1/v < 0 \). Thus

\[
\log(a_0^2 + 1/8)/\sigma + 1 + \epsilon \mu^{1/3} - 1/(\sigma \mu^{1/3}) < 0.
\]

By Lemma 3.2 there exists a positive integer \( r \leq 1 + 1/(\sigma \mu^{1/3}) \) such that

\[
\log \mu^{1/3} \psi(r - 1) \leq \sigma \left( 1 + \epsilon \mu^{1/3} - 1/(\sigma \mu^{1/3}) \right).
\]

Then

(3.3)

\[
\log(a_0^2 + 1/8) + \log \left( \mu^{1/3} \psi(r - 1) \right) < 0.
\]

Put \( \delta_n = a_n - b_{n-1} \) and set \( y_0 = 1, y_1 = -\delta, \) and \( y_{-1} = 0 \) when \( i \geq 2 \).

Define \( x_{n+1} = \sum_{k=0}^{n} y_k a_{n+k} \). Then \( x_n = a_0, x_{n-1} = z_{n} = \delta_n \) when \( n = 1, \) and \( x_{n+1} = \delta_{n+1} - \theta \delta_{n+2} \) when \( n = 2 \). Next, if \( m, n \geq 1 \),

\[
\delta_{m,n} = \delta_{m,n-1} - 2 \delta_{m,n-2} + \delta_{m-1} \delta_{n-1} - \delta_{m-1} \delta_{n-2}.
\]

Now let \( p_0, p_1, \ldots, p_{r-1} \) be an increasing sequence of positive integers and define the matrix \( A_{r-1} = A_{r-1}(p_0, p_1, \ldots, p_{r-1}) \) to be the \( r \times r \) matrix whose \((h, i)\) entry is \( s_{h,i} \) for \( 0 \leq h, i \leq r - 1 \). Let \( B_{r-1} = B_{r-1}(p_0, p_1, \ldots, p_{r-1}) \) be the \( r \times r \) matrix obtained from \( A_{r-1} \) by multiplying the first column by \((1 + \theta)^{r-1}\) and dividing the first row by \((1 + \theta)^{r-1}\). Call the elements of the \( i \)th row of \( B_{r-1} \), \( y_0, y_1, \ldots, y_{i-1} \). Then if \( p_1 > 0 \)

\[
\sum_{j=0}^{r-1} y_j^2 < \sum_{j=0}^{r-1} \left( \delta_{p_1,j} + \theta \delta_{p_2,j} \right)^2 < \mu x^2.
\]

Similarly, if \( p_0 = 0 \),

\[
\sum_{j=0}^{r-1} y_j^2 < \sum_{j=0}^{r-1} \left( \delta_{p_1,j} + \theta \delta_{p_2,j} \right)^2 < \mu x^2.
\]

Since \( \log(a_0^2 + 1/8) > 0 \), we see from (3.3) that \( \mu x^2 \leq 1/8 \), hence

\[
\mu x^2/(1 + \theta)^2 < 1/8 \quad \text{and} \quad \sum_{j=0}^{r-1} y_j^2 < a_0^2 + \mu x^2/(1 + \theta)^2.
\]

By Hadamard’s inequality, if \( p_0 = 0 \), then

\[
|\det(A_{r-1})|^2 = |\det(B_{r-1})|^2 < (a_0^2 + 1/8)(\mu x)^{-1},
\]

while if \( p_0 > 0 \), then

\[
|\det(A_{r-1})|^2 < (\mu x)^{r-1}.
\]
By (3.3) \(|\det(A_{-\infty})| < 1\) and by Lemma 3.1, \(a(s) = \sum_{n=0}^{\infty} a_n s^n\) is a rational function whose denominator has degree \(\leq r - 1 < 1/(e\mu^2)<\) (since \(r\) as used in Lemma 3.1 is less than \(r\) as used in this proof). By the Fatou-Hurwitz Lemma [7], we may write \(a(s) = P(s)/Q(s)\) where \(P(s)\) and \(Q(s)\) are relatively prime polynomials with integral coefficients satisfying \(Q(0) = 1\) and \(\deg Q(s) \leq 1/(e\mu^2)<\). Now \(P(s)/Q(s) - \lambda(1 - \theta s)^{-1} = \sum_{n=0}^{\infty} a_n s^n\) is analytic in \(|s| < 1\). Since \(\lambda \neq 0, 1/\theta\) is a pole of \(P(s)/Q(s)\) and \(Q(1/\theta) = 0\). Since \(Q(0) = 1\), \(\theta\) is an algebraic integer. As all other roots of \(Q(s)\) lie in \(|s| > 1\), all conjugates of \(\theta\) (except \(\theta\) itself) lie in \(|s| < 1\) and \(\theta \in \mathbb{R} \cup \mathbb{T}\).

3.4. Lemma. Suppose \(y_1, y_1, y_2, \ldots\) is a sequence of positive numbers satisfying

\[ \sum_{i=1}^{n} y_i \to 0 \quad \text{as} \quad n \to \infty. \]

Put \(d_n = \sup_{n \geq m \geq n} \sum_{i=m}^{n} y_i\). Then if \(i_1 < i_2 < \ldots < i_r\) is a strictly increasing sequence of positive integers, we have

\[ \sum_{j=1}^{r} \sum_{k=1}^{i_j+k} y_{j+k} < \frac{4}{r} \sum_{j=1}^{r} d_j. \]

Proof. Choose \(j \leq r\) and let \(h\) be the largest power of 2 which is \(\leq r/d_j\); let \(l\) be the largest power of 2 which is \(\leq j\); let \(g\) be the largest power of 2 which is \(\leq r\). Since \(i_j \geq j \geq h\),

\[ \sum_{k=1}^{i_j+k} y_{j+k} \leq \delta_j + \delta_{2j} + \delta_{4j} + \ldots + \delta_{2^l j} \leq \delta_j + \delta_{2j} + \delta_{4j} + \ldots + \delta_{2^l j} \]

\[ \leq \delta_j + \delta_{2j} + \delta_{4j} + \ldots + \delta_{g j}. \]

Hence

\[ \sum_{j=1}^{r} \sum_{k=1}^{i_j+k} y_{j+k} \leq \delta_1 + \delta_2 + \delta_4 + \ldots + \delta_{g j} +
\]

\[ + 2(\delta_{2j} + \delta_{4j} + \delta_{8j} + \ldots + \delta_{2^l j}) +
\]

\[ + 4(\delta_{4j} + \delta_{8j} + \delta_{16j} + \ldots + \delta_{2^l j}) +
\]

\[ + 4^2(\delta_{8j} + \delta_{16j} + \ldots + \delta_{2^l j}) +
\]

\[ + \ldots + g \delta_{g j} \]

\[ \leq \delta_1 + \delta_2 + 2 \delta_4 + \ldots + (2g-1) \delta_{g j} \]

\[ \leq \delta_1 + 3 \delta_2 + 4(2 \delta_4 + 4 \delta_8 + \ldots + (g/2) \delta_{g j}) \]

\[ \leq \delta_1 + (\delta_1 + 2 \delta_2) + 4 ((\delta_1 + \delta_3) + (\delta_1 + \delta_5) + \ldots + (\delta_{2g-1} + \ldots + \delta_{2^l j})) \]

\[ \leq 4(\delta_1 + \delta_2 + \ldots + \delta_{g j}). \]

3.5. Lemma. If \(X = (x_{ij})\) is an \(r \times r\) matrix and \(\sum_{i=1}^{r} \sum_{j=1}^{r} |x_{ij}|^2 < r\), then \(|\det(X)| < 1\).

Proof. By Hadamard's inequality and the arithmetic-geometric mean inequality,

\[ |\det(X)|^2 \leq \prod_{i=1}^{r} \sum_{j=1}^{r} |x_{ij}|^2 \leq \left[ \frac{1}{r} \right] \sum_{i=1}^{r} \sum_{j=1}^{r} |x_{ij}|^2 < 1. \]

Proof of Theorem 2.7. By subtracting a constant multiple of \(1/(1 - z)\) from \(a(z)\), if necessary, we may assume that some of the \(a_n\) are \(0 \mod 1\). This will not change the number of distinct values (mod 1) assumed by the \(a_n\). It will replace \(a(z)\) by \((1 - z)\tilde{a}(z)\) and \(\tilde{a}(z)\) by \((1 - z)\tilde{a}(z)\), where \(\tau\) is a constant. Since \(\alpha \leq \beta\), (2.8), (2.9), and (2.10) will remain satisfied. We may assume that \(a(z)\) is not constant. Since \(a(z)\) is regular in a neighborhood of 0 we may divide \(a(z)\) and \(\tilde{a}(z)\) by the same integral power of \(z\) and assume that \(t_0 = 1\). By further dividing \(a(z)\) by an integral power of \(z\), if necessary, we may assume that \(t_0 \neq 0\). We may multiply \(a(z)\) and \(\tilde{a}(z)\) by the same non-zero constant and assume that \(t_0 = 1\). Thus, after appropriate renaming, we may assume the hypotheses of Theorem 2.7 and, in addition, that \(t_0 \neq 0\), \(t_0 = 1\), \(a(z)\) is not constant, and that one of the \(\leq f\) distinct values (mod 1) assumed by the \(a_n\) is 0. If \(N\) is a positive integer we may write

\[ N \alpha_n = c_n + e_n \quad \text{where} \quad \alpha_n \in \mathbb{Z} \quad \text{and} \quad -1/2 < e_n < 1/2. \]

By Dirichlet's theorem [2], p. 14, Th. VII) there are infinitely many choices for \(N\) for which

\[ |e_n| \leq 1/N^{2+1/(r-1)}, \quad n = 0, 1, 2, \ldots \]

where the right hand side of (3.6) is interpreted as 0 if \(f = 1\). We may decrease \(\alpha\) and \(\beta\), if necessary, and assume that \(0 \leq \alpha \leq \beta \leq 1\) and \(\alpha + \beta = 2 - 1/f\).

In the preceding lemmas we use the following notation:

\[ T = \sum_{k=0}^{\infty} |a_k|; \]

\[ N \quad \text{is a positive integer satisfying (3.6)}; \]

\[ \alpha_{m,n} = N \sum_{k=0}^{n} t_k \alpha_{m+n-k}; \]

\[ \beta_{m,n} = -N \sum_{k=0}^{n} t_k \beta_{m+n-k}; \]

\[ \psi_{m,n} = -N \sum_{k=0}^{n} t_k \psi_{m+n-k}; \]

\[ \phi_{m,n} = -N \sum_{k=0}^{n} t_k \phi_{m+n-k}. \]
We shall let \( p_0 < p_1 < \ldots < p_r \) be a strictly increasing, finite sequence of non-negative integers. Define \( f_r = A_r(p_0, p_1, \ldots, p_r) \) to be the \((r+1) \times (r+1)\) matrix whose \((m, n)\) entry is \( \alpha_{mn} \), \( 0 \leq m, n \leq r \).

### 3.7. Lemma
We have \( x_{mn} = u_{m,n} + v_{m,n} + w_{m,n} \).

**Proof.**

\[
\begin{align*}
x_{mn} &= \sum_{k=0}^{n} \sum_{k=0}^{m} t_k t_l c_{m+n-k-l} = \sum_{k=0}^{n} \sum_{k=0}^{m} t_k t_l (N \alpha_{m+n-k-l} - \xi_{m+n-k-l}) \\
&= w_{mn} + N \sum_{k=0}^{n} \sum_{k=0}^{m} t_k \alpha_{m+n-k} - \sum_{k=0}^{m+n-l} t_k \xi_{m+n-k-l} \\
&= w_{mn} + N \sum_{k=0}^{n} \sum_{k=0}^{m} t_k (s_{m+n-k} - \sum_{k=0}^{m+n-l} t_k \xi_{m+n-k-l}),
\end{align*}
\]

since \( \sum_{k=0}^{m+n-l} t_k \xi_{m+n-k-l} = s_{m+n-k-l} \). Continuing,

\[
x_{mn} = w_{mn} + u_{mn} - N \sum_{k=0}^{n} \sum_{k=0}^{m} t_k \alpha_{m+n-l-k} = w_{mn} + u_{mn} + v_{mn}. \quad \blacksquare
\]

### 3.8. Lemma
We have \( \sum_{k=0}^{r} |s_k|^2 = o(\theta^2) \).

**Proof.** By the Cauchy–Schwarz inequality,

\[
\sum_{k=0}^{r} |s_k|^2 \leq \left( \sum_{k=0}^{r} |s_k| \right)^2 \leq \left( \sum_{k=0}^{r} 1 \right) \sum_{k=0}^{r} |s_k|^2 = o(\theta^2).
\]

Let \( \theta \) be the largest power of 2 which is \( \leq r \). Then

\[
\sum_{k=0}^r |s_k| \leq |s_0| + \sum_{k=0}^{r-1} |s_k| \leq |s_0| + \sum_{k=0}^{r-1} 2^{k+1} = o(2^{(r+1)/2}) = o(\theta^2),
\]

since \( \theta < 1 \). \( \blacksquare \)

### 3.9. Lemma
We have

\[
\begin{align*}
&\sum_{m=0}^{r} \sum_{n=0}^{r} |\varphi_m \varphi_n|^2 \leq (r+1)^2 \frac{T^2}{N^2(r+2)(r+1)}; \\
&\sum_{m=0}^{r} \sum_{n=0}^{r} |\psi_m \psi_n|^2 = N^2 o(\theta^2); \\
&\sum_{m=0}^{r} \sum_{n=0}^{r} |\psi_m \psi_n|^2 = N^2 o(\theta^2).
\end{align*}
\]

**Proof.** Since

\[
|\varphi_m \varphi_n|^2 \leq \sum_{k=0}^{r} \sum_{k=0}^{r} |s_k|^2 |s_{m+n-k-l}| \leq T^2 N^2(r+2)(r+1),
\]

the first inequality is clear. Next, since \( n \leq r \), \( \varphi_m \varphi_n \) is the coefficient of \( \varphi_m \varphi_n \) in

\[
N \sum_{k=0}^{r} \sum_{k=0}^{r} t_k t_l s_k s_{l}. \quad \exists \epsilon \in \mathbb{C},
\]

By Parseval's equality with \( \epsilon = e^{i \theta} \), in the following integrals

\[
\sum_{n=0}^{r} |\varphi_m \varphi_n|^2 \leq \frac{N^2}{\theta} \int \left| s_k \right|^2 \left| \sum_{k=0}^{r} t_k s_k \sum_{l=0}^{r} t_l s_l \right|^2 \theta \]

\[
\leq \frac{N^2}{\theta} \int \left| s_k \right|^2 \left| \sum_{k=0}^{r} t_k s_k \right|^2 \left| \sum_{l=0}^{r} t_l s_l \right|^2 \theta \]

\[
\leq \frac{N^2}{\theta} \int \left| s_k \right|^2 \left| \sum_{k=0}^{r} \left| s_k \right|^2 \right|^2 \theta \]

\[
\leq \frac{N^2}{\theta} \int \left| s_k \right|^2 \theta \]

Put \( c_\theta = \sup_{|\theta| < 1} \sum_{k=0}^{r} |s_k|^2 \). Since \( 0 < a \leq \frac{1}{2} \), and \( c_\theta = o(1/\theta^2) \), we have

\[
\sum_{k=0}^{r} c_\theta = o(\theta^{-2}).
\]

Next, using Lemma 4.5 with \( \theta = |s_0|^2 \)

\[
\sum_{n=0}^{r} \sum_{n=0}^{r} \left| \sum_{k=0}^{r} \sum_{k=0}^{r} \left| s_k \right|^2 \right|^2 \leq N^2 T^2 \sum_{n=0}^{r} \sum_{n=0}^{r} \left| s_k \right|^2 \left| \sum_{k=0}^{r} t_k s_k \left| \sum_{l=0}^{r} t_l s_l \right|^2 \theta \]

\[
\leq N^2 T^2 \sum_{n=0}^{r} c_\theta = N^2 o(\theta^{-2}) = N^2 o(\theta^{-2}),
\]

since \( \alpha + \beta = 2 - 1/\theta \) and \( \beta \leq 1 \) implies \( \alpha > 1 - 1/\theta \) or \( 1 - \alpha > 1/\theta \). This proves the second inequality. Similarly, \( \psi_{m,n} \) is the coefficient of \( \varphi_{m,n} \) in

\[
-N \sum_{k=0}^{r} \sum_{k=0}^{r} t_k t_l s_k s_{l} \theta \]

and proceeding as for \( \psi_{m,n} \), using Lemmas 4.5 and 3.8, we obtain

\[
\sum_{n=0}^{r} \sum_{n=0}^{r} |\varphi_m \varphi_n|^2 \leq N^2 \left( \sum_{k=0}^{r} |s_k|^2 \right)^2 \sum_{k=0}^{r} |s_k|^2 \leq N^2 (o(\theta^2))^2 o(\theta^{-2}) \]

\[
= N^2 (r^2 - \theta^2) = N^2 o(\theta^{-2}). \quad \blacksquare
\]
Proof of Theorem 2.7. (Completed.) If \( f \gg 2 \), put
\[
X_r = \sum_{i=0}^r \sum_{m=0}^r a^2_{m,n}, \quad \delta = 1/(36N^4) \quad \text{and} \quad r = \lceil 3N^{2/3} \rceil
\]
where \( N \) will be chosen later and chosen so large that \( r \gg 2 \). Since
\[
2a^2_{m,n} \leq 3(w^2_{m,n} + v^2_{m,n} + w^2_{m,n}),
\]
we have by Lemma 3.9
\[
X_r \leq 6 \delta T^* \{N^{2/3} \delta^{-1} \} + N^2 \hat{o}(r^{1/2}) \leq 6 \{N^{2/3} \delta^{-1} \} + \hat{o}(N^2 \delta^{-1/2}) \leq 12 \delta T^* \delta^{-1/2} + \hat{o}(r^{1/2}).
\]
Thus \( X_r \), which is the sum of the squares of \( A_n \), is \( \ll \tau^2 \delta^{-1} \) for all sufficiently large \( r \). Choose \( N \) satisfying (3.6) so large that \( X_r \ll \tau^2 \delta^{-1} \) and that \( r \gg 2 \).

Then by Lemma 3.5,
\[
| \text{det} \{A, (p_0, p_1, \ldots, p_f) \}| < 1,
\]
and this is true for all \( p_0 < p_1 < \ldots < p_f \). By Lemma 3.1, \( c(x) = \sum_{n=0}^\infty c_n x^n \) is rational. If \( f = 1 \), we choose \( N = 1 \) and then by Lemma 3.9, \( X_r = o(r) \).

Then exactly as above, using Lemmas 3.5 and 3.1, \( c(x) = \sum_{n=0}^\infty c_n x^n \) is rational.

Put \( e(x) = \sum_{n=0}^\infty e_n x^n \) so that \( N a(x) = o(x) + e(x) \). If \( c(x) = P(x)/Q(x) \) where \( P(x) \) and \( Q(x) \) are polynomials with \( Q(0) = 1 \), then
\[
e(x) = d(x) e(x)
\]
where
\[
d(x) = N a(x) Q(x) - i(x) P(x) \quad \text{and} \quad e(x) = i(x) Q(x).
\]
If \( d(x) = \sum_{n=0}^\infty d_n x^n \) and \( e(x) = \sum_{n=0}^\infty e_n x^n \), then
\[
\lim_{n \to \infty} d_n = 0, \quad e_0 = 1 \quad \text{and} \quad \sum_{n=0}^\infty |e_n| < \infty.
\]
There are \( f \) distinct \( e_n \). Put \( \sigma = \min |e_n - e_0| \) and \( \tau = \max |e_n| \). Choose \( n_0 \) so large that \( \tau \sum_{n=n_0+1}^\infty |e_n| < \sigma/4 \) and \( |d_n| < \sigma/4 \) for all \( n \geq n_0 \). Now
\[
d_n = \sum_{m=0}^n e_m e_{n-m} \quad \text{and} \quad n \geq n_0 \sum_{m=0}^n |e_m e_{n-m} - d_n| < \sigma/4, \quad \text{hence}
\]
\[
|e_n - \sum_{m=0}^{n_0} (-e_m) e_{n-m}| < \sigma/2.
\]

Thus \( e_n \) is determined by the previous \( (n_0+1) \) \( e_i \) and hence the sequence \( e_0, e_1, e_2, \ldots \) is eventually periodic with period \( \ll f^{n_0 + 1} \). Thus there exists \( q \) so that \( e_n = e_{n-q} \) for all sufficiently large \( n \) and \( \varepsilon(x) = q(x)/(1-x^q) \) where \( q(x) \) is a polynomial. Hence \( \varepsilon(x) \) and \( a(x) = (\varepsilon(x) + \varepsilon(1-x))/N \) are rational functions. By the Fakou–Hurwitz lemma [7] we may assume that \( Q(z) \) has integral coefficients, hence that \( a(z) \) may be written in the form \( a(z) = \psi(z)/v(z) \) where \( \psi(z) = (1-z^q)Q(z) \) has integral coefficients and \( v(0) = 1 \). Write \( v(z) = \sum_{i=0}^l \bar{v}_i z^i \) with \( v_0 = 1 \). Then
\[
\sum_{i=0}^l \bar{v}_i a_{n-i} = 0 \quad \text{for all} \quad n > n_1 = \deg(\psi(z)).
\]

Write \( a_n = [a_n] + \gamma_n \). There are \( \leq f \) distinct \( \gamma_n \) and
\[
\sum_{i=0}^l \bar{v}_i \gamma_{n-i} = 0 \pmod{1} \quad \text{for} \quad n > n_1.
\]

Thus \( \gamma_n = -\sum_{i=1}^l \bar{v}_i \gamma_{n-i} \pmod{1} \) if \( n > n_1 \). Hence \( \gamma_n \) is determined by the previous \( l \) \( \gamma \)'s and the sequence \( \gamma_0, \gamma_1, \gamma_2, \ldots \) is ultimately periodic. Hence \( \sum_{n=0}^\infty [a_n] z^n = a(z) - \sum_{n=0}^\infty [a_n] z^n \) is a rational function. \( \blacksquare \)

References


UNIVERSITY OF CALIFORNIA
Los Angeles, California, U.S.A.

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