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On theta functions of real algebraic number fields

by

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Introduction. Such theta functions appear at first in Hecke's early work. A more general and systematic treatment was begun by Kloosterman [4]. But he did not succeed in finding the largest groups under which the theta functions are invariant. Moreover he only treated theta functions of quadratic modules which are free over the ring of integers of the ground field. To our surprise the subject does not seem to have been taken up since then. So it is time to make a new start of the theory, and we do this with inclusion of Schoeneberg's generalisation of theta series [6]. We feel it both justified and practical to attach the names of Kloosterman and Schoeneberg to these functions. To deal with them is the object of the first chapter.

In the second chapter we want to consider the behaviour of Hilbert modular forms and especially of the Kloosterman–Schoeneberg theta series under the Hecke operators. This was first done by Herrmann [3]. But his approach is rather complicated, and we prefer to take up the subject from the beginning instead of translating Herrmann's results into our language. However, one merit of Herrmann has to be noted, that he stressed the necessity of considering h groups of transformations commensurable with the Hilbert modular group, where h is the ideal class number of the underlying field. We may justly call these more general groups also Hilbert modular groups.

Another treatment of the Hecke operators, and even for more general fields has been given by Weil [7]. While his adelic method comprises the ideal classes of the ground field in the characters employed, we proceed in a more earthbound way, aiming at a knowledge of the modular forms as explicit as possible. This will prove useful in applications which we have in mind. On the other hand, the connection between modular forms and Dirichlet series which is Weil's chief concern, will not be touched here.

In § 7 we introduce the Brandt matrices of definite quaternion algebras over real number fields. They enjoy the same basic features with the Hecke operators. There is little to be said which is not yet contained in our earlier treatment [2] of Brandt matrices of quaternion algebras over the rational field. In § 8 we study the symmetry properties of Hecke operators resulting from Petersson's scalar product. In § 9 we consider matrices of Kloosterman–Schoeneberg theta functions whose Fourier coefficients are Brandt matrices. As in the case of the rational ground field [2], the Brandt matrices turn out to be representations of the Hecke operators. The final result, Theorem 7, can be used to prove that the Brandt matrices even yield a faithful representation of the Hecke operators in the spaces of Hilbert modular forms of weight $k > 2$. Indeed, M. F. Vigneras has informed us that she has a proof for this which will be published elsewhere.

We intend to continue this work. It opens a way to studying the symmetric Hilbert modular forms in a narrow sense, namely those which remain symmetric after application of the Hecke operators. The existence of these has first become apparent by the papers of Doi and Naganuma [1], [5]. Also the problem of finding bases of the spaces of modular forms in one variable of real "Nebentype" (in Hecke's sense) seems closely connected with the Naganuma-map.

Chapter I

The Kloosterman–Schoeneberg theta series

§ 1. Notations concerning the metric space

k a totally real algebraic number field of degree $[k:\boldsymbol{Q}] = n$. Its elements will be written with greek letters $\alpha, \beta, \ldots,$ and $\alpha^i, \beta^i, \ldots$ $(i = 1, \ldots, n)$ mean its conjugates in $\boldsymbol{R}$. Trace and norm with respect to $\boldsymbol{Q}$ are $\operatorname{tr}(\alpha)$, $n(\alpha)$. Rational numbers will be denoted by italics.

V a vector space over k of dimension m. The vectors will be denoted by small german letters. If necessary the vectors corresponding to an $\mathfrak{f} \in V$ in the conjugate spaces V^i will be written $\mathfrak{f}^i$.

$q(\mathfrak{x})$ a totally positive quadratic form on V with coefficients in k. The conjugates with respect to $\boldsymbol{Q}$ are $q^i(\mathfrak{x})$, where the superscript of $\mathfrak{x}$ may be left out for sake of simplicity.

$q(\mathfrak{x}, \mathfrak{y}) = \frac{1}{2}\big(q(\mathfrak{x}+\mathfrak{y}) - q(\mathfrak{x}) - q(\mathfrak{y})\big)$ the corresponding bilinear form such that

$$q(\mathfrak{x}, \mathfrak{x}) = q(\mathfrak{x}).$$

$\mathfrak{b}_\mu$ $(\mu = 1, \ldots, m)$ a basis of V/k such that the generic vector is

$$\mathfrak{x} = \sum \xi_\mu \mathfrak{b}_\mu.$$

$F = \big(q(\mathfrak{b}_\mu, \mathfrak{b}_\nu)\big)$ a symmetric matrix. With the row vector $\{\xi\} = \{\xi_1, \ldots, \xi_m\}$ we can write

$$q(\mathfrak{x}) = \{\xi\} F \{\xi\}^t = F[\xi] \quad (t = \text{transpose}).$$

S^i for $i = 1, \ldots, n$ a set of arbitrary real matrices satisfying

$$(S^i)^t S^i = F^i \quad (i = 1, \ldots, n).$$

They will be kept fixed throughout Chapter I.

$p^i(\eta) = p^i(\eta_1, \ldots, \eta_m)$ a set of arbitrary homogeneous polynomials of degrees l_i which satisfy the Laplacian

$$\Delta p^i(\eta) = \sum_\mu \frac{\partial^2 p^i(\eta)}{\partial \eta_\mu^2} = 0.$$

In other words, the $p^i(\eta)$ are spherical harmonics.

$$P(\eta_1^1, \ldots, \eta_m^1, \ldots, \eta_1^n, \ldots, \eta_m^n) = \prod_{i=1}^n p^i(\eta_1^i, \ldots, \eta_m^i)$$

with mn independent variables η_μ^i is also a spherical harmonic.

γ_ν $(\nu = 1, \ldots, n)$ a basis of $k/\boldsymbol{Q}$.

$$G = (\gamma_\nu^i) = \begin{pmatrix} \gamma_1^1 & \cdots & \gamma_1^n \\ \cdot & \cdot & \cdot \\ \gamma_n^1 & \cdots & \gamma_n^n \end{pmatrix}$$

with the upper index i counting the number of the column and the lower index counting the number of the row.

$$G_m = G \times E_m.$$

The "blown up" matrix G with $\gamma_\nu^i E_m$ instead of γ_ν^i, E_m the m-rowed unit matrix.

z^i $(i = 1, \ldots, n)$ independent complex variables taking values only in the upper half plane.

$$Z_0(z) = \begin{pmatrix} z^1 F^1 & & \\ & \ddots & \\ & & z^n F^n \end{pmatrix}$$

with n m-rowed blocks along the diagonal, otherwise 0.

$$Z_1(z) = G_m Z_0(z) G_m^t.$$

$Z_0(z)$ and $Z_1(z)$ are complex mn-rowed symmetric matrices lying in the upper Siegel half-space.

We decompose the generic vector $\mathfrak{x}$ of V in the way

$$\mathfrak{x} = \sum x_{\mu\nu}\mathfrak{b}_\mu\gamma_\nu = \sum \xi_\mu \mathfrak{b}_\mu$$

with $x_{\mu\nu} \in \boldsymbol{Q}$. Then, with the row vector

$$\{x\} = \{x_{11}, \ldots, x_{m1}, \ldots, x_{1n}, \ldots, x_{mn}\}$$

we have

$$\operatorname{tr}\big(zq(\mathfrak{x})\big) = \sum_i z^i q^i(\xi^i) = \{x\}Z_1(z)\{x\}^t,$$

where $\operatorname{tr}\big(zq(\mathfrak{x})\big)$ is a reasonable abbreviation although the z^i are not numbers in k^i.

§ 2. Notations concerning the lattice

$\mathfrak{o}$ the maximal order of k, $\mathfrak{d}$ the different of k.

$\mathfrak{L}$ a lattice, i.e. a finite $\mathfrak{o}$-module of rank m.
$\mathfrak{l}_1, \ldots, \mathfrak{l}_{mn}$ a basis of $\mathfrak{L}$ with respect to $\boldsymbol{Z}$,

$$\mathfrak{l}_\varrho = \sum_{\mu\nu} t_{\varrho,\mu\nu}\mathfrak{b}_\mu\gamma_\nu$$

where the pairs of indices $\mu\nu$ are ordered $11, \ldots, m1, \ldots, 1n, \ldots, mn$. Then

$$T = (t_{\varrho,\mu\nu})$$

is a non-singular square matrix with rational coefficients.

$$Z(z) = TZ_1(z)T^t.$$

With the generic vector $\mathfrak{x} = \sum x_\varrho \mathfrak{l}_\varrho$ of $\mathfrak{L}$ and $\{x\}$ the row vector $\{x_1, \ldots, x_{mn}\}$ we have

$$\operatorname{tr}\big(zq(\mathfrak{x})\big) = \{x\}Z(z)\{x\}^t.$$

The *norm* of the lattice $\mathfrak{L}$ is defined by (mind the factor $\frac{1}{2}$)

$$N(\mathfrak{L}) = \tfrac{1}{2}\big(\text{g.c.d.}\, q(\mathfrak{x}) \text{ for all } \mathfrak{x} \in \mathfrak{L}\big).$$

The *complement* of $\mathfrak{L}$ is the greatest lattice $\tilde{\mathfrak{L}}$ defined by

$$q(\mathfrak{L}, \tilde{\mathfrak{L}}) \subseteq \mathfrak{d}^{-1},$$

where $\mathfrak{d}$ is the different of k.

LEMMA 1. *If $\mathfrak{x} = \sum x_\varrho \mathfrak{l}_\varrho$ lies in $\mathfrak{L}$, the vector $\mathfrak{y} = \sum y_\varrho \mathfrak{l}_\varrho$, defined by*

$$Z(1)\{y\}^t = \{x\}^t,$$

lies in $\tilde{\mathfrak{L}}$, and vice versa.

Proof. The statement is equivalent to: $\operatorname{tr}\big(q(\mathfrak{x}, \mathfrak{y})\big) \in \boldsymbol{Z}$ for all $\mathfrak{x} \in \mathfrak{L}$ iff (= if and only if) $\mathfrak{y} \in \tilde{\mathfrak{L}}$ and vice versa. This is indeed so because

$$\operatorname{tr}\big(q(\mathfrak{x}, \mathfrak{y})\big) = \sum x_\varrho y_\sigma t_{\varrho,\mu\nu} t_{\sigma,\varkappa\lambda} \operatorname{tr}\big(q(\mathfrak{b}_\mu, \mathfrak{b}_\varkappa)\gamma_\nu\gamma_\lambda\big) = \sum x_\varrho y_\sigma z_{\varrho\sigma}$$

with $(z_{\varrho\sigma}) = Z(1)$.

LEMMA 2. *The ideal*

$$\mathfrak{n}(\mathfrak{L}) = N(\mathfrak{L})^{-1}N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-2}$$

is integral. It is called the level of the lattice.

Proof. From the definition follows: $\mathfrak{K} = \mathfrak{d}\tilde{\mathfrak{L}}$ is the greatest lattice such that $q(\mathfrak{L}, \mathfrak{K}) \subseteq \mathfrak{o}$. On the other hand, we have $q\big(\mathfrak{L}, N(\mathfrak{L})^{-1}\mathfrak{L}\big) \subseteq \mathfrak{o}$, and therefore

$$N(\mathfrak{L})^{-1}\mathfrak{L} \subseteq \mathfrak{d}\tilde{\mathfrak{L}}.$$

This implies

$$N\big(N(\mathfrak{L})^{-1}\mathfrak{L}\big) = N(\mathfrak{L})^{-1} \subseteq N(\mathfrak{d}\tilde{\mathfrak{L}}) = N(\tilde{\mathfrak{L}})\mathfrak{d}^2,$$

whence the contention.

LEMMA 3. $N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}\tilde{\mathfrak{L}} \subseteq \mathfrak{L}$.

Proof. The relation between $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ is involutorial. Thus the greatest lattice $\mathfrak{K}$ with $q(\mathfrak{K}, \tilde{\mathfrak{L}}) \subseteq \mathfrak{d}^{-1}$ is $\mathfrak{K} = \mathfrak{L}$. Now $q\big(N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}\tilde{\mathfrak{L}}, \tilde{\mathfrak{L}}\big) = \mathfrak{d}^{-1}$ and therefore $N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}\tilde{\mathfrak{L}} \subseteq \mathfrak{L}$.

LEMMA 4. *If $N(\mathfrak{L})$ is integral, $\mathfrak{L} \subseteq \tilde{\mathfrak{L}}$ and $\mathfrak{n}'(\mathfrak{L}) = N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}$ is integral.*

LEMMA 5. *Let $\gamma \neq 0$. Then the complements of $\mathfrak{L}$ with respect to the quadratic forms $q(\mathfrak{x})$ and $\gamma q(\mathfrak{x})$ are connected by the equation*

$$\tilde{\mathfrak{L}}_{\gamma q} = \gamma^{-1}\tilde{\mathfrak{L}}_q.$$

Proof. $\tilde{\mathfrak{L}}_{\gamma q}$ and $\tilde{\mathfrak{L}}_q$ are the largest lattices with

$$\gamma q(\mathfrak{L}, \tilde{\mathfrak{L}}_{\gamma q}) = q(\mathfrak{L}, \gamma\tilde{\mathfrak{L}}_{\gamma q}) = q(\mathfrak{L}, \tilde{\mathfrak{L}}) = \mathfrak{d}^{-1}.$$

§ 3. The theta functions, first properties.

From now on we assume the dimension

$$m = 2k$$

as even. We also make the assumption that $N(\mathfrak{L})$ be integral, but this will be dropped at the end. With the matrices introduced in § 1 and 2 we form

$$U(z) = \begin{pmatrix} \sqrt{-iz^1}\,S^1 & & \\ & \ddots & \\ & & \sqrt{-iz^n}\,S^n \end{pmatrix} G_m^t T^t;$$

the signs of the square roots are arbitrary, but they are uniquely fixed since the z^i lie in the upper half plane. These matrices satisfy

$$(1)\qquad -iZ(z) = U(z)^t\, U(z).$$

Let $P(\eta_\mu^1, \ldots, \eta_\mu^n)$ be the spherical harmonic polynomial of § 1. Then, with $\{x\} = \{x_1, \ldots, x_{mn}\}$ as in § 2,

$$(2)\qquad \begin{aligned} Q(x) &= (-iz^1)^{-l_1/2} \ldots (-iz^n)^{-l_n/2} P\big(U(z)\{x\}^t\big) \\ &= n(-iz)^{-(l/2)} P\big(U(z)\{x\}^t\big) \end{aligned}$$

(where a reasonable abbreviation is used) is a homogeneous polynomial in the x_ϱ not depending on the variables z^i. With these definitions we form

$$(3)\qquad \vartheta(z, \mathfrak{r}) = \vartheta(z, \mathfrak{r}, \mathfrak{L}, P) = \sum_{\mathfrak{x}\in\mathfrak{L}} Q(\mathfrak{x}+\mathfrak{r})\, e^{\pi i \operatorname{tr}(zq(\mathfrak{x}+\mathfrak{r}))}$$

where $\mathfrak{r}$ is a vector in $\tilde{\mathfrak{L}}$. This can be written as well as

$$\vartheta(z, \mathfrak{r}) = n(-iz)^{-(l/2)} \sum_{\{x\}\in \mathbf{Z}^{mn}} P\big(U(z)\{x+r\}^t\big) e^{2\pi i Z(z)[x+r]}$$

where $Z(z)[x]$ is the quadratic form with the matrix $Z(z)$, the sum extended over all vectors $\{x\} \in \mathbf{Z}^{mn}$.

The following functional equation is obvious

$$(4)\qquad \vartheta(z+\beta, \mathfrak{r}) = e^{\pi i \operatorname{tr}(\beta q(\mathfrak{r}))}\, \vartheta(z, \mathfrak{r}) \qquad \text{for } \beta \in \mathfrak{o}.$$

The inversion formula

$$(5)\qquad \vartheta(z, \mathfrak{r}) = \frac{i^{-kn}}{\sqrt{\Phi}}\, n(-z)^{-k-(l)} \sum_{\{x\}\in\mathbf{Z}^{mn}} Q\big(Z(1)^{-1}\{x\}^t\big) e^{-\pi i Z(z)^{-1}[x] + 2\pi i \sum x_\varrho r_\varrho}$$

with

$$\Phi = \det\big(\operatorname{tr}\big(q(\mathfrak{l}_\mu \mathfrak{l}_\nu)\big)\big)$$

we derive from [2], formula (8) on p. 81. Proof: We let all z^i be purely imaginary. Then $U(z)$ and $-iZ(z)$ become real, the latter positive definite. In that formula we insert $-iZ(z)$ and $U(z)$ for F and S and put $z = i$. Then

$$\begin{aligned} \Theta(i, \mathfrak{r}) &= \sum_{\{x\}} P\big(U(z)\{x+r\}^t\big) e^{2\pi i Z(z)[x+r]} \\ &= \frac{i^{-kn-l_1-\ldots-l_n}}{\sqrt{\Phi}} \sum_{\{x\}} P\big(U(z)^{-t}\{x\}^t\big) e^{-\pi i Z(z)^{-1}[x] + 2\pi i \sum x_\varrho r_\varrho} \end{aligned}$$

which can be analytically continued to all z^i in the upper half plane. From the definitions we take

$$U(z)^{-t} = U\left(\frac{-1}{z}\right) Z(1)^{-1}$$

and from (2), (3)

$$\Theta(i, \mathfrak{r}) = n(-iz)^{(l/2)}\, \vartheta(z, \mathfrak{r}).$$

With this our last formula becomes

$$\vartheta(z, \mathfrak{r}) = \frac{i^{-kn}}{\sqrt{\Phi}}\, n(-z)^{-k-(l)} \sum_{\{x\}} Q\big(Z(1)^{-1}\{x\}^t\big) e^{-\pi i Z(z)^{-1}[x] + 2\pi i \sum x_\varrho r_\varrho}$$

or (5).

In (5) we write

$$Z(1)^{-1}\{x\}^t = \{x'\}^t + \{s\}^t$$

such that, if $\{x\}$ runs over $\mathbf{Z}^{mn}$, also $\{x'\}$ runs over $\mathbf{Z}^{mn}$ and $\{s\}$ runs over a system of representatives of $Z(1)^{-1}\mathbf{Z}^{mn} \bmod \mathbf{Z}^{mn}$ (remember that $N(\mathfrak{L})$ has been assumed integral and therefore $Z(1)$ has integral coefficients). From Lemma 1 we take now: if $\mathfrak{x} = \sum x_\varrho \mathfrak{l}_\varrho \in \mathfrak{L}$, then $\sum x'_\varrho \mathfrak{l}_\varrho + \sum s_\varrho \mathfrak{l}_\varrho = \mathfrak{x}' + \mathfrak{s}$ runs over all vectors of $\tilde{\mathfrak{L}}$, and especially $\mathfrak{x}'$ again over $\mathfrak{L}$ and $\mathfrak{s}$ over a set of representatives of $\tilde{\mathfrak{L}} \bmod \mathfrak{L}$. With this in mind we can rewrite (5) as

$$(6)\qquad \vartheta(z, \mathfrak{r}) = \frac{i^{-kn}}{\sqrt{\Phi}}\, n(z)^{-k-(l)} \sum_{\mathfrak{s}\in\tilde{\mathfrak{L}}/\mathfrak{L}} e^{2\pi i \operatorname{tr}(q(\mathfrak{r},\mathfrak{s}))}\, \vartheta\left(\frac{-1}{z}, \mathfrak{s}\right).$$

§ 4. The final functional equations. We want to know the behaviour of $\vartheta(z, \mathfrak{r})$ under the transformations

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta}$$

with

$$\alpha\delta - \beta\gamma = 1; \quad \alpha, \delta \in \mathfrak{o}; \quad \beta \in N(\mathfrak{L})^{-1}\mathfrak{d}^{-1}, \quad \gamma \in N(\mathfrak{L})\mathfrak{d}.$$

Pursuing this aim we follow the well-known procedure shown by Hermite. At first we assume γ integral and totally positive and put

$$\frac{\alpha z + \beta}{\gamma z + \delta} = \frac{\alpha}{\gamma} - \frac{1}{\gamma(\gamma z + \delta)}.$$

Now we decompose the summation vector $\mathfrak{x}$ in (3) in the way

$$\mathfrak{x} = \mathfrak{x}_0 + \gamma\mathfrak{y}$$

and write

$$\mathfrak{x}+\mathfrak{r} = \gamma(\mathfrak{y}+\mathfrak{s}) \quad \text{with } \mathfrak{s} = \frac{1}{\gamma}(\mathfrak{x}_0+\mathfrak{r}).$$

Here $\mathfrak{y}$ also runs over $\mathfrak{L}$ and $\mathfrak{x}_0$ over a system of representatives of the residue classes of $\mathfrak{L} \bmod \gamma\mathfrak{L}$. This yields

$$\vartheta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \mathfrak{r}\right) = \sum_{\mathfrak{x}_0} e\left(\frac{\alpha}{\gamma} q(\mathfrak{x}_0+\mathfrak{r})\right) \vartheta_{q\gamma}\left(\frac{-1}{\gamma z+\delta}, \mathfrak{s}\right)$$

where the following abbreviation has been used:

$$e(x) = e^{\pi i \operatorname{tr}(x)};$$

furthermore, the theta function on the right has been formed with the quadratic form $\gamma q(\mathfrak{x})$ instead of $q(\mathfrak{x})$. Due to Lemma 5, $\mathfrak{s}$ is a vector in $\tilde{\mathfrak{L}}_{\gamma q}$. Thus (6) can be applied to transform the right hand side into (simultaneously with $q \to \gamma q$ we have $\Phi \to n(\gamma)^m \Phi$)

$$\frac{i^{-kn}}{n(\gamma)^k \sqrt{\Phi}} n(\gamma z+\delta)^{k+(l)} \sum_{\mathfrak{x}_0, \mathfrak{t}} e\left(\frac{\alpha}{\gamma} q(\mathfrak{x}_0+\mathfrak{r}) + 2q(\mathfrak{x}_0+\mathfrak{r}, \mathfrak{t})\right) \vartheta_{\gamma q}(\gamma z+\delta, \mathfrak{t})$$

where $\mathfrak{t}$ runs over $\tilde{\mathfrak{L}}_{\gamma q}/\mathfrak{L} = \gamma^{-1}\tilde{\mathfrak{L}}/\mathfrak{L}$ (Lemma 5). Equation (4) yields

$$\begin{aligned}\vartheta_{\gamma q}(\gamma z+\delta, \mathfrak{t}) &= e(\gamma\delta q(\mathfrak{t}))\, \vartheta_{\gamma q}(\gamma z, \mathfrak{t}) \\ &= e(\gamma\delta q(\mathfrak{t})) \sum_{\mathfrak{x} \in \mathfrak{L}} Q(\mathfrak{x}+\mathfrak{t})\, e(z\gamma^2 q(\mathfrak{x}+\mathfrak{t})).\end{aligned}$$

Here we put

$$\mathfrak{x}+\mathfrak{t} = \frac{1}{\gamma}(\mathfrak{y}+\mathfrak{u})$$

and let $\mathfrak{y}, \mathfrak{u}$ run over $\mathfrak{L}, \tilde{\mathfrak{L}}/\mathfrak{L}$ respectively. Then both sides run over $\gamma^{-1}\tilde{\mathfrak{L}}$. Furthermore we have

$$q(\mathfrak{x}_0+\mathfrak{r}, \mathfrak{t}) \equiv q(\mathfrak{x}_0+\mathfrak{r}, \mathfrak{x}+\mathfrak{t}) \equiv \frac{1}{\gamma} q(\mathfrak{x}_0+\mathfrak{r}, \mathfrak{y}+\mathfrak{u}) \bmod 2\mathfrak{d}^{-1}$$

and

$$\gamma q(\mathfrak{t}) \equiv \gamma q(\mathfrak{x}+\mathfrak{t}) \equiv \frac{1}{\gamma} q(\mathfrak{y}+\mathfrak{u}) \bmod 2\mathfrak{d}^{-1}.$$

We now assume γ prime to $\mathfrak{n}'(\mathfrak{L}) = N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}$ which is integral because of Lemma 4. We choose $\gamma' \in \mathfrak{o}$ such that $\gamma\gamma' \equiv 1 \bmod \mathfrak{n}'(\mathfrak{L})$. Then the vectors $\mathfrak{r}' = \gamma\gamma'\mathfrak{r}$ and $\mathfrak{u}' = \gamma\gamma'\mathfrak{u}$ also run over $\tilde{\mathfrak{L}}/\mathfrak{L}$. Using the last results

and

$$\begin{aligned}&\frac{\alpha}{\gamma} q(\mathfrak{x}_0+\mathfrak{r}') + \frac{2}{\gamma} q(\mathfrak{x}_0+\mathfrak{r}', \mathfrak{y}+\mathfrak{u}') + \frac{\delta}{\gamma} q(\mathfrak{y}+\mathfrak{u}') \\ &\qquad \equiv \alpha\gamma' q(\mathfrak{r}) + 2\gamma' q(\mathfrak{r}, \mathfrak{u}) + \delta\gamma' q(\mathfrak{u}) + \frac{\delta}{\gamma} q(\alpha\mathfrak{x}_0+\mathfrak{y}) \bmod 2\mathfrak{d}^{-1}\end{aligned}$$

we arrive at

$$(7) \quad \begin{aligned}&\vartheta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \mathfrak{r}\right) n(\gamma z+\delta)^{-k-(l)} \\ &\qquad = w(\gamma) \sum_{\mathfrak{u}} e\big(\alpha\gamma' q(\mathfrak{r}) + 2\gamma' q(\mathfrak{r}, \mathfrak{u}) + \delta\gamma' q(\mathfrak{u})\big)\, \vartheta(z, \mathfrak{u})\end{aligned}$$

with

$$(8) \quad w(\gamma) = \frac{i^{-kn}}{n(\gamma)^k \sqrt{\Phi}} \sum_{\mathfrak{x}_0 \in \mathfrak{L}/\gamma\mathfrak{L}} e^{\pi i \operatorname{tr}\left(\frac{\alpha}{\gamma} q(\mathfrak{x}_0)\right)}.$$

This is a Gaussian sum which can be evaluated by well known means. We state here only so much that it takes only values ± 1, that it is 1 if $q(\mathfrak{x})$ has square discriminent in k, and that it does not depend on α.

Now we insert $\alpha \equiv \delta \equiv 0 \bmod \mathfrak{n}'(\mathfrak{L}) = N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1}$ into (7) and compare the result with (6). This leads to

$$\vartheta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \mathfrak{r}\right) n(\gamma z+\delta)^{-k-(l)} = w(\gamma)\, \vartheta\left(\frac{-1}{z}, \delta\mathfrak{r}\right) n(-z)^{-k-(l)}$$

or, with an obvious change of notations,

$$(9) \quad \vartheta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, \mathfrak{r}\right) \prod_{i=1}^{n} (\gamma^i z^i + \delta^i)^{-k-l_i} = w(\delta)\, \vartheta(z, \delta\mathfrak{r})$$

under the following conditions:

$$\alpha\delta - \beta\gamma = 1; \quad \alpha, \delta \in \mathfrak{o}; \quad \beta, \gamma \in \mathfrak{n}'(\mathfrak{L}); \quad \delta \gg 0.$$

The condition $\delta \gg 0$ can be omitted if we can prove: every such transformation is a product

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha_1 & \beta_1 \\ \gamma_1 & \delta_1\end{pmatrix}\begin{pmatrix}\alpha_2 & \beta_2 \\ \gamma_2 & \delta_2\end{pmatrix} \quad \text{with } \delta_1, \delta_2 \gg 0.$$

We have to find $\alpha_1, \ldots, \delta_1$ such that

$$\alpha_1\delta - \gamma_1\beta = \delta_2 \gg 0, \quad \alpha_1\delta_1 - \beta_1\gamma_1 = 1$$

with $\beta_1 \equiv \gamma_1 \equiv 0 \bmod \mathfrak{n}'(\mathfrak{L})$, $\delta_1 \gg 0$. That this is always possible is an easy exercise.

We are particularly interested in the case $\mathfrak{r} = 0$. It is evident that then $\vartheta(z+\beta) = \vartheta(z)$ for $\beta \in N(\mathfrak{L})^{-1}\mathfrak{d}^{-1}$. Therefore β can always be allowed such. Also the condition that $N(\mathfrak{L})$ be integral can be relaxed. If $N(\mathfrak{L})$ is not integral, we take an integral $\mu \gg 0$ such that $\mu N(\mathfrak{L}) \subseteq \mathfrak{o}$. Then transformations of $\frac{z}{\mu}$ with the properties

$$\alpha,\delta \in \mathfrak{o};\quad \beta \in \mu^{-1}N(\mathfrak{L})^{-1}\mathfrak{d}^{-1};\quad \gamma \in \mu N(\tilde{\mathfrak{L}})^{-1}\mathfrak{d}^{-1} = \mu\mathfrak{n}(\mathfrak{L})N(\mathfrak{L})\mathfrak{d}$$

are allowed or transformations of z with

$$(10)\qquad \alpha\delta-\beta\gamma = 1;\quad \alpha,\delta \in \mathfrak{o};\quad \beta \in N(\mathfrak{L})^{-1}\mathfrak{d}^{-1};\quad \gamma \in N(\mathfrak{L})\mathfrak{d}\mathfrak{n}(\mathfrak{L}).$$

So we have proved

THEOREM 1. *The theta function*

$$\vartheta(z) = \sum_{\mathfrak{x}\in\mathfrak{L}} Q(\mathfrak{x})e^{\pi i\,\mathrm{tr}(zq(\mathfrak{x}))}$$

satisfies the functional equations (9) *with* $\mathfrak{r} = 0$ *under the conditions* (10), *where* $\mathfrak{d}$ *is the different of* k, $N(\mathfrak{L})$ *the norm of* $\mathfrak{L}$, *and* $\mathfrak{n}(\mathfrak{L})$ *the level of* $\mathfrak{L}$, *defined in* § 2.

The character is a Gaussian sum

$$w(\delta) = \frac{i^{-mn/2}}{n(\gamma)^{m/2}\sqrt{\Phi}} \sum_{\mathfrak{x}\in\mathfrak{L}/\delta\mathfrak{L}} e^{\pi i\,\mathrm{tr}(\delta^{-1}q(\mathfrak{x}))},\qquad \Phi = \det\big(\mathrm{tr}\,(q(\mathfrak{l}_\mu\mathfrak{l}_\nu))\big).$$

It is quadratic character modulo the level and is equal to 1 *if the discriminant of* $\mathfrak{L}$ *is a square in* k.

Chapter II

Hecke operators and Brandt matrices

§ 5. Notations in Chapters II

k, as in § 1 and 2.

K a totally definite quaternion algebra over k.

$\mathfrak{M}_1, \ldots, \mathfrak{M}_H$ a system of representatives of left ideal classes with a common maximal left order $\mathfrak{O}_1$.

H the number of left ideal classes of K.

$\mathfrak{O}_1, \ldots, \mathfrak{O}_H$ the right orders of the $\mathfrak{M}$; some of which are in general isomorphic.

$\mathfrak{d}_K^2$ the discriminant of (each) $\mathfrak{O}_i$ with respect to $\mathfrak{o}$.

$\mathfrak{a} \sim \mathfrak{b}$: Two ideals $\mathfrak{a}$, $\mathfrak{b}$ of k are called *equivalent* if $\mathfrak{b} = \varrho\mathfrak{a}$ with a totally positive ϱ.

$\mathfrak{J}$ an ideal class of k. For an ideal $\mathfrak{a}$, the symbol $\mathfrak{J}\mathfrak{a}$ cannot be misunderstood.

$\mathfrak{g}_\mathfrak{J}$ a system of representatives of all ideal classes of k.

Ideals in K and k will be denoted by big and small german letters, elements in K by big italics. Norms from K to k are

$$N(M) = N_{K/k}(M),\qquad N(\mathfrak{M}) = N_{K/k}(\mathfrak{M}).$$

$\Gamma_0(\mathfrak{n},\mathfrak{a})$ for an integral ideal $\mathfrak{n}$ and an arbitrary ideal $\mathfrak{a}$ is the generalized Hilbert modular group of fractional linear substitutions

$$z \to \frac{\alpha z+\beta}{\gamma z+\delta}\quad\text{or}\quad z^i \to \frac{\alpha^i z^i+\beta^i}{\gamma^i z^i+\delta^i}$$

with

$$\alpha\delta-\beta\gamma = 1;\quad \alpha,\delta\in\mathfrak{o},\quad \beta\in\mathfrak{a}^{-1},\quad \gamma\in\mathfrak{n}\mathfrak{a}.$$

Elements of such groups will be denoted by big italics.

$G_k(\Gamma)$ the space of integral Hilbert modular forms of weight k with respect to the group Γ:

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\prod_{i=1}^{n}(\gamma^i z^i+\delta^i)^{-k} = f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)n(\gamma z+\delta)^{-k} = f(z).$$

A confusion between the two meanings of k (field and weight) has not to be feared.

We only deal with modular forms of character 1.

§ 6. Hecke operators. With an ideal $\mathfrak{m}$ and the different $\mathfrak{d}$ of k we form the order

$$M(\mathfrak{m}) = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}:\ \alpha,\beta\in\mathfrak{o},\ \beta\in\mathfrak{m}^{-1}\mathfrak{d}^{-1},\ \gamma\in\mathfrak{n}\mathfrak{m}\mathfrak{d}\right\}$$

in the matrix algebra $M_2(k)$. The *level* $\mathfrak{n}$ will be kept fixed throughout. With $\mathfrak{m} = \mathfrak{g}_\mathfrak{J}$, a system of representatives of the ideal classes of k, the $M(\mathfrak{g}_\mathfrak{J})$ represent all types of orders which are locally isomorphic with $M(\mathfrak{o})$. But they do not represent all types uniquely. This fact allows certain reductions of Hecke operators. But we will not deal with them here.

The groups $\Gamma_0(\mathfrak{n},\mathfrak{g}_\mathfrak{J}\mathfrak{d})$ are the groups of units of norm 1 of these orders. The notation seems inconsequent, but it is motivated by the applications to the theta functions later.

As usual we attach to a matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with totally positive determinant a linear operator $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}_k$ acting on modular forms $f(z)$ of weight k

as

$$(11) \qquad f(z)\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}_k = f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\frac{n(\alpha\delta-\beta\gamma)^{k/2}}{n(\gamma z+\delta)^k}.$$

We select a system of ideals

$$(12) \qquad \mathfrak{N}_{\mathfrak{J}} = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}(\mathfrak{o})\begin{pmatrix}\nu_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}$$

of norms

$$N(\mathfrak{N}_{\mathfrak{J}}) = \mathfrak{g}_{\mathfrak{J}} = \bigcap_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}\nu_{\mathfrak{p}}.$$

Their right orders are $M(\mathfrak{g}_{\mathfrak{J}})$. Now, for a given integral ideal $\mathfrak{m}$ of k, we consider all integral ideals $\mathfrak{M}_\nu$ $(\nu = 1, 2, \ldots)$ of norm $\mathfrak{m}$ and left order $M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}})$ which are left equivalent with $\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}^{-1}\mathfrak{N}_{\mathfrak{J}}$. They can be written

$$(13) \qquad \mathfrak{M}_\nu = \mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}^{-1}\mathfrak{N}_{\mathfrak{J}}N_\nu$$

with matrices N_ν whose determinants (norms) are totally positive and *equal among each other*. Forming the norms in (13) one gets because of the choice of the $\mathfrak{N}_{\mathfrak{J}}$

$$\mathfrak{m} \sim \frac{N(\mathfrak{N}_{\mathfrak{J}})}{N(\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}})} = \frac{\mathfrak{g}_{\mathfrak{J}}}{\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}}} \sim \frac{\mathfrak{J}}{\mathfrak{J}\mathfrak{m}^{-1}}.$$

With the matrices N_ν we define *Hecke operators* on $G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}}))$

$$(14) \qquad T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}) = n(\mathfrak{m})^{k/2-1}\sum_\nu [N_\nu]_k.$$

The N_ν are not uniquely determined by $\mathfrak{m}$. But a unit from $\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}})$ as a left factor of N_ν does not change the action of N_ν on an $f(z) \in G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}}))$. One can also multiply N_ν by $\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}$, with a totally positive unit ε of k, and thus (14) by $\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}_k$. If ε is a square of a unit η, the action on $f(z)$ is as

$$\begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}_k = \begin{bmatrix}\eta & 0\\ 0 & \eta^{-1}\end{bmatrix}_k \in \Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}}).$$

Otherwise

$$(15) \qquad U_k(\varepsilon) = \begin{bmatrix}\varepsilon & 0\\ 0 & 1\end{bmatrix}_k$$

is a unit operator, not the identity on $f(z)$. Obviously it commutes with the operators (14):

$$(16) \qquad U_k(\varepsilon)T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}) = T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})U_k(\varepsilon).$$

PROPOSITION 1. *$T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})$ maps $G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}}))$ into $G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}}))$.*

Proof. Multiplication of the $\mathfrak{M}_\nu$ on the right by elements $G \in \Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}})$ permutes them:

$$N_\nu G = G' N_{\nu'}$$

with a unit G' of $M(\mathfrak{g}_{\mathfrak{J}})$. Because $|G| = 1$ and the $|N_\nu|$ are equal, $|G'| = 1$ and thus $G' \in \Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}})$. For a function $f(z) \in G_k(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}})$ we have

$$f(z)T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})[G]_k = f(z)T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}).$$

PROPOSITION 2. *For relatively prime $\mathfrak{m}_1, \mathfrak{m}_2$ the products of the operators are*

$$T_k(\mathfrak{m}_1, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-1})T_k(\mathfrak{m}_2, \mathfrak{J}\mathfrak{m}_1^{-1}/\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}) = T_k(\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}).$$

Proof. Let $\mathfrak{M}_{1\nu}, \mathfrak{M}_{2\mu}$ be the ideals (13) with norms $\mathfrak{m}_1, \mathfrak{m}_2$ and left orders $M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}_1^{-1}})$, $M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}})$. Then

$$\mathfrak{M}_{3\mu\nu} = \mathfrak{N}_{\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}_1^{-1}}N_{2\mu}N_{2\mu}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}_1^{-1}}^{-1}\mathfrak{N}_{\mathfrak{J}}N_{1\nu}N_{2\mu}$$

are all these ideals with norms $\mathfrak{m}_1\mathfrak{m}_2$. Hence

$$T_k(\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}) = n(\mathfrak{m}_1\mathfrak{m}_2)^{k/2-1}\sum_{\mu,\nu}[N_{1\nu}]_k[N_{2\mu}]_k,$$

q.e.d.

For the extension of Proposition 2 to not relatively prime $\mathfrak{m}_1, \mathfrak{m}_2$ we need yet another operator

$$(17) \qquad V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2}) = [N]_k$$

for

$$(18) \qquad M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-2}})\mathfrak{m} = \mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-2}}^{-1}\mathfrak{N}_{\mathfrak{J}}N.$$

Also this is only defined up to a factor $U_k(\varepsilon)$. We have yet to show that the ideals on both sides of (18) are equivalent. This follows from the equivalence of their norms and the equality of their left orders.

PROPOSITION 3. *The $V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2})$ depend only on the class of $\mathfrak{m}$. They map $G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}}))$ on $G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-2}}))$, and they satisfy*

$$V_k(\mathfrak{m}_1, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-2})V_k(\mathfrak{m}_2, \mathfrak{J}\mathfrak{m}_1^{-2}/\mathfrak{J}\mathfrak{m}_1^{-2}\mathfrak{m}_2^{-2}) = V_k(\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-2}\mathfrak{m}_2^{-2}).$$

The proof is obvious.

PROPOSITION 4. *If $\mathfrak{p}$ is a prime ideal not dividing $\mathfrak{n}$, the following products are*

$$T_k(\mathfrak{p}^\varrho, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho})T_k(\mathfrak{p}^\sigma, \mathfrak{J}\mathfrak{p}^{-\varrho}/\mathfrak{J}\mathfrak{p}^{-\varrho-\sigma})$$
$$= \sum_{\tau=0}^{\min(\varrho,\sigma)} n(\mathfrak{p})^{(k-1)\tau}T_k(\mathfrak{p}^{\varrho+\sigma-2\tau}, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{2\tau-\varrho-\sigma})V_k(\mathfrak{p}^\tau, \mathfrak{J}\mathfrak{p}^{2\tau-\varrho-\sigma}/\mathfrak{J}\mathfrak{p}^{-\varrho-\sigma})$$

where the factors $U_k(\varepsilon)$ which were arbitrary in all operators now must be fixed in a suitable way.

Proof. We begin with the case $\sigma = 1$. Then $T_k(\mathfrak{p}^\varrho, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho})$ can be split up in the way

$$(19)\quad T_k(\mathfrak{p}^\varrho, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho}) = T'_k(\mathfrak{p}^\varrho, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho}) + n(\mathfrak{p})^{k-2} T_k(\mathfrak{p}^{\varrho-2}, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{2-\varrho}) V_k(\mathfrak{p}, \mathfrak{J}\mathfrak{p}^{2-\varrho}/\mathfrak{J}\mathfrak{p}^{-\varrho})$$

where T'_k means the sum (14) restricted to primitive ideals (13), i.e. those not divisible by $\mathfrak{p}$, and the rest. Of course, $T_k(\mathfrak{p}^{\varrho-2}, \ldots) = 0$ for $\varrho < 2$. Now let $\mathfrak{M}'_\nu$ run over all primitive ideals (13) of norm $\mathfrak{p}^\varrho$ and $\mathfrak{P}_\mu$ over all ideals of norm $\mathfrak{p}$ such that the products $\mathfrak{M}'_\nu\mathfrak{P}_\mu$ exist. Among the products occur also imprimitive ideals

$$\mathfrak{M}'_\nu\mathfrak{P}_\mu = \mathfrak{M}''_\varrho\mathfrak{p}.$$

In these cases we have $(\mathfrak{p}) = \mathfrak{P}'_\sigma\mathfrak{P}_\mu$ where $(\mathfrak{p})$ means the two-sided ideal with the appropriate left order. There exist exactly $n(\mathfrak{p})+1$ integral ideals of norm $\mathfrak{p}$ and given left order. Therefore, if $\varrho = 1$, we have $n(\mathfrak{p})+1$ imprimitive products $\mathfrak{M}'_\nu\mathfrak{P}_\mu$. But if $\varrho > 1$, we must exclude the cases when $\mathfrak{M}''_\varrho\mathfrak{P}'_\sigma$ is imprimitive. By a given $\mathfrak{P}_\mu$, $\mathfrak{P}'_\sigma$ is uniquely determined as the left factor of $(\mathfrak{p}) = \mathfrak{P}'_\sigma\mathfrak{P}_\mu$; and by a given $\mathfrak{M}''_\varrho$, $\mathfrak{P}'_\sigma$ is also uniquely determined such that $\mathfrak{M}''_\varrho\mathfrak{P}_\sigma$ is imprimitive. Hence, for $\varrho > 1$, there exist only $n(\mathfrak{p})$ imprimitive products $\mathfrak{M}'_\nu\mathfrak{P}_\mu$. These considerations, combined with the proof of Proposition 2 lead to

$$T'_k(\mathfrak{p}^\varrho, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho}) T_k(\mathfrak{p}, \mathfrak{J}\mathfrak{p}^{-\varrho}/\mathfrak{J}\mathfrak{p}^{-\varrho-1}) = T'_k(\mathfrak{p}^{\varrho+1}, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{-\varrho-1}) + \\ + \big(n(\mathfrak{p})+\varepsilon\big) n(\mathfrak{p})^{k-2} T'_k(\mathfrak{p}^{\varrho-1}, \mathfrak{J}/\mathfrak{J}\mathfrak{p}^{1-\varrho}) V_k(\mathfrak{p}, \mathfrak{J}\mathfrak{p}^{1-\varrho}/\mathfrak{J}\mathfrak{p}^{-\varrho-1})$$

with $\varepsilon = 1$ for $\varrho = 1$ and $\varepsilon = 0$ for $\varrho > 1$. If $\varrho = 1$, this formula is already that of Proposition 4. For $\varrho > 1$ we use induction on ϱ, leading to the claimed formula for $\sigma = 1$. At last we apply induction on σ. In both induction proofs the following is needed:

PROPOSITION 5. *The $U_k(\varepsilon)$ commute with all $T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})$, $V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2})$, and*

$$T_k(\mathfrak{m}_1, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_1^{-1}) V_k(\mathfrak{m}_2, \mathfrak{J}\mathfrak{m}_1^{-1}/\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}) = V_k(\mathfrak{m}_2, \mathfrak{J}/\mathfrak{J}\mathfrak{m}_2^{-2}) T_k(\mathfrak{m}_1, \mathfrak{J}\mathfrak{m}_2^{-2}/\mathfrak{J}\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-2}).$$

Proof. The first statement follows immediately from the definition. The second is the expression of the identity

$$\mathfrak{M}_1\mathfrak{m}_2 = \mathfrak{m}_2\mathfrak{M}_1.$$

The $T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})$, $V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2})$ are now ordered in quadratic schemes, indexed by the ideal classes of k:

$$(20)\qquad T_k(\mathfrak{m}) = \big(T_{\mathfrak{J}\mathfrak{K}}(m)\big), \qquad V_k(\mathfrak{m}) = \big(V_{\mathfrak{J}\mathfrak{K}}(\mathfrak{m})\big)$$

with

$$T_{\mathfrak{J}\mathfrak{K}}(\mathfrak{m}) = \begin{cases} T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}) & \text{if } \mathfrak{K} = \mathfrak{J}\mathfrak{m}^{-1}, \\ 0 & \text{otherwise;} \end{cases}$$

$$V_{\mathfrak{J}\mathfrak{K}}(\mathfrak{m}) = \begin{cases} V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2}) & \text{if } \mathfrak{K} = \mathfrak{J}\mathfrak{m}^{-2}, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2. *For relatively prime $\mathfrak{m}_1, \mathfrak{m}_2$ the product scheme is*

$$T_k(\mathfrak{m}_1) T_k(\mathfrak{m}_2) = T_k(\mathfrak{m}_1\mathfrak{m}_2)$$

and for a prime ideal $\mathfrak{p}$ not dividing the level $\mathfrak{n}$

$$T_k(\mathfrak{p}^\varrho) T_k(\mathfrak{p}^\sigma) = \sum_{\tau=0}^{\min(\varrho,\sigma)} n(\mathfrak{p})^{(k-1)\tau} T_k(\mathfrak{p}^{\varrho+\sigma-2\tau}) V_k(\mathfrak{p}^\tau)$$

if the factors $U_k(\varepsilon)$ which were arbitrary are now suitably fixed.

This is a consequence of Propositions 2–4. Formula (16) implies

THEOREM 3. *The operators $T_k(\mathfrak{m})$, $V_k(\mathfrak{m})$ map modular forms invariant under $U_k(\varepsilon)$ among each other.*

At last we consider the case of a square-free level $\mathfrak{n}$. Let $\mathfrak{q}$ be a prime ideal dividing $\mathfrak{n}$. There exists exactly one two-sided integral ideal $\mathfrak{Q}$ of norm $\mathfrak{q}$ with left order $M(\mathfrak{g}_\mathfrak{J})$. Its $\mathfrak{p}$-adic components are

$$\mathfrak{Q}_\mathfrak{p} = M_\mathfrak{p}(\mathfrak{g}_\mathfrak{J}) \text{ for } \mathfrak{p} \neq \mathfrak{q}, \quad \mathfrak{Q}_\mathfrak{q} = M_\mathfrak{q}(\mathfrak{g}_\mathfrak{J}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with

$$\alpha\delta - \beta\gamma \equiv 0 \bmod \mathfrak{q}, \ \not\equiv 0 \bmod \mathfrak{q}^2, \quad \text{and } \alpha \equiv \gamma \equiv \delta \equiv 0 \bmod \mathfrak{q}.$$

The sums (13), restricted to these ideals will be called $\hat{T}_k(\mathfrak{q}, \mathfrak{J}/\mathfrak{J}\mathfrak{q}^{-1})$ and the schemes (20) formed with them by $\hat{T}_k(\mathfrak{q})$.

THEOREM 4. *If the level $\mathfrak{n}$ is square-free, these operators $\hat{T}_k(\mathfrak{q})$ for prime ideals $\mathfrak{q}$ dividing $\mathfrak{n}$ can be formed. They commute among each other and with the $T_k(\mathfrak{m})$, $V_k(\mathfrak{m})$ with $\mathfrak{m}$ prime to $\mathfrak{n}$. Their squares are*

$$\hat{T}_k(\mathfrak{q})^2 = V_k(\mathfrak{q}).$$

§ 7. Brandt matrices. The elements M of a totally definite quaternion algebra K over k can be represented by matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha, \ldots \in k(\sqrt{-1})$. Let ξ_0, ξ_1 be the basis of a two-dimensional space over $k(\sqrt{-1})$, then

$$M\xi_0 = \alpha\xi_0 + \gamma\xi_1, \quad M\xi_1 = \beta\xi_0 + \delta\xi_1$$

defines an inverse representation $r_1(M) = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$. We now consider the action of M on the polynomials

$$\eta_\lambda = \frac{1}{\sqrt{\lambda!(l-\lambda)!}}\,\xi_0^{l-\lambda}\xi_1^\lambda \quad (\lambda = 0, \ldots, l).$$

This gives an inverse representation $r_l(M)$ of degree $l+1$.

The $r_l^i(M)$ denote the representations obtained from $r_l(M)$ by replacing the coefficients by its conjugates in the fields $k^i(\sqrt{-1})$. Then also

$$R_l(M) = r_l^1(M) \times \ldots \times r_l^n(M) \tag{21}$$

is an inverse representation of the multiplicative group of K.

Proposition 6. *The coefficients of $R_l(M)$ are homogeneous spherical harmonic polynomials of degree ln with respect to the variables η_μ^i ($i = 1, \ldots, n$; $\mu = 1, \ldots, 4$) defined by*

$$N_{K^i/k^i}(M^i) = (\eta_1^i)^2 + \ldots + (\eta_4^i)^2.$$

If ε is a unit of k of norm $n_{k/Q}(\varepsilon) = 1$,

$$R_l(\varepsilon M) = R_l(M). \tag{22}$$

The proof of the first statement has been given in [2] for $k = \boldsymbol{Q}$; it carries over to $k \supset \boldsymbol{Q}$ because the variables occurring in the $r_l^i(M)$ are independent. The second statement is obvious.

For a given pair ij and a given integral ideal $\mathfrak{m}$ of k we consider all integral ideals $\mathfrak{A}$ of the form

$$\mathfrak{A} = \mathfrak{M}_i^{-1}\mathfrak{M}_j A, \quad N(\mathfrak{A}) = \mathfrak{m}. \tag{23}$$

(Due to two-sided factors of $\mathfrak{A}$ the same ideal may occur with different pairs ij.) The element $A \in K$ is defined by a given $\mathfrak{A}$ up to a unit U_j of $\mathfrak{O}_j$. Now, because K is totally definite, every unit is

$$U_j = \varepsilon U_{j\nu} \tag{24}$$

with ε a unit of k of norm 1 and a unit $U_{j\nu}$ of $\mathfrak{O}_j$ from a finite set of, say, e_j elements. We define the *Brandt matrix* as

$$B_l(\mathfrak{m}) = {}^{*}\big(B_{ij}(\mathfrak{m})\big) \quad (i, j = 1, \ldots, H) \tag{25}$$

where the coefficients are $(l+1)^n$-rowed matrices

$$B_{ij}(\mathfrak{m}) = \sum R_l(A)\frac{1}{e_j}, \tag{26}$$

the sum extended over all integral ideals $\mathfrak{A}$ of norm $\mathfrak{m}$ and left order $\mathfrak{O}_i$ which are left equivalent with $\mathfrak{M}_i^{-1}\mathfrak{M}_j$, and for each $\mathfrak{A}$ over all A satisfying (23); but from a set εA with units ε of k of norm 1, only one such A is taken. Which A is taken does not matter because of (22). e_j means the index of the unit group of norm 1 of k in the unit group of $\mathfrak{O}_j$.

Similarly we define analogous matrices

$$A_{ij}(\mathfrak{a}) = n(\mathfrak{a})^{-l}\sum R_l(A)\frac{1}{e_j}, \quad A_l(\mathfrak{a}) = \big(A_{ij}(\mathfrak{a})\big) \tag{27}$$

for

$$\mathfrak{O}_i\mathfrak{a} = \mathfrak{M}_i^{-1}\mathfrak{M}_j A.$$

Proposition 7. *The $A_l(\mathfrak{a})$ depend only on the class of $\mathfrak{a}$, and $A_l(1)$ is idempotent. Their products are*

$$A_l(\mathfrak{a}_1)A_l(\mathfrak{a}_2) = A_l(\mathfrak{a}_1\mathfrak{a}_2). \tag{28}$$

Proof. The first statement is obvious. Secondly we have

$$A_l(1) = \operatorname{diag}\Big(\sum_\nu R_l(U_{i\nu})\frac{1}{e_i}\Big)$$

where diag means the diagonal matrix with these entries and $U_{i\nu}$ are as in (24). Now it is clear that $A_l^2(1) = A_l(1)$. The product (28) is obvious.

Theorem 5. *The Brandt matrices behave under multiplication as*

$$B_l(\mathfrak{m}_1)B_l(\mathfrak{m}_2) = B_l(\mathfrak{m}_1\mathfrak{m}_2)$$

for relatively prime $\mathfrak{m}_1, \mathfrak{m}_2$, and

$$B_l(\mathfrak{p}^\varrho)B_l(\mathfrak{p}^\sigma) = \sum_{\tau=0}^{\min(\varrho,\sigma)} n(\mathfrak{p})^{(l+1)\tau}B_l(\mathfrak{p}^{\varrho+\sigma-2\tau})A_l(\mathfrak{p}^\tau)$$

for prime ideals $\mathfrak{p}$ not dividing the discriminant $\mathfrak{d}_K^2$ of K, and lastly

$$B_l(\mathfrak{p})^2 = n(\mathfrak{p})^l A_l(\mathfrak{p})$$

for prime ideals $\mathfrak{p}$ dividing $\mathfrak{d}_K^2$.

The $A_l(\mathfrak{m}_1)$ and $B_l(\mathfrak{m}_2)$ commute.

Proof. The last statement is an expression of

$$\mathfrak{M}_1\mathfrak{m}_2 = \mathfrak{m}_2\mathfrak{M}_1.$$

The first can be literally taken over from [2], Ch. II, § 6, where $k = \boldsymbol{Q}$. This applies also to the second statement with a slight modification concerning the factors $\mathfrak{p}^\tau$ occurring in the multiplication: they are counted with the weights $n(\mathfrak{p})^{(l+1)\tau}A_l(\mathfrak{p}^\tau)$. (A similar situation occurred in Proposition 4.)

The fourth statement is equivalent to the fact that the square of a two-sided prime ideal $\mathfrak{P}$ of norm $\mathfrak{p}$ is equal to $\mathfrak{p}$.

§ 8. Symmetry properties. For two modular forms $f(z), g(z) \in G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_\mathfrak{J}))$ the scalar product is

(29) $$(f(z), g(z)) = \int f(z)\overline{g(z)} n(y^{2k-2} dx\, dy) \qquad (z = x+iy)$$

the integral extended over a fundamental domain of $\Gamma_0(\mathfrak{n}, \mathfrak{g}_\mathfrak{J})$; it converges if at least one of the $f(z)$, $g(z)$ is a cusp form.

For a $f(z) \in G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_\mathfrak{J}))$, $g(z) \in G_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}}))$ we form

$$(f(z)T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}), g(z)) = n(\mathfrak{m})^{k-1} \sum_\nu f(N_\nu(z))\overline{g(z)} n\left(\frac{y^{2k-2}dx\,dy}{(\gamma_\nu z+\delta_\nu)^k}\right)$$

integrated over the fundemental domain $\mathfrak{F}$ of $\Gamma_0(\mathfrak{n}, \mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}})$, where the N_ν are taken from (14). In each integral we substitute $N_\nu(z) = w$, and w now runs over $N_\nu^{-1}\mathfrak{F}$. With the abbreviation $w = u+iv$ the result is

$$\int_\mathfrak{F} f(N_\nu(z))\overline{g(z)} n\left(\frac{y^{2k-2}dx\,dy}{(\gamma_\nu z+\delta_\nu)^k}\right) = \int_{N_\nu^{-1}\mathfrak{F}} f(w)\overline{g(N_\nu^{-1}(w))} n\left(\frac{v^{2k-2}du\,dv}{(-\gamma_\nu w+\alpha_\nu)^k}\right).$$

Hence the adjoint operator is

(30) $$T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})^* = n(\mathfrak{m})^{k/2-1} \sum [N_\nu^{-1}]_k.$$

Similarly

(31) $$V_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-2})^* = [N^{-1}]_k$$

where the N_ν and N are taken from (13) and (18). It remains to express the right hand sides by the Hecke operators of § 6. It suffices to do this for (30) and "primitive" $T_k(\mathfrak{m})$ (in which the ideals (13) are primitive), because all $T_k(\mathfrak{m})$ are sums of primitive ones and such divisible by some $V_k(\mathfrak{m})$, as in (19).

With the ideals (13) the ideals

$$\mathfrak{M}'_\nu = M^{-1}\mathfrak{m} = N_\nu^{-1}\mathfrak{N}_\mathfrak{J}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}\mathfrak{m}$$

run over all integral ideals of norm $\mathfrak{m}$ and right order $M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}})$ which are right equivalent with $\mathfrak{N}_\mathfrak{J}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}\mathfrak{m}$. Then also the ideals

$$\mathfrak{M}''_\nu = N_\nu\mathfrak{M}'_\nu N_\nu^{-1} = \mathfrak{N}_\mathfrak{J}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}\mathfrak{m} N_\nu^{-1}$$

are integral, have norm $\mathfrak{m}$ and left order $M(\mathfrak{g}_\mathfrak{J})$, and are left equivalent with $\mathfrak{N}_\mathfrak{J}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}\mathfrak{m}$. The N_ν can be chosen in such a way that they represent all ideals (13) and simultaneously all $\mathfrak{M}''_\nu$ of these properties, one-to-one. Indeed, all N_ν in (13) can be obtained from one by multiplication on the right by units of $M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}})$

$$N_\nu = N_1 U_\nu, \qquad U_\nu \text{ units of } M(\mathfrak{g}_{\mathfrak{J}\mathfrak{m}^{-1}}).$$

For all these N_ν the $\mathfrak{M}''$ are equal. On the other hand, one can generate all N_ν^{-1} in $\mathfrak{M}''_\nu$ by multiplication on the right by units of $M(\mathfrak{g}_\mathfrak{J})$:

$$N_\nu^{-1} = N_1^{-1}V_\nu^{-1}, \qquad V_\nu \text{ units of } M(\mathfrak{g}_\mathfrak{J})$$

or in other words $N_\nu = V_\nu N_1$. All these N_ν yield the same ideal $\mathfrak{M}$. So we take

$$N_\nu = V_\nu N_1 U_\nu$$

and let U_ν, V_ν run simultaneously over the appropriate set of units. (For the existence of the U_ν, V_ν one uses the fact that units of a prescribed local behaviour at finitely many places exist [2a], Satz 5.)

Lastly let

$$\mathfrak{N}_{\mathfrak{J}\mathfrak{m}^{-1}}\mathfrak{m} = \mathfrak{N}_{\mathfrak{J}\mathfrak{m}} M.$$

Then

$$\mathfrak{M}'' = \mathfrak{N}_\mathfrak{J}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{m}} M N_\nu^{-1}$$

represents the same ideals as (13), with $\mathfrak{J}$ instead of $\mathfrak{J}\mathfrak{m}^{-1}$, and (30) means

$$V_k(\mathfrak{m}, \mathfrak{J}\mathfrak{m}/\mathfrak{J}\mathfrak{m}^{-1})T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})^* = T_k(\mathfrak{m}, \mathfrak{J}\mathfrak{m}/\mathfrak{J})$$

or

(32) $$T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})^* = V_k(\mathfrak{m}^{-1}, \mathfrak{J}\mathfrak{m}^{-1}/\mathfrak{J}\mathfrak{m})T_k(\mathfrak{m}, \mathfrak{J}\mathfrak{m}/\mathfrak{J}).$$

Eventually we use orthonormal bases $f_{\mathfrak{J},\nu}(z)$ of the spaces $S_k(\Gamma_0(\mathfrak{n}, \mathfrak{g}_\mathfrak{J}))$ of cusp forms and represent the $T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1})$ in these spaces

(33) $$f_{\mathfrak{J},\mu}(z)T_k(\mathfrak{m}, \mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1}) = \sum_{\mathfrak{K},\nu} f_{\mathfrak{K},\nu}(z)t_{\mathfrak{J}/\mathfrak{K},\mu\nu},$$

with

$$t_{\mathfrak{J}/\mathfrak{K},\mu\nu} = \begin{cases} t_{\mathfrak{J}/\mathfrak{J}\mathfrak{m}^{-1},\mu\nu} & \text{for } \mathfrak{K} = \mathfrak{J}\mathfrak{m}^{-1}, \\ 0 & \text{otherwise.}\end{cases}$$

The $t_{\mathfrak{J}\mathfrak{K},\mu\nu}$ form matrices

$$\mathfrak{T}_k(\mathfrak{m}) = (t_{\mathfrak{J}\mathfrak{K},\mu\nu})$$

representing the $T_k(\mathfrak{m})$ in the spaces of cusp forms. Similarly we form matrices $\mathfrak{B}_k(\mathfrak{m})$.

THEOREM 6. *The matrices* $\mathfrak{T}_k(\mathfrak{m})$, $\mathfrak{B}_k(\mathfrak{m})$ *enjoy the same properties as the operators* $T_k(\mathfrak{m})$, $V_k(\mathfrak{m})$ *in Theorem 2. Moreover their Hermitean adjoints satisfy*

(34) $$\mathfrak{T}_k(\mathfrak{m})^* = \mathfrak{B}_k(\mathfrak{m}^{-1})\mathfrak{T}_k(\mathfrak{m}), \qquad \mathfrak{B}_k(\mathfrak{m})^* = \mathfrak{B}_k(\mathfrak{m}^{-1}).$$

They can be transformed simultaneously into diagonal form, and the eigenvalues of the $\mathfrak{T}_k(\mathfrak{m})$ *are real if and only if the square of each ideal of* k *is principal.*

Proof. The second formula (34) is obtained in the same way, with obvious simplifications. Because of this formula and the commutativity of the $\mathfrak{B}_k(\mathfrak{m})$, they can be transformed simultaneously into diagonal form with an unitary matrix $\mathfrak{C}$. Put

$$\mathfrak{C}^{-1}\mathfrak{B}_k(\mathfrak{m})\mathfrak{C} = \mathfrak{D}_k(\mathfrak{m}), \quad \mathfrak{C}^{-1}\mathfrak{T}_k(\mathfrak{m})\mathfrak{C} = \mathfrak{T}'_k(\mathfrak{m})$$

where $\mathfrak{D}_k(\mathfrak{m}) = \operatorname{diag}(\varrho(\mathfrak{m}))$, and the $\varrho(\mathfrak{m})$ are roots of unity. Since the $\mathfrak{D}_k(\mathfrak{m})$, $\mathfrak{T}'_k(\mathfrak{m})$ commute, also the $\mathfrak{D}'_k(\mathfrak{m}) = \operatorname{diag}(\sqrt{\varrho(\mathfrak{m})})$ commute with all $\mathfrak{T}'_k(\mathfrak{m})$. With these matrices we form

$$\mathfrak{T}''_k(\mathfrak{m}) = \mathfrak{T}'_k(\mathfrak{m})\mathfrak{D}'_k(\mathfrak{m})^{-1}.$$

The adjoint matrices are

$$\mathfrak{T}''_k(\mathfrak{m})^* = \mathfrak{D}'_k(\mathfrak{m})\mathfrak{T}'_k(\mathfrak{m})^*$$

which is because of (34)

$$= \mathfrak{D}'_k(\mathfrak{m})^{-1}\mathfrak{T}'_k(\mathfrak{m}) = \mathfrak{T}''_k(\mathfrak{m}).$$

Thus the $\mathfrak{T}''_k(\mathfrak{m})$ are Hermitean and can be transformed simultaneously with the $\mathfrak{D}_k(\mathfrak{m})$ into diagonal form.

The eigenvalues of the $\mathfrak{T}_k(\mathfrak{m})$ are those of the $\mathfrak{T}''_k(\mathfrak{m})$ multiplied by the $\sqrt{\varrho(\mathfrak{m})}^{-1}$. They are real iff $\varrho(\mathfrak{m}) = 1$, and this is the case iff $m^2 \sim 1$. This completes the proof.

§ 9. Brandt matrices and theta functions. In § 9 we will show that the Brandt matrices are the representations of the Hecke operators in the spaces of Kloosterman–Schoeneberg theta functions. This is known in the case $k = \boldsymbol{Q}$, and the proof follows the same line (see [2], Ch. II, § 6).

Referring to the representatives $\mathfrak{M}_i$ of the left ideal classes of K and the representatives $\mathfrak{g}_{\mathfrak{J}}$ of the ideal classes of k we put

$$\mu_{ij}\frac{N(\mathfrak{M}_i)}{N(\mathfrak{M}_j)} = \mathfrak{g}_{ij} \tag{35}$$

where $\mathfrak{g}_{ij}$ is the class representative so defined. The numbers μ_{ij} are not uniquely determined by (35). Indeed, they can be multiplied by any totally positive units of k. But we fix them for all pairs ij. Due to Proposition 6 (§ 7) the Fourier series

$$\Theta_{ij}(z) = \sum_{A\in\mathfrak{M}_j^{-1}\mathfrak{M}_i} R_l(A)\frac{1}{e_j}e\big(\mu_{ij}N(A)\big) \tag{36}$$

(where the abbreviation

$$e(X) = e^{2\pi i\operatorname{tr}(zX)}$$

different as in § 4, is used) are theta functions, and Theorem 1 (§ 4) states that they are modular forms of weight $l+2$:

$$\Theta_{ij}(z)\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}_{l+2} = \Theta_{ij}(z) \quad \text{for} \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{ij}\mathfrak{d}) \tag{37}$$

where $\mathfrak{d}_K^2$ is the discriminant of K/k ($\mathfrak{d}_K$ is the level of the norm form of K) and $\mathfrak{d}$ the different of k.

With

$$\mathfrak{A} = \mathfrak{M}_i^{-1}\mathfrak{M}_j A, \quad N(\mathfrak{A}) = \mathfrak{m}, \quad \big(N(A)\big) = \mathfrak{m}N(\mathfrak{M}_j^{-1}\mathfrak{M}_i)$$

as in (23), we can write

$$\mu_{ij}N(A) = \mathfrak{g}_{ij}\mathfrak{m} \tag{38}$$

and thus

$$\Theta_{ij}(z) = \sum B_{ij}(\mathfrak{m})e(\mathfrak{g}_{ij}\mathfrak{m}) \tag{39}$$

summed over all integral $\mathfrak{m}$ in the class of $\mathfrak{g}_{ij}^{-1}$. The expression $\mathfrak{g}_{ij}\mathfrak{m}$ is not unique, and it has to be interpreted in the sense of (38), with μ_{ij} fixed. In the sum (39) we must also include $\mathfrak{m} = 0$. For $l > 0$, the definition gives $B_l(0) = 0$. For $l = 0$ we supplement the definition by

$$B_{ij}(0) = \begin{Bmatrix}\dfrac{1}{e_i} & \text{for} & i = j\\ 0 & \text{for} & i \neq j\end{Bmatrix} \quad \text{if} \quad l = 0. \tag{40}$$

Eventually the $\Theta_{ij}(z)$ are grouped together in the matrix

$$\Theta(z) = \big(\Theta_{ij}(z)\big) \tag{41}$$

of $H(l+1)^n$ rows.

This matrix is in turn split up into the sum

$$\Theta(z) = \sum_{\mathfrak{J}}\Theta_{\mathfrak{J}}(z) \tag{42}$$

with

$$\Theta_{\mathfrak{J}}(z) = \sum_{\mathfrak{m}\in\mathfrak{J}^{-1}} B_{\mathfrak{J}}(\mathfrak{m})e(\mathfrak{g}_{\mathfrak{J}}\mathfrak{m}) \tag{43}$$

where the $B_{\mathfrak{J}}(\mathfrak{m})$ has the ($(l+1)^n$ rowed) coefficients

$$B_{\mathfrak{J},ij}(\mathfrak{m}) = \begin{cases}B_{ij}(\mathfrak{m}) & \text{if } \mathfrak{g}_{ij} = \mathfrak{g}_{\mathfrak{J}},\\ 0 & \text{otherwise}.\end{cases} \tag{44}$$

The $\Theta_{\mathfrak{J}}(z)$ are invariant under $\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$.

Now we study the behaviour of $\Theta(z)$ under multiplication by Brandt matrices $B(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ not dividing the level and under the action of the Hecke operator $T(\mathfrak{p})$. The subscript l indicating the weight will be omitted in § 9, but we need a subscript indicating the class to which the argument is supposed to belong. This means that we have to write $B_{\mathfrak{J}^{-1}}(\mathfrak{m})$ if we want $\mathfrak{m}$ to belong to $\mathfrak{J}^{-1}$.

The multiplicative properties of Theorem 5 are

$$B(\mathfrak{p})B(\mathfrak{m}) = B(\mathfrak{m}\mathfrak{p}) + n(\mathfrak{p})^{l+1}A(\mathfrak{p})B(\mathfrak{m}\mathfrak{p}^{-1})$$

where the second term is only present if $\mathfrak{m}$ is divisible by $\mathfrak{p}$. In our new notation we write this

$$B_{\mathfrak{p}-1}(\mathfrak{p})B_{\mathfrak{J}}(\mathfrak{m}) = B_{\mathfrak{J}\mathfrak{p}-1}(\mathfrak{m}\mathfrak{p})+n(\mathfrak{p})^{l+1}A_{\mathfrak{p}-2}(\mathfrak{p})B_{\mathfrak{J}\mathfrak{p}}(\mathfrak{m}\mathfrak{p}^{-1})$$

and

(45) $$B_{\mathfrak{p}-1}(\mathfrak{p})\Theta_{\mathfrak{J}}(z) = \Theta_{\mathfrak{J}}^1(z)+n(\mathfrak{p})^{l+1}A_{\mathfrak{p}-2}(\mathfrak{p})\Theta_{\mathfrak{J}}^2(z)$$

with

$$\Theta_{\mathfrak{J}}^1(z) = \sum_{\mathfrak{m}\in\mathfrak{J}^{-1}} B_{\mathfrak{J}\mathfrak{p}-1}(\mathfrak{m}\mathfrak{p})e(\mathfrak{g}_{\mathfrak{J}}\mathfrak{m}),$$

$$\Theta_{\mathfrak{J}}^2(z) = \sum_{\mathfrak{m}\in\mathfrak{J}^{-1}} B_{\mathfrak{J}\mathfrak{p}}(\mathfrak{m}\mathfrak{p}^{-1})e(\mathfrak{g}_{\mathfrak{J}}\mathfrak{m}).$$

Both functions can be written in another way. Put

(46) $$(\pi_{\mathfrak{J}}) = \frac{\mathfrak{p}\,\mathfrak{g}_{\mathfrak{J}}}{\mathfrak{g}_{\mathfrak{J}\mathfrak{p}}}, \quad \pi_{\mathfrak{J}} \gg 0$$

and $\mathfrak{m}\mathfrak{p} = \mathfrak{p}'$. Then

$$\mathfrak{g}_{\mathfrak{J}}\mathfrak{m} = \pi_{\mathfrak{J}\mathfrak{p}-1}^{-1}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}\mathfrak{m}'.$$

The left term is more exactly $\mu_{ij}N(A)$, and the right one accordingly. So the $\pi_{\mathfrak{J}}$ are uniquely fixed by the μ_{ij}. The first summand in (45) is

(47) $$\Theta_{\mathfrak{J}}^1(z) = n(\mathfrak{p})^{-1}n(\pi_{\mathfrak{J}\mathfrak{p}-1})^{l/2+1}\Theta_{\mathfrak{J}\mathfrak{p}-1}(z)\sum_{\nu}[P_\nu]_{l+2}$$

with

$$P_\nu = \begin{pmatrix}1 & \nu\\ 0 & \pi_{\mathfrak{J}\mathfrak{p}-1}\end{pmatrix}, \quad \nu \in \mathfrak{g}_{\mathfrak{J}}^{-1}\mathfrak{d}^{-1} \bmod \mathfrak{g}_{\mathfrak{J}}^{-1}\mathfrak{d}^{-1}\mathfrak{p}.$$

The second summand is, up to the factor $n(\mathfrak{p})^{l+1}A_{\mathfrak{p}-2}(\mathfrak{p})$

(48) $$\Theta_{\mathfrak{J}}^2(z) = n(\pi_{\mathfrak{J}})^{-l/2-1}\Theta_{\mathfrak{J}\mathfrak{p}}(z)[Q']_{l+2}, \quad Q' = \begin{pmatrix}\pi_{\mathfrak{J}} & 0\\ 0 & 1\end{pmatrix}.$$

It remains to express (45) by Hecke operators. In the following considerations we use the orders $M(\mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$ in the matrix algebra $M_2(k)$, defined in the beginning of § 6, with level $\mathfrak{n} = \mathfrak{d}_K$. The ideals $\mathfrak{N}_{\mathfrak{J}}$ are defined in (12). At first we construct a matrix $P = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with the following properties ([2a], Satz 5):

$$\alpha \in \mathfrak{g}_{\mathfrak{J}}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}}^{-1}, \quad \beta \in \mathfrak{g}_{\mathfrak{J}}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}}^{-1}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}^{-1}\mathfrak{d}^{-1}, \quad (|P|) = \mathfrak{g}_{\mathfrak{J}}^2\mathfrak{g}_{\mathfrak{J}\mathfrak{p}}^{-1}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}^{-1}$$

$$\gamma \in \mathfrak{d}_K\mathfrak{g}_{\mathfrak{J}}\mathfrak{d}, \quad \delta \in \mathfrak{g}_{\mathfrak{J}}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}^{-1}, \quad \text{and } |P| \gg 0.$$

We verify by easy local considerations

(49) $$M(\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}\mathfrak{d})\mathfrak{g}_{\mathfrak{J}}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}^{-1} = \mathfrak{N}_{\mathfrak{J}\mathfrak{p}-1}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{p}}P,$$

and we form the ideals

(50) $$\mathfrak{M}_\nu = \mathfrak{N}_{\mathfrak{J}}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{p}}PP_\nu$$

for the different P_ν in (47). They are different, integral, have norm $\mathfrak{p}$ and left order $M(\mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$, and they are left equivalent with $\mathfrak{N}_{\mathfrak{J}}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{p}}$. The last ideal of this kind is

(51) $$\mathfrak{M}' = \mathfrak{N}_{\mathfrak{J}}^{-1}\mathfrak{N}_{\mathfrak{J}\mathfrak{p}}Q'$$

with Q' from (48). Therefore the $\mathfrak{M}_\nu$ and $\mathfrak{M}'$ make up all ideals used in the definition of the Hecke operator $T(\mathfrak{p}, \mathfrak{J}\mathfrak{p}/\mathfrak{J})$, and we have

$$T(\mathfrak{p}, \mathfrak{J}\mathfrak{p}/\mathfrak{J}) = n(\mathfrak{p})^{l/2}\Big(\sum [PP_\nu]_{l+2}+[Q']_{l+2}\Big).$$

This equation is multiplied on the left by $V(\mathfrak{p}^{-1}, \mathfrak{J}\mathfrak{p}^{-1}/\mathfrak{J}\mathfrak{p})$. Because this operator only depends on the class of $\mathfrak{p}$, and $\mathfrak{p} \sim \mathfrak{g}_{\mathfrak{J}}\mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}^{-1}$,

$$V(\mathfrak{p}^{-1}, \mathfrak{J}\mathfrak{p}^{-1}/\mathfrak{J}\mathfrak{p}) = [P^{-1}]_{l+2}$$

with P from (49). Thus we have

$$V(\mathfrak{p}^{-1}, \mathfrak{J}\mathfrak{p}^{-1}/\mathfrak{J}\mathfrak{p})T(\mathfrak{p}, \mathfrak{J}\mathfrak{p}/\mathfrak{J}) = n(\mathfrak{p}^{l/2})\Big(\sum_\nu [P_\nu]_{l+2}+[P']_{l+2}\Big)$$

with $P' = P^{-1}Q'$, an operator which transforms a modular form for $\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}\mathfrak{d})$ into one for $\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$. Equations (45), (47), (48) allow the following expression:

$$B_{\mathfrak{p}-1}(\mathfrak{p})\Theta_{\mathfrak{J}}(z)-n(\pi_{\mathfrak{J}\mathfrak{p}-1})^{l/2+1}\Theta_{\mathfrak{J}\mathfrak{p}-1}(z)V(\mathfrak{p}^{-1}, \mathfrak{J}\mathfrak{p}^{-1}/\mathfrak{J}\mathfrak{p})T(\mathfrak{p}, \mathfrak{J}\mathfrak{p}/\mathfrak{J})$$
$$= n(\pi_{\mathfrak{J}})^{-l/2-1}A_{\mathfrak{p}-2}(\mathfrak{p})n(\mathfrak{p})^{l+1}\Theta_{\mathfrak{J}\mathfrak{p}}(z)[Q']_{l+2}-n(\mathfrak{p})^{-1}n(\pi_{\mathfrak{J}\mathfrak{p}-1})^{l/2+1}\Theta_{\mathfrak{J}\mathfrak{p}-1}(z)[Q']_{l+2}.$$

Both functions on the left are invariant under $\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$. On the right hand side, both functions are invariant under $Q'^{-1}\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}\mathfrak{p}}\mathfrak{d})Q'$ (note that $P\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}\mathfrak{p}-1}\mathfrak{d})P^{-1} = \Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}\mathfrak{p}}\mathfrak{d})$ as results from (49).) Since the composite of the groups $\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}}\mathfrak{d})$ and $Q'^{-1}\Gamma_0(\mathfrak{d}_K, \mathfrak{g}_{\mathfrak{J}\mathfrak{p}}\mathfrak{d})Q'$ is not discontinuous, both differences vanish, and we have proved

THEOREM 7. *If the factors $U_{l+2}(\varepsilon)$, which were arbitrary in the definition of the Hecke operators in § 6, are properly adjusted, the Brandt matrices and the Hecke operators are linked by*

(52) $$B_l(\mathfrak{p})\Theta_l(z) = \Theta_l(z)G_{l+2}V_{l+2}(\mathfrak{p}^{-1})T_{l+2}(\mathfrak{p})G_{l+2}^{-1},$$

(53) $$A_l(\mathfrak{p})\Theta_l(z) = \Theta_l(z)G_{l+2}V_{l+2}(\mathfrak{p}^{-1})G_{l+2}^{-1}$$

with

$$G_{l+2} = \operatorname{diag}\left(n(\mathfrak{g}_3)^{l/2+1}\right)$$

for a prime ideal $\mathfrak{p}$ *not dividing the discriminant* $\mathfrak{d}_K$ *of* K/k.

Equation (52) also holds for the prime divisors of $\mathfrak{d}_K$. We leave the proof to the reader, if he is interested.

The $U_{l+2}(\varepsilon)$ are often equal to the identity, namely either if the totally positive units of k are squares of other units, or if they are norms of units of every maximal order of K.

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Corrigendum to the paper "Elementary methods in the theory of L-functions, VII. Upper bound for $L(1,\chi)$"

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by

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In the course of the proof of Lemma 1 the inequality

$$(1)\qquad c(n) = 1 - \sum_{j=1}^{s} \frac{\beta_j}{\beta_j+1} \geqslant \prod_{j=1}^{s} \frac{1}{\beta_j+1} = \frac{1}{d(b)}$$

on p. 401, line (-4) is obviously incorrect and also Lemma 1 is not valid in the given form (it is only valid for the special case $\theta(d) \neq 0$ which corresponds to the case $D = p$ prime). This fault may be corrected in the following way.

We shall show that on adding to the right side of (2.4)

$$(2)\qquad \frac{1}{2}\sum_{q\in Q}\sum_{\substack{q\mid n\\ n\leqslant x}} d\left(\frac{n}{q}\right) + \frac{1}{2}\sum_{q\in Q}\sum_{\substack{q'\in Q\\ q'\neq q}}\sum_{\substack{qq'\mid n\\ n\leqslant x}} d\left(\frac{n}{qq'}\right)$$

Lemma 1 is already valid.

After Lemma 1 we added the sum

$$(3)\qquad \frac{1}{2}\sum_{q\in Q}\sum_{\substack{q'\in Q\\ qq'\leqslant x}} \frac{x}{qq'}\log\frac{x}{qq'}$$

to the right of (2.4) (p. 403, line 3) to make more handy the expression and now the same sum occurs on the right of (2.4) (apart from an insignificant error term $o(x\log x)$), thus the Corollary and all the later parts of the proof remain unchanged.